A NOTE ON COMMUTATIVITY UP TO A FACTOR
OF BOUNDED OPERATORS

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Abstract. In this note, we explore commutativity up to a factor \( AB = \lambda BA \)
for bounded operators \( A \) and \( B \) in a complex Hilbert space. Conditions on
possible values of the factor \( \lambda \) are formulated and shown to depend on spectral
properties of the operators. Commutativity up to a unitary factor is consid-
ered. In some cases, we obtain some properties of the solution space of the
operator equation \( AX = \lambda XA \) and explore the structures of \( A \) and \( B \) that
satisfy \( AB = \lambda BA \) for some \( \lambda \in \mathbb{C} \setminus \{0\} \). A quantum effect is an operator \( A \)
on a complex Hilbert space that satisfies \( 0 \leq A \leq I \). The sequential product
of quantum effects \( A \) and \( B \) is defined by \( A \circ B = A^{\frac{1}{2}}BA^{\frac{1}{2}} \). We also obtain
properties of the sequential product.

1. Introduction

Commutation relations between selfadjoint operators in a complex Hilbert space
play an important role in the interpretation of quantum mechanical observables
and analysis of their spectra. For related works refer to [2], [3], [4], [6], [8], [9]
and [12]. Accordingly, such relations have been extensively studied in the math-
ematical literature (see, for example, the classic study of C. R. Putnam in [11]).
An interesting, related aspect concerns the commutativity up to a factor for a pair
of operators. Certain forms of non-commutativity can be conveniently phrased in
this way. In [1], J. A. Brooke, P. Busch and D. B. Pearson gave some examples to
illustrate this. A quantum effect is a yes-no measurement. An effect is represented
by an operator \( A \) on a Hilbert space that satisfies \( 0 \leq A \leq I \). A sharp effect is
represented by a selfadjoint projection operator on a Hilbert space. The sequential
product of quantum effects \( A \) and \( B \) is defined by \( A \circ B = A^{\frac{1}{2}}BA^{\frac{1}{2}} \). Sequential
measurements are very important in quantum mechanics. For detailed works refer
to [8], [9] and [12]. Let \( H \) be a complex Hilbert space, \( B(H) \) be the Banach algebra
of bounded linear operators on \( H \), \( \varepsilon(H) \) be the set of quantum effects on \( H \), \( P(H) \)
be the set of sharp effects on \( H \), \( I \) be the identity operator on some Hilbert space
and \( M_{n \times m} \) be the set of \( n \times m \) matrices. For an operator \( A \in B(H) \), denote by
\( N(A) \), \( R(A) \), \( \sigma(A) \), \( r(A) \) the null space, the range, the spectra and the spectral ra-
dius of \( A \), respectively; \( \text{dim} N(A) \) denotes the dimension of \( N(A) \). Recall that, for

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A, B ∈ B(H), A and B commute up to a factor means that \( AB = \lambda BA \), for some \( \lambda \in \mathbb{C} \setminus \{0\} \), and A and B commute up to a unitary factor means that \( AB = UBA \), where \( U \) is a unitary operator in \( B(H) \). For each \( 0 \leq A \in B(H), B \in B(H) \), we also define \( A \circ B = A^{1/2}BA^{1/2} \). The main results shown by J. A. Brooke, P. Busch and D. B. Pearson in [1] are the following two theorems.

**Theorem 1.1** ([1]). Let \( A, B \in B(H) \) such that \( AB = \lambda BA \neq 0, \lambda \in \mathbb{C} \). Then

(i) if A or B is selfadjoint, then \( \lambda \in \mathbb{R} \);
(ii) if both A and B are selfadjoint, then \( \lambda \in \{-1,1\} \); and
(iii) if A and B are selfadjoint and one of them is positive, then \( \lambda = 1 \).

**Theorem 1.2** ([1]). Let \( A, B \in B(H) \) be selfadjoint operators. The following statements are equivalent.

(i) \( AB = UBA \) for some unitary operator \( U \).
(ii) \( AB^2 = B^2A \) and \( BA^2 = A^2B \).

In [9] S. Gudder and G. Nagy gave the following result on sequential measurement.

