PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 6, Pages 1775–1778
S 0002-9939(03)07297-6
Article electronically published on October 15, 2003

EXPONENTIAL NONNEGATIVITY

HERBERT WEIGEL

(Communicated by Joseph A. Ball)

Abstract. Let A be a Banach algebra, $a \in A$, $\sigma(a)$ the spectrum of a and $\tau(a)$ the spectral abscissa of a. If $\tau(a) \in \sigma(a)$, then we show that there exists an algebra cone $C \subseteq A$ such that a is exponentially nonnegative with respect to C and the spectral radius is increasing on C.

Let A be a complex Banach algebra with unit e, $||e|| = 1$. A closed convex and positively homogeneous set $C \subseteq A$ is called an algebra cone if $C \cdot C \subseteq C$ and $e \in C$. If, in addition, $C \cap (-C) = \{0\}$ is valid, then the cone C is called proper. The order induced in A by C is denoted by $\leq$. We denote the spectrum of $a \in A$ by $\sigma(a)$ or occasionally by $\sigma(a; A)$, the spectral radius of a by $r(a)$ and the spectral abscissa of a by $\tau(a) := \max\{Re\,\lambda : \lambda \in \sigma(a)\}$. If $a \in Q(C) := \{u \in A : e^{tu} \geq 0, t \in \mathbb{R}^+\}$, then a is called exponentially nonnegative or quasipositive.

In [5], Theorem 2, Herzog and Schmoeger proved the converse of a Banach algebra version of the Perron-Frobenius Theorem: Let $a \in A$, $r(a) \in \sigma(a)$. Then there exists a cone $C \subseteq A$ with the property: $a \in C$ and the spectral radius is increasing on C. In the matrix case, the converse of the Perron-Frobenius Theorem was already treated in 1968 by Vandergraft (cf. [8] and [2], p. 8).

In the present paper we extend the result of Herzog and Schmoeger to exponentially nonnegative elements of a Banach algebra A. The matrix version of this extension was proved in 1970 by Elsner (cf. [3] and [1], p. 80).

I thank the reviewer for his suggestion to treat Example 3.

Theorem. *Let $a \in A$. Then the following conditions α) and β) are equivalent.*

- α) *There exists an algebra cone $C \subseteq A$ such that $a \in Q(C)$ and the spectral radius is increasing on C: $0 \leq u \leq v \Rightarrow r(u) \leq r(v)$.*
- β) $\tau(a) \in \sigma(a)$.

Proof. "$\alpha) \Rightarrow \beta)$" (cf. [4], Theorem 1, (12)): Since $Q(C)$, $\sigma(a)$ and $\tau(a)$ are positively homogeneous, it is no restriction to assume $\sigma(a) \subseteq K := \{z \in \mathbb{C} : |z| \leq 1\}$. Then $\tau(a) \in K$ follows. Because $a \in Q(C)$ and according to the definition of $Q(C)$ we have $\exp(a) \geq 0$ and therefore $r(\exp(a)) \in \sigma(\exp(a)) = \exp(\sigma(a))$ is valid (cf. [6], Theorem 5.2). From this we conclude that $\exp(\tau(a)) = r(\exp(a)) \in \exp(\sigma(a))$ is true. Since the exponential function is injective on K, $\tau(a) \in \sigma(a)$ follows.

"$\beta) \Rightarrow \alpha)$": Let $a \neq 0$ and $b := \frac{1}{||a||}a$. From $\tau(a) \in \sigma(a)$, hence $\tau(b) \in \sigma(b)$, it follows that $\exp(\tau(b)) \in \exp(\sigma(b)) = \sigma(\exp(b))$ and thus $r(\exp(b)) = \exp(\tau(b)) \in$

Received by the editors October 25, 2002 and, in revised form, February 14, 2003.
2000 *Mathematics Subject Classification.* Primary 45H05.

$\sigma(\exp(b))$ are valid. Then according to [5], Theorem 2 there exists an algebra cone $C \subseteq A$ possessing the properties:

1) $\exp(b) \in C$;

2) the spectral radius is increasing on C.

It remains to show that $b \in Q(C)$ or equivalently $\exp(tb) \in C$ if $t \in \mathbb{R}^+$. Then, since $Q(C)$ is positively homogeneous, $a = ||a|| \cdot b \in Q(C)$ is proved.

