EXPONENTIAL NONNEGATIVITY

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Abstract. Let $A$ be a Banach algebra, $a \in A$, $\sigma(a)$ the spectrum of $a$ and $\tau(a)$ the spectral abscissa of $a$. If $\tau(a) \in \sigma(a)$, then we show that there exists an algebra cone $C \subseteq A$ such that $a$ is exponentially nonnegative with respect to $C$ and the spectral radius is increasing on $C$.

Let $A$ be a complex Banach algebra with unit $e$, $\|e\| = 1$. A closed convex and positively homogeneous set $C \subseteq A$ is called an algebra cone if $C \cdot C \subseteq C$ and $e \in C$. If, in addition, $C \cap (-C) = \{0\}$ is valid, then the cone $C$ is called proper. The order induced in $A$ by $C$ is denoted by $\leq$. We denote the spectrum of $a \in A$ by $\sigma(a)$ or occasionally by $\sigma(a; A)$, the spectral radius of $a$ by $r(a)$ and the spectral abscissa of $a$ by $\tau(a) := \max\{\Re \lambda : \lambda \in \sigma(a)\}$. If $a \in Q(C) := \{u \in A : e^{tu} \geq 0, t \in \mathbb{R}^+\}$, then $a$ is called exponentially nonnegative or quasipositive.

In [3], Theorem 2, Herzog and Schmoeger proved the converse of a Banach algebra version of the Perron-Frobenius Theorem: Let $a \in A$, $r(a) \in \sigma(a)$. Then there exists a cone $C \subseteq A$ with the property: $a \in C$ and the spectral radius is increasing on $C$. In the matrix case, the converse of the Perron-Frobenius Theorem was already treated in 1968 by Vandergraft (cf. [8] and [2], p. 8).

In the present paper we extend the result of Herzog and Schmoeger to exponentially nonnegative elements of a Banach algebra $A$. The matrix version of this extension was proved in 1970 by Elsner (cf. [3] and [1], p. 80).

I thank the reviewer for his suggestion to treat Example 3.

Theorem. Let $a \in A$. Then the following conditions $\alpha)$ and $\beta)$ are equivalent.

$\alpha)$ There exists an algebra cone $C \subseteq A$ such that $a \in Q(C)$ and the spectral radius is increasing on $C$: $0 \leq u \leq v \Rightarrow r(u) \leq r(v)$.

$\beta)$ $\tau(a) \in \sigma(a)$.

Proof. “$\alpha) \Rightarrow \beta)$” (cf. [1], Theorem 1, (12)): Since $Q(C)$, $\sigma(a)$ and $\tau(a)$ are positively homogeneous, it is no restriction to assume $\sigma(a) \subseteq K := \{z \in \mathbb{C} : |z| \leq 1\}$. Then $\tau(a) \in K$ follows. Because $a \in Q(C)$ and according to the definition of $Q(C)$ we have $\exp(a) \geq 0$ and therefore $r(\exp(a)) \in \sigma(\exp(a)) = \exp(\sigma(a))$ is valid (cf. [3], Theorem 5.2). From this we conclude that $\exp(r(a)) = r(\exp(a)) \in \exp(\sigma(a))$ is true. Since the exponential function is injective on $K$, $\tau(a) \in \sigma(a)$ follows.

“$\beta) \Rightarrow \alpha)$” Let $a \neq 0$ and $b := \frac{1}{\|a\|}a$. From $\tau(a) \in \sigma(a)$, hence $\tau(b) \in \sigma(b)$, it follows that $\exp(\tau(b)) \in \exp(\sigma(b)) = \sigma(\exp(b))$ and thus $r(\exp(b)) = \exp(\tau(b)) \in$
Lemma. Let $a \in \mathbb{C}^{d \times d}$. Then $\alpha$ and $\beta$ are equivalent. 

$\alpha$) The eigenvalues of $a$ are simple. 

We only sketch the steps of the elementary proof.

$\alpha \Rightarrow \beta$): From $\lambda \in \sigma(a) : ax = \lambda x$, $x \neq 0$ we get $a \cdot nx = n \cdot ax = \lambda nx$ and since $\lambda$ is simple $nx = gx$, $g \in \mathbb{R}$ follows. Now $n$ being nilpotent, the equality $0 = n^l x = g^l x$ for a suitable $l \in \mathbb{N}$ and therefore $g = 0$ result. Hence $nx = 0$ and since $\mathbb{C}^d$ is the linear hull of eigenvectors corresponding to the $d$ different eigenvalues we conclude that $n = 0$.