**Theorem 1.3** ([9]). For \( A, B \in \mathcal{S}(H) \), if \( A \circ B \in \mathcal{P}(H) \), then \( AB = BA \).

They put forward an open problem: if \( A, B \in \mathcal{S}(H) \) with \( \dim H = \infty \) and \( A \circ B \geq B \), does \( AB = BA = B \) hold?

In this paper, firstly we give simple proofs of the theorems above and generalizations of them. Secondly, we show a further relation between the spectra of \( AB \) and the factor \( \lambda \). In the case that \( H \) is finite dimensional, we obtain a property of the solution space of the operator equation \( AX = \lambda XA \). Also, if \( A \) has finite rank and is normal, we explore the structure of \( A \) and \( B \) which commute up to a factor. Thirdly, we give a generalization of Theorem 1.3 in [9] with proof different from [9] and answer the open question raised by S. Gudder and G. Nagy. This question was also independently answered by A. Gheondea and S. Gudder in [7].

We will use repeatedly the Fuglede-Putnam Theorem.

**Theorem 1.4** (Fuglede-Putnam Theorem ([11])). If \( N \) and \( M \) are normal operators on \( H \) and \( K \), and \( B : K \to H \) is an operator such that \( NB = BM \); then \( N^*B = BM^* \).

2. Pairs commuting up to a factor

**Theorem 2.1.** Let \( A, B \in B(H) \) such that \( AB = \lambda BA \neq 0, \lambda \in \mathbb{C} \). Then

(i) if A or B is selfadjoint, then \( \lambda \in \mathbb{R} \);
(ii) if either A or B is selfadjoint and the other is normal, then \( \lambda \in \{-1,1\} \); and
(iii) if both A and B are normal, then \( |\lambda| = 1 \).

**Proof.** (i) Suppose that \( A \) is selfadjoint; then \( \lambda A \) is normal. By the Fuglede-Putnam Theorem, we have \( AB = \overline{\lambda}BA \). Hence \( (\lambda - \overline{\lambda}) = 0 \). This implies \( \lambda \in \mathbb{R} \).

(ii) Suppose that \( A \) is normal and \( B \) is selfadjoint. By the Fuglede-Putnam Theorem, we have \( A^*B = \overline{\lambda}BA^* \). Then \( AA^*B = \overline{\lambda}ABA^* = |\lambda|^2BA^* \). From (i) and (iii) of Theorem 1.1, we have \( \lambda \in \mathbb{R} \) and \( |\lambda|^2 = 1 \). Hence \( \lambda \in \{-1,1\} \).

(iii) From \( AB = \lambda BA \), we have \( B^*A^* = \lambda A^*B^* \). By the Fuglede-Putnam Theorem, we get \( A^*B = \overline{\lambda}BA^* \) and \( BA^* = \lambda A^*B \). Hence \( A^*B = |\lambda|^2A^*B \). Since \( AB \neq 0 \) and \( A \) is normal, we can get \( A^*B \neq 0 \). Therefore \( |\lambda| = 1 \).
In fact, with similar deduction, we can generalize (i) of Theorem 2.1 in the following way:

**Corollary 2.2.** Suppose \( A, B, C \in B(H) \) and \( A, C \) are selfadjoint operators with 
\[ AB = \lambda BC \neq 0, \lambda \in \mathbb{C}; \text{ then } \lambda \in \mathbb{R}. \]

The following result is a generalization of Theorem 1.2 in [1].

**Theorem 2.3.** Let \( A, B \in B(H) \). Then the following statements are equivalent.

(i) \( AB^* = B^* BA \) and \( BAA^* = A^* AB \).

(ii) Both \( AB \) and \( BA \) are normal.

(iii) There exist unitary operators \( U \) and \( V \) in \( B(H) \) such that \( AB = UB^* A^* \) and \( BA = VA^* B^* \).

**Proof.** The equivalence of (i) and (ii) was proved by S. Gudder and G. Nagy in [2].

(ii) \( \Rightarrow \) (iii) By the fact that an operator \( N \in B(H) \) is normal if and only if there exists a unitary operator \( U \) such that \( N = UN^* \), the proof is trivial.