For this aim we first describe the cone C (cf. [5], Theorem 2). Let $B := \Gamma(\Gamma(a))$ be the second commutant of $a \in A$, which is a closed commutative subalgebra of A. Additionally, the property $\sigma(u; A) = \sigma(u; B), u \in B$ is satisfied. Of course $\Gamma(b) \subseteq \Gamma(e^b)$, where $b := \frac{1}{\|a\|}a$. Since $\Gamma(e^b) \subseteq \Gamma(b)$ (cf. [7], proof of the Theorem), $\Gamma(e^b) = \Gamma(b)$ and therefore $\Gamma(\Gamma(e^b)) = \Gamma(\Gamma(b)) = \Gamma(\Gamma(a)) = B$ follows. Moreover, let $\varphi \neq 0$ be a multiplicative linear functional on B satisfying $\varphi(\exp(b)) = r(\exp(b))$. Such a φ exists, since we showed before that $r(\exp(b)) \in \sigma(\exp(b); A) = \sigma(\exp(b); B)$. Now the cone C is defined by

$$C := \{u \in B : \varphi(u) = r(u)\} \subseteq B.$$

According to $\varphi(\exp(b)) = r(\exp(b))$ we have $\exp(b) \in C$. Now we show that $\exp(tb) \in C$ if $t \in \mathbb{R}^+$. Since the multiplicative functional φ is continuous, it follows by an easy computation that $\varphi(\exp(tb)) = \exp(t\,\varphi(b))$. From $\exp(\varphi(b)) = \varphi(\exp(b)) = r(\exp(b)) \in \mathbb{R}$ we conclude that there exists a $k \in \mathbb{Z}$ such that $\varphi(b) = Re\,\varphi(b) + 2k\pi i$. Since $\|\varphi\| = 1$, we get $2|k| \cdot \pi \leq |\varphi(b)| \leq \|\varphi\| \cdot \|b\| = 1$. Thus $k = 0$ and therefore $\varphi(b) \in \mathbb{R}$ follows. If $t \in \mathbb{R}^+$, we now have $\varphi(\exp(tb)) = \exp(t\,\varphi(b)) = (\exp(\varphi(b)))^t = (\varphi(\exp(b)))^t = (r(\exp(b)))^t = (\exp(\tau(b)))^t = \exp(t\,\tau(b)) = r(\exp(tb))$. According to the definition of the cone C we finally have $\exp(tb) \in C$ if $t \in \mathbb{R}^+$.

Up to now we treated the case $a \neq 0$. Now let $a = 0$. Then condition β) is true: $\tau(a) = 0 \in \{0\} = \sigma(a)$ and the cone $C := \{te : t \in \mathbb{R}^+\}$ satisfies condition α). $\square$

In the course of the proof of the foregoing theorem a commutative algebra B and a cone $C \subseteq B$ were defined. In [5], Theorem 3 it was proved that

$$\mathrm{rad}(B) = C \cap (-C),$$

where $\mathrm{rad}(B)$ is the radical of the commutative algebra B. Hence if B is semisimple, $\mathrm{rad}(B) = \{0\}$. Then the cone C is a proper cone. The following examples 1 and 2 illustrate this situation.

Example 1.

Lemma. *Let $a \in \mathbb{C}^{d\times d}$. Then α) and β) are equivalent. α) The eigenvalues of a are simple. β) $a \cdot n = n \cdot a$, n nilpotent $\Rightarrow n = 0$.*

We only sketch the steps of the elementary proof.

"$\alpha) \Rightarrow \beta)$": From $\lambda \in \sigma(a) \ : \ ax = \lambda x, \ x \neq 0$ we get $a \cdot nx = n \cdot ax = \lambda nx$ and since λ is simple $nx = \varrho x, \ \varrho \in \mathbb{R}$ follows. Now n being nilpotent, the equality $0 = n^l\, x = \varrho^l\, x$ for a suitable $l \in \mathbb{N}$ and therefore $\varrho = 0$ result. Hence $nx = 0$ and since $\mathbb{C}^d$ is the linear hull of eigenvectors corresponding to the d different eigenvalues we conclude that $n = 0$.

"$\beta) \Rightarrow \alpha$": Assume a not to satisfy α). If

$$J := \begin{pmatrix} \lambda & \delta & & & \\ & \lambda & \delta & 0 & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & \delta \\ & & & & \lambda \end{pmatrix}, \ \delta = 0 \ \text{ or } \ \delta = 1$$

is a $k \times k$ Jordan block ($k > 1$) corresponding to $\lambda \in \sigma(a)$, then the $k \times k$ matrix

$$n_k := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ & 0 & & \vdots \\ & & & 0 \end{pmatrix}$$

is nilpotent and commutes with J. Using this observation it is easy to construct a nilpotent $n \in \mathbb{C}^{d\times d}$, $n \neq 0$ that commutes with a, a contradiction to assumption β).