$\beta \Rightarrow \alpha$): Assume $a$ not to satisfy $\alpha$). If
\[ J := \begin{pmatrix} \lambda & \delta & 0 & \cdots & 0 \\ \lambda & \delta & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \delta & \lambda \end{pmatrix}, \quad \delta = 0 \text{ or } \delta = 1 \]

is a \( k \times k \) Jordan block (\( k > 1 \)) corresponding to \( \lambda \in \sigma(a) \), then the \( k \times k \) matrix

\[ n_k := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & \ddots & \vdots & \vdots \\ & 0 & \cdots & \cdots \\ & & 0 & \end{pmatrix} \]

is nilpotent and commutes with \( J \). Using this observation it is easy to construct a nilpotent \( n \in \mathbb{C}^{d \times d} \), \( n \neq 0 \) that commutes with \( a \), a contradiction to assumption \( \beta \).

**Corollary.** If the eigenvalues of \( a \in \mathbb{C}^{d \times d} \) are simple, then the commutative algebra \( B := \Gamma(a) \subseteq \mathbb{C}^{d \times d} \) is semisimple and the cone \( C \subseteq B \) is proper. \[ \square \]

**Example 2.** Let \( a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{C}^{2 \times 2} \) satisfy \( \beta \cdot \gamma \neq 0 \). Using the abbreviations

\[ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \beta \\ \gamma & \delta - \alpha \end{pmatrix} \quad \text{and} \quad D = (\alpha - \delta)^2 + 4\beta \gamma \]

the following is true: \( B := \Gamma(a) \) is a two-dimensional subalgebra of \( \mathbb{C}^{2 \times 2} \):

\[ B = \{ xe + yb : x, y \in \mathbb{C} \} \]

and

\[ D \neq 0 \Rightarrow (\sigma(xe + yb) = \{0\} \Leftrightarrow x = y = 0) \]

\[ \Rightarrow C \cap (-C) = \text{rad}(B) = \{0\}, \]

hence the cone \( C \) is proper;

\[ D = 0 \Rightarrow (\sigma(xe + yb) = \{0\} \Leftrightarrow y = \frac{2}{\alpha - \delta} x) \]

\[ \Rightarrow C \cap (-C) = \text{rad}(B) = \{x(e + \frac{2}{\alpha - \delta} b) : x \in \mathbb{C} \} \neq \{0\}, \]

hence the cone \( C \) is not proper. \[ \square \]

**Example 3.** Let \( A := \mathbb{C}^{3 \times 3} \) and

\[ a = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Then \( \sigma(a) = \{-i, 0, i\} \). According to Theorem 5.2 in \([4]\) there is no algebra cone in \( A \) on which the spectral radius is increasing and which contains \( a \). Because of \( \tau(a) = 0 \in \sigma(a) \) it follows from the theorem treated in the present paper, that there is an algebra cone \( C \subseteq A \) such that \( a \in Q(C) \) and the spectral radius is increasing on \( C \). Moreover, by the corollary of the lemma in Example 1 of the present paper, \( C \) is a proper cone.
Using the following notation,
\[ e_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
we find that
\[ \alpha) \ B = \Gamma(\Gamma(a)) \] is a three-dimensional subalgebra of \( A = \mathbb{C}^{3\times3} \):
\[ B = \{ xe_1 + ye_2 + xa : x, y, z \in \mathbb{C} \} \subseteq A \]
and
\[ e^{ta} = \cos t \cdot e_1 + e_2 + \sin t \cdot a \in B, \ t \in \mathbb{R}^+; \]
\( \beta) \) the multiplicative functional \( \varphi \) on \( B \) defined by
\[ \varphi(e_1) = 0, \ \varphi(e_2) = 1, \ \varphi(a) = 0 \]
has the property
\[ \varphi(e^a) = e^{\varphi(a)} = 1 = r(e^a) \]
(the last equality follows from \( \sigma(e^a) = e^{\sigma(a)} \));
\( \gamma) \) the cone \( C = \{ u \in B : \varphi(u) = r(u) \} \subseteq B \) is described explicitly by
\[ C = \{ u = xe_1 + ye_2 + za \in B : x, y, z \in \mathbb{C} \text{ and } y \geq |x \pm iz| \}. \]
Since \( e^{ta} = \cos t \cdot e_1 + e_2 + \sin t \cdot a, \ t \in \mathbb{R}^+ \), it follows that \( 1 = y \geq |\cos t \pm i \cdot \sin t| = |e^{\pm it}| = 1 \), and therefore \( e^{ta} \in C, \ t \in \mathbb{R}^+. \)

References

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