(iii) \( \Rightarrow \) (ii) Observe that \( B^* A^* = ABU^* \), and so \( AB = UABU^* \). Hence \( AB \) commutes with \( U \) and \( U^* \), and similarly for \( B^* A^* \). Thus we get \( ABB^* A^* = UBA^* U^* AB = BAA^* AB \). Hence \( AB \) is normal. Similarly, from \( BA = VA^* B^* \), we can get \( BA \) is normal.

If both \( A \) and \( B \) are selfadjoint, then Theorem 2.3 becomes Proposition 4.1 and Proposition 4.2 in [1].

For given \( 0 \neq A \in B(H) \), let
\[ S_A^n = \{ B \in B(H) : \sigma(AB) \text{ has exactly } n \text{ distinct nonzero values} \}. \]
Then we have

**Theorem 2.4.** For each \( 0 < n < \infty \) and \( B \in S_A^n \), if \( AB = \lambda BA \) for some \( \lambda \in \mathbb{C} \), then \( \lambda^n = 1 \).

**Proof.** Suppose \( B \in S_A^n \) with \( AB = \lambda BA \). Let \( \sigma(AB) \setminus \{0\} = \{\alpha_1, \ldots, \alpha_n\} \). Then \( \{\alpha_1, \ldots, \alpha_n\} = \{\lambda \alpha_1, \ldots, \lambda \alpha_n\} \), since \( \sigma(AB) \setminus \{0\} = \sigma(AB) \setminus \{0\} \). Clearly, for each \( \alpha_i \in \sigma(AB) \setminus \{0\} \), there exist \( 1 \leq p \leq n \) and \( \alpha_j \in \sigma(AB) \setminus \{0\}, 1 < j \leq p \), with \( \alpha_i \neq \alpha_j \), such that \( \alpha_i = \lambda \alpha_j \). If \( \alpha_i = \lambda \alpha_j \), then \( p = q \) and \( \alpha_i \neq \alpha_j \). In fact, we assume that \( p < q \). Then \( \lambda^p = \lambda^q = 1 \) and \( \lambda^{q-p} = 1 \); hence \( \alpha_{ji} = \alpha_{ji-p+1} \). It is a contradiction.

Later we have \( \frac{m}{p} \in \mathbb{N} \). Hence \( \lambda^m = 1 \).

The next result gives some restrictions that must be satisfied by a pair of operators commuting up to a scalar factor.

**Theorem 2.5.** Let \( A, B \in B(H) \) such that \( AB = \lambda BA \neq 0 \), for some \( \lambda \in \mathbb{C} \). Then

(i) \( AB \) is bounded below if and only if both \( A \) and \( B \) are bounded below;

(ii) if \( A \) is normal and \( R(B) \) is dense, then \( AB \) is not nilpotent.

**Proof.** (i) If \( A \) and \( B \) are bounded below, clearly, we have that \( AB \) is bounded below. Conversely, if \( AB \) is bounded below, then \( B \) is bounded below. We also have that \( A \) is bounded below, since \( AB = \lambda BA \).

(ii) If \( AB = \lambda BA \) and \( A \) is normal, then \( N(A) \) and \( R(A) \) are reducing subspaces of \( A \) and \( B \). Thus \( A \) and \( B \) have forms \( A = \text{diag}(A_1, 0) \) and \( B = \text{diag}(B_1, B_2) \), with respect to \( H = N(A) \oplus N(A) \). Since \( AB = \text{diag}(A_1 B_1, 0) \), we have that \( AB \) is
nilpotent if and only if $A_1B_1$ is nilpotent. Hence, without loss of generality, we may assume that $A$ is injective. Then $R(A)$ is dense, since $A$ is normal and injective. If there exists $n \in \mathbb{N}$ such that $(AB)^n = 0$, then $(AB)^{n-1} = 0$, since $R(A)$ and $R(B)$ are dense. By the same deduction, we can get $AB = 0$. It is a contradiction. \qed

Next, we turn to the study of properties of pairs of operators commuting up to a scalar factor with one of the operators having finite rank.