Corollary. *If the eigenvalues of $a \in \mathbb{C}^{d\times d}$ are simple, then the commutative algebra $B := \Gamma(\Gamma(a)) \subseteq \mathbb{C}^{d\times d}$ is semisimple and the cone $C \subseteq B$ is proper.* □

Example 2. Let $a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{C}^{2\times 2}$ satisfy $\beta \cdot \gamma \neq 0$. Using the abbreviations

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 0 & \beta \\ \gamma & \delta - \alpha \end{pmatrix} \ \text{ and } \ D = (\alpha - \delta)^2 + 4\beta\gamma$$

the following is true: $B := \Gamma(\Gamma(a))$ is a two-dimensional subalgebra of $\mathbb{C}^{2\times 2}$:

$$B = \{xe + yb \ : \ x, y \in \mathbb{C}\}$$

and

$$\begin{aligned} D \neq 0 \quad &\Rightarrow \quad (\sigma(xe + yb) = \{0\} \Leftrightarrow x = y = 0) \\ &\Rightarrow \quad C \cap (-C) = \mathrm{rad}(B) = \{0\}, \\ &\qquad\ \ \text{hence the cone } C \text{ is proper;} \end{aligned}$$

$$\begin{aligned} D = 0 \quad &\Rightarrow \quad \left(\sigma(xe + yb) = \{0\} \Leftrightarrow y = \tfrac{2}{\alpha-\delta}x\right) \\ &\Rightarrow \quad C \cap (-C) = \mathrm{rad}(B) = \{x(e + \tfrac{2}{\alpha-\delta}b) : \ x \in \mathbb{C}\} \neq \{0\}, \\ &\qquad\ \ \text{hence the cone } C \text{ is not proper.} \end{aligned}$$

□

Example 3. Let $A := \mathbb{C}^{3\times 3}$ and

$$a = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\sigma(a) = \{-i, 0, i\}$. According to Theorem 5.2 in [6] there is no algebra cone in A on which the spectral radius is increasing and which contains a. Because of $\tau(a) = 0 \in \sigma(a)$ it follows from the theorem treated in the present paper, that there is an algebra cone $C \subseteq A$ such that $a \in Q(C)$ and the spectral radius is increasing on C. Moreover, by the corollary of the lemma in Example 1 of the present paper, C is a proper cone.

Using the following notation,

$$e_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find that

α) $B = \Gamma(\Gamma(a))$ is a three-dimensional subalgebra of $A = \mathbb{C}^{3\times 3}$:

$$B = \{xe_1 + ye_2 + xa : x, y, z \in \mathbb{C}\} \subseteq A$$

and

$$e^{ta} = \cos t \cdot e_1 + e_2 + \sin t \cdot a \in B,\ t \in \mathbb{R}^+;$$

β) the multiplicative functional φ on B defined by

$$\varphi(e_1) = 0,\ \varphi(e_2) = 1,\ \varphi(a) = 0$$

has the property

$$\varphi(e^a) = e^{\varphi(a)} = 1 = r(e^a)$$

(the last equality follows from $\sigma(e^a) = e^{\sigma(a)}$);

γ) the cone $C = \{u \in B : \varphi(u) = r(u)\} \subseteq B$ is described explicitly by

$$C = \{u = xe_1 + ye_2 + za \in B : x, y, z \in \mathbb{C} \text{ and } y \geq |x \pm iz|\}.$$

Since $e^{ta} = \cos t \cdot e_1 + e_2 + \sin t \cdot a,\ t \in \mathbb{R}^+$, it follows that $1 = y \geq |\cos t \pm i \cdot \sin t| = |e^{\pm it}| = 1$, and therefore $e^{ta} \in C,\ t \in \mathbb{R}^+$. □

References

[1] A. Berman, M. Neumann and R. J. Stern, *Nonnegative matrices in dynamic systems*, Pure and Applied Mathematics, John Wiley and Sons, New York, 1989. MR **90j:**93030

[2] A. Berman and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Academic Press, 1979. MR **82b:**15013

[3] L. Elsner, *Monotonie und Randspektrum bei vollstetigen Operatoren*, Arch. Rational Mech. Anal. 36 (1970), 356-365. MR **40:**4804

[4] G. Herzog and R. Lemmert, *On quasipositive elements in ordered Banach algebras*, Studia Math. 129 (1998), 59-65. MR **99g:**46061

[5] G. Herzog and Ch. Schmoeger, *A note on a Theorem of Raubenheimer and Rode*, Proc. Amer. Math. Soc. 131 (2003), 3507–3509.

[6] H. Raubenheimer and S. Rode, *Cones in Banach algebras*, Indag. Math. 7 (1996), 489-502. MR **99i:**46035

[7] C. Schmoeger, *Remarks on commuting exponentials in Banach algebras*, Proc. Amer. Math. Soc. 127 (1999), 1337-1338. MR **99h:**46090

[8] J. S. Vandergraft, *Spectral properties of matrices which have invariant cones*, SIAM J. Appl. Math. 16 (1968), 1208-1222. MR **39:**5599

Fakultät für Mathematik, Universität Karlsruhe, D-76128 Karlsruhe, Germany
E-mail address: herbert.weigel@math.uni-karlsruhe.de