As in the proof of Theorem 2.5, if $AB = \lambda BA$ and $A$ is normal, then $N(A)$ and $R(A)$ are reducing subspaces of $A$ and $B$. Thus $A$ and $B$ have forms $A = \text{diag}(A_1, 0)$ and $B = \text{diag}(B_1, B_2)$, with respect to $H = N(A)^\perp \oplus N(A)$. Suppose that $A$ has finite rank and $\{a_1, \cdots, a_m\}$ are the distinct nonzero eigenvalues of $A$. Then $A = \sum_{i=1}^m a_iP_i$, where $P_i$ is the orthogonal projection from $H$ onto $N(A - a_iI)$, for $1 \leq i \leq m$ and $P_iP_j = P_jP_i = 0$, $i \neq j$. Denote $H_i = P_iH$. Then $A_1$ and $B_1$ have operator matrix forms $A_1 = \text{diag}(a_1I, \cdots, a_mI)$ and $B_1 = (B_{ij})_{m \times m}$, with respect to the space decomposition $N(A)^\perp = \bigoplus_{i=1}^m H_i$, respectively. With the symbols above, we have the following three results.

**Proposition 2.6.** Suppose that $A \in B(H)$ has finite rank and is normal. If $\sigma(A) \cap \sigma(\lambda A)$ has no nonzero element, then there does not exist nonzero $B \in B(H)$ such that $AB = \lambda BA \neq 0$. If $\sigma(A) \cap \sigma(\lambda A)$ has nonzero elements, then there exists $B \in B(H)$ such that $AB = \lambda BA \neq 0$ if and only if either $m = 1, \lambda = 1$ or $m > 1, \lambda \in \mathbb{C} \setminus \{0\}$.

**Proof.** If $A = 0$, it is trivial. If $A \neq 0$, then we have $AB = \lambda BA \neq 0$ if and only if $A_1B_1 = \lambda B_1A_1 \neq 0$, since $AB = \text{diag}(A_1B_1, 0)$. Hence, without loss of generality, we may assume that $A$ is injective. Also, $AB = \lambda BA$ if and only if $a_iB_{ij} = \lambda a_jB_{ij}$, for $1 \leq i, j \leq m$. If $\sigma(A) \cap \sigma(\lambda A)$ has no nonzero element, then $a_i \neq \lambda a_j$, for $i, j$. Hence there does not exist $B_{ij}$ such that $a_iB_{ij} = \lambda a_jB_{ij} \neq 0$, for $i, j$. Clearly, if $m = 1$ and $\lambda = 1$, then we have $AB = \lambda BA \neq 0$, for each $0 \neq B \in B(H)$. If $m > 1$ and $\sigma(A) \cap \sigma(\lambda A)$ has nonzero elements, then there exists $1 \leq i, j \leq m$ such that $a_i = \lambda a_j$. Hence, for each $B$ with $B_{ij} \neq 0$ and $B_{st} = 0, s \neq i, t \neq j$, we have $AB = \lambda BA \neq 0$. On the other hand, if $AB = \lambda BA \neq 0$ and $m = 1$, then we have $\lambda = 1$, since $\sigma(A) \cap \sigma(\lambda A)$ has nonzero element $a_1$. \qed

If, with respect to a suitable decomposition of the space, $B$ has a form $B = \bigoplus_{i=1}^s I_k^{(i)} \bigoplus_{i=s+1}^t M_{j_i}^{(i)} (\oplus 0) \oplus B_2$, then we call it a standard form, where the $k \times k$ block operator matrix $J_k^{(i)}$ and the $j_i \times j_i$ block operator matrix $M_{j_i}^{(i)}$ are defined by

$$J_k^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ B_{k1}^{(i)} & \cdots & 0 & 0 \end{pmatrix}, \quad M_{j_i}^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & B_{(j_i-1)j_i}^{(i)} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

$s, t \in \mathbb{N} \cup \{0\}, k, j_i \in \mathbb{N}$ and $B_{(\_\_)}^{(\_\_)}$ are exactly the nonzero entries of $(B_{ij})_{m \times m}$. In this case, if there exist $a_i^{(i)} \in \sigma(A) \setminus \{0\}, 1 \leq i \leq s + t$ with $a_i^{(p)} \neq a_i^{(l)}$ for $1 \leq p \neq l \leq s + t$, such that $A$ has a form $A = \bigoplus_{i=1}^s I_k^{(i)} \bigoplus_{i=s+1}^t N_{j_i}^{(i)} \oplus A'$, $A'$ is a diagonal operator and the $k \times k$ block
operator matrix $I_k^{(i)}$ and the $j_i \times j_i$ block operator matrix $N_{j_i}^{(i)}$ are defined by

$$I_k^{(i)} = \begin{pmatrix} a^{(i)}I & \lambda a^{(i)}I & \cdots & \lambda^{k-1}a^{(i)}I \\ \vdots & \ddots & \ddots & \vdots \\ \lambda a^{(i)}I & \cdots & \cdots & \cdots \\ \lambda^{k-1}a^{(i)}I & \cdots & \cdots & \cdots \end{pmatrix}, \quad N_{j_i}^{(i)} = \begin{pmatrix} a^{(i)}I & \lambda a^{(i)}I & \cdots & \lambda^{j_i-1}a^{(i)}I \\ \vdots & \ddots & \ddots & \vdots \\ \lambda a^{(i)}I & \cdots & \cdots & \cdots \\ \lambda^{j_i-1}a^{(i)}I & \cdots & \cdots & \cdots \end{pmatrix},$$

respectively, then we say $A$ and $B$ are compatible.

**Theorem 2.7.** Suppose that $0 \neq A \in B(H)$ has finite rank and is normal. Then $AB = \lambda BA$ if and only if one of the following conditions holds:

(i) there exists a $B_{ii} \neq 0$, $1 \leq i \leq m$, $\lambda = 1$ and $B = \text{diag}(B_{11}, \ldots, B_{mm}, B_2)$;

(ii) $B_{ii} = 0$, for $1 \leq i \leq m$, and $A$ and $B$ are compatible.

**Proof.** It is easy to show that (i) and (ii) imply $AB = \lambda BA$, respectively.

Conversely, from $AB = \lambda BA$, we have $a_i B_{ij} = \lambda a_j B_{ij}$. If there exists a $B_{ii} \neq 0$, for some $1 \leq i \leq m$, then $\lambda = 1$ and $AB = BA$. Since $(A - a_i I)B = B(A - a_i I)$, by the Fuglede-Putnam Theorem, $H_i$ is a reducing subspace of $B$. Then $B_{ij} = 0, i \neq j$. Hence $B = \text{diag}(B_{11}, \ldots, B_{mm})$. If $B_{ii} = 0$ for $1 \leq i \leq m$, then $AB = \lambda BA$ implies that $a_i B_{ij} = \lambda a_j B_{ij}$. Since $a_i \neq a_j, i \neq j$, we obtain that there is at most one nonzero entry in each row and column of $(B_{ij})_{m \times m}$. It is easy to see that with respect to a suitable decomposition of the space, $B$ has a form $B = \bigoplus_{i=1}^{s} J_k^{(i)} \bigoplus_{i=s+1}^{s+t} M_j^{(i)} \oplus B_2$, where

$$J_k^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_{(k-1)k}^{(i)} & 0 \\ B_{k1}^{(i)} & \cdots & \cdots & 0 \end{pmatrix}, \quad M_j^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_{(j-1)j}^{(i)} & 0 \\ 0 & \cdots & \cdots & 0 \end{pmatrix},$$

$s, t \in \mathbb{N} \cup \{0\}$, $k_i, j_i \in \mathbb{N}$ and $J_k^{(i)}, M_j^{(i)}$ are exactly the nonzero entries of $(B_{ij})_{m \times m}$. Also, simple computation, we can get that there exists $a^{(i)} \in \sigma(A) \setminus \{0\}, 1 \leq i \leq s + t$ with $a^{(p)} \neq a^{(l)}$ for $1 \leq p \neq l \leq s + t$ such that $A$ has a form $A = \bigoplus_{i=1}^{s} J_k^{(i)} \bigoplus_{i=s+1}^{s+t} M_j^{(i)} \oplus A'$, with respect to the same decomposition of the space, where $A'$ is a diagonal operator and the $k_i \times k_i$ block operator matrix $I_{k_i}^{(i)}$ and the $j_i \times j_i$ block operator matrix $N_{j_i}^{(i)}$ are defined by

$$I_{k_i}^{(i)} = \begin{pmatrix} a^{(i)}I & \lambda a^{(i)}I & \cdots & \lambda^{k_i-1}a^{(i)}I \\ \vdots & \ddots & \ddots & \vdots \\ \lambda a^{(i)}I & \cdots & \cdots & \cdots \\ \lambda^{k_i-1}a^{(i)}I & \cdots & \cdots & \cdots \end{pmatrix}, \quad N_{j_i}^{(i)} = \begin{pmatrix} a^{(i)}I & \lambda a^{(i)}I & \cdots & \lambda^{j_i-1}a^{(i)}I \\ \vdots & \ddots & \ddots & \vdots \\ \lambda a^{(i)}I & \cdots & \cdots & \cdots \\ \lambda^{j_i-1}a^{(i)}I & \cdots & \cdots & \cdots \end{pmatrix}.$$

We will show that if $s > 1$, then $k_1 = \cdots = k_s$. We assume that $k_i < k_j, 1 \leq i, j \leq s$; then $\lambda^{k_i} = 1$ and $\lambda^{k_j} = 1$. Hence $\lambda^{k_j - k_i} = 1$. Therefore $a^{(i)} = \lambda^{k_j - k_i} a^{(i)}$. It is a contradiction with $a_i \neq a_j, i \neq j$. Hence $k_1 = \cdots = k_s$. Let $k = k_i$; then, clearly we have $\lambda^k = 1$ and the $j_i \leq k_i$ for $s + 1 \leq i \leq s + t$. That is to say, $A$ and $B$ are compatible.

**Remark.** If $A$ is compact and normal, a result similar to Theorem 2.7 can be obtained by the same deduction. Its representation is more complex, and so it is omitted.
For given $0 \neq A \in B(H)$, let $S^\lambda_A = \{ B \in B(H) : AB = \lambda BA \}$. Then $S^\lambda_A$ is a closed subspace of $B(H)$. If $\lambda^n = 1$, for some $n \in \mathbb{N}$, then for each $B \in S^\lambda_A$, we have $B^{n+1} \in S^\lambda_A$. In fact, $AB^{n+1} = \lambda^{n+1}B^{n+1}A = \lambda B^{n+1}A$. If $H$ is finite dimensional, by Theorem 4.4.6 in [10], we have $S^\lambda_A \neq \{0\}$ if and only if $\sigma(A) \cap \sigma(\lambda A) \neq \phi$.

**Proposition 2.8.** Suppose that $H$ is finite dimensional and $0 \neq A \in B(H)$ is normal. Then

$$\dim S^\lambda_A = (\dim N(A))^2 + \sum_{1 \leq i,j \leq n} \dim H_i \dim H_j \psi(i,j),$$

where $\psi(i,j) = \begin{cases} 1 & a_i = \lambda a_j \\ 0 & a_i \neq \lambda a_j \end{cases}$. 

**Proof.** Suppose that $A$ is normal. Then $AB = \lambda BA$ if and only if $A_1B_1 = \lambda B_1A_1$. Hence $\dim S^\lambda_A = \dim S^\lambda_A + (\dim N(A))^2$. Next, we assume that $A$ is injective. We will show that $\dim S^\lambda_A = \sum_{1 \leq i,j \leq n} \dim H_i \dim H_j \psi(i,j)$. In this case, $AB = \lambda BA$ if and only if $a_iB_{ij} = \lambda a_jB_{ij}$, for $1 \leq i,j \leq m$. If $a_i \neq \lambda a_j$, i.e. $\psi(i,j) = 0$, for some $1 \leq i,j \leq m$, then $AB = \lambda BA$ implies $B_{ij}$ must be zero. If $a_i = \lambda a_j$, i.e. $\psi(i,j) = 1$, for some $1 \leq i,j \leq m$, then for each $B_{ij} \in M_{\dim H_i \times \dim H_j}$, we have $a_iB_{ij} = \lambda a_jB_{ij}$. Thus $\dim S^\lambda_A = \sum_{1 \leq i,j \leq m} \dim H_i \dim H_j \psi(i,j)$. □

3. **Sequential quantum measure**

We first prove a lemma.

**Lemma 3.1.** Suppose that $A$ is an injective positive operator and $B \in B(H)$ with dense range. If $A \circ B \in P(H)$, then $AB = BA = I$.

**Proof.** Since $R(A)$ is dense in $H$, we have that $R(A^{\frac{1}{2}})$ is dense. That is, $\overline{R(A^{\frac{1}{2}})} = H$. Hence $R(BA^{\frac{1}{2}}) = H$. This implies that $R(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = H$. By the assumption $A \circ B \in P(H)$, we have $A^{\frac{1}{2}}BA^{\frac{1}{2}} = I$. Therefore $R(A^{\frac{1}{2}}) = H$. This shows that $A$ is invertible. Hence $B = A^{-1}$, i.e., $AB = BA = I$. □

If $A$ is positive and $B \in B(H)$, then $A$ and $B$ have operator matrices $A = \text{diag}(A_1,0)$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with respect to the space decomposition $H = \overline{R(A)} \oplus N(A)$, respectively. Since

\[
\begin{pmatrix} A_1^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} A_1^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1^{\frac{1}{2}}B_{11}A_1^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix},
\]

$A \circ B \in P(H)$ is equivalent to $A_1 \circ B_{11} \in P(\overline{R(A)})$. With the symbols above we have

**Theorem 3.2.** Suppose that $A \in B(H)$ is positive and that $B \in B(H)$ is selfadjoint. Then $N(A)$ is an invariant subspace of $B$, $N(B|_{\overline{R(A)}})$ is an invariant subspace of $A$, and $A \circ B \in P(H)$ if and only if $AB = BA$ and $A|_{M_{\overline{R(A)}}}B|_{M_{\overline{R(A)}}} \in P(\overline{R(B|_{\overline{R(A)}})})$.

**Proof.** Suppose that $A \in B(H)$ is positive, $B \in B(H)$ is selfadjoint, $N(A)$ is an invariant subspace of $B$, $N(B|_{\overline{R(A)}})$ is an invariant subspace of $A$ and $A \circ B \in P(H)$. Then we have $A = \text{diag}(A_1,0), B = \text{diag}(B_{11},B_{22})$ and $A_1 \circ B_{11} \in P(H)$.
If $A, B \in \mathcal{E}(H)$, then we can easily get that the part of the necessity condition of Theorem 3.2 is Theorem 1.3 in [3]. The virtue of our proof is that we can know the structure of $A, B$ and $AB$ clearly. One can use this method to prove Corollary 2.4, Theorem 2.5 and Theorem 2.6 in [3].

The following theorem is an answer to the question raised by S. Gudder and G. Nagy in [3], which is also a generalization of Theorem 2.6 (c) in [3]. This question was also independently answered by A. Gheondea and S. Gudder in [7]. Though our proof is essentially the same as A. Gheondea and S. Gudder’s, maybe our presentation of it is better and simpler. We still retain the proof here.

**Theorem 3.3.** If $A, B \in \mathcal{E}(H)$, then we have $A \circ B \geq B$ if and only if $AB = BA = B$.

**Proof.** It is clear that $AB = BA = B$ implies $A \circ B \geq B$. Conversely, if

$$A \circ B = \begin{pmatrix} A_1^\frac{1}{2} B_{11} A_1^\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix},$$

then $B_{12} = B_{22} = 0$. Hence $B = \text{diag}(B_{11}, 0)$ and $A_1^\frac{1}{2} B_{11} A_1^\frac{1}{2} \geq B_{11}$. This implies that $A_1^\frac{n}{2} B_{11} A_1^\frac{n}{2} \geq B_{11}$, for $n \in \mathbb{N}$. Suppose that $A_1 = \int_0^1 \lambda dE_\lambda$ is the spectral calculus of $A_1$. Let $H_1 = E(1)$ and $H_2 = E([0,1))$. Then for each $x \in H_2$, we have $A_1^\frac{n}{2} x \rightarrow 0$ as $n \rightarrow \infty$. This implies $B_{11} x = 0$, for $x \in H_2$. Hence $A = \text{diag}(I, A_1', 0)$ and $B = \text{diag}(B_{11}', 0, 0)$, with respect to $H = H_1 \oplus H_2 \oplus N(A)$, where $A_1' = A|_{H_2}$.
and $B_{11}' = B|_{H_1}$. Therefore,

$$AB = \begin{pmatrix}
B_{11}' & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = BA = B.$$ 

\[\square\]

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**References**


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