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A NOTE ON COMMUTATIVITY UP TO A FACTOR OF BOUNDED OPERATORS

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ABSTRACT. In this note, we explore commutativity up to a factor $AB = \lambda BA$ for bounded operators A and B in a complex Hilbert space. Conditions on possible values of the factor λ are formulated and shown to depend on spectral properties of the operators. Commutativity up to a unitary factor is considered. In some cases, we obtain some properties of the solution space of the operator equation $AX = \lambda XA$ and explore the structures of A and B that satisfy $AB = \lambda BA$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. A quantum effect is an operator A on a complex Hilbert space that satisfies $0 \leq A \leq I$. The sequential product of quantum effects A and B is defined by $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. We also obtain properties of the sequential product.

1. Introduction

Commutation relations between selfadjoint operators in a complex Hilbert space play an important role in the interpretation of quantum mechanical observables and analysis of their spectra. For related works refer to [2], [3], [4], [6], [8], [9] and [12]. Accordingly, such relations have been extensively studied in the mathematical literature (see, for example, the classic study of C. R. Putnam in [11]). An interesting, related aspect concerns the commutativity up to a factor for a pair of operators. Certain forms of non-commutativity can be conveniently phrased in this way. In [1], J. A. Brooke, P. Busch and D. B. Pearson gave some examples to illustrate this. A quantum effect is a yes-no measurement. An effect is represented by an operator A on a Hilbert space that satisfies $0 \leq A \leq I$. A sharp effect is represented by a selfadjoint projection operator on a Hilbert space. The sequential product of quantum effects A and B is defined by $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. Sequential measurements are very important in quantum mechanics. For detailed works refer to [8], [9] and [12]. Let H be a complex Hilbert space, $B(H)$ be the Banach algebra of bounded linear operators on H, $\varepsilon(H)$ be the set of quantum effects on H, $P(H)$ be the set of sharp effects on H, I be the identity operator on some Hilbert space and $M_{n\times m}$ be the set of $n \times m$ matrices. For an operator $A \in B(H)$, denote by $N(A)$, $R(A)$, $\sigma(A)$, $r(A)$ the null space, the range, the spectra and the spectral radius of A, respectively; $\dim N(A)$ denotes the dimension of $N(A)$. Recall that, for

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$A, B \in B(H)$, A and B commute up to a factor means that $AB = \lambda BA$, for some $\lambda \in \mathbb{C} \setminus \{0\}$, and A and B commute up to a unitary factor means that $AB = UBA$, where U is a unitary operator in $B(H)$. For each $0 \leq A \in B(H), B \in B(H)$, we also define $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. The main results shown by J. A. Brooke, P. Busch and D. B. Pearson in [1] are the following two theorems.

Theorem 1.1 ([1])**.** *Let $A, B \in B(H)$ such that $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}$. Then*
(i) *if A or B is selfadjoint, then $\lambda \in \mathbb{R}$;*
(ii) *if both A and B are selfadjoint, then $\lambda \in \{-1, 1\}$; and*
(iii) *if A and B are selfadjoint and one of them is positive, then $\lambda = 1$.*

Theorem 1.2 ([1])**.** *Let $A, B \in B(H)$ be selfadjoint operators. The following statements are equivalent.*
(i) *$AB = UBA$ for some unitary operator U.*
(ii) *$AB^2 = B^2 A$ and $BA^2 = A^2 B$.*

In [9] S. Gudder and G. Nagy gave the following result on sequential measurement.

Theorem 1.3 ([9])**.** *For $A, B \in \varepsilon(H)$, if $A \circ B \in P(H)$, then $AB = BA$.*

They put forward an open problem: if $A, B \in \varepsilon(H)$ with $\dim H = \infty$ and $A \circ B \geq B$, does $AB = BA = B$ hold?

In this paper, firstly we give simple proofs of the theorems above and generalizations of them. Secondly, we show a further relation between the spectra of AB and the factor λ. In the case that H is finite dimensional, we obtain a property of the solution space of the operator equation $AX = \lambda XA$. Also, if A has finite rank and is normal, we explore the structure of A and B which commute up to a factor. Thirdly, we give a generalization of Theorem 1.3 in [9] with proof different from [9] and answer the open question raised by S. Gudder and G. Nagy. This question was also independently answered by A. Gheondea and S. Gudder in [7].

We will use repeatedly the Fuglede-Putnam Theorem.

Theorem 1.4 (Fuglede-Putnam Theorem ([11]))**.** *If N and M are normal operators on H and K, and $B : K \to H$ is an operator such that $NB = BM$; then $N^* B = BM^*$.*

2. Pairs commuting up to a factor

Theorem 2.1. *Let $A, B \in B(H)$ such that $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}$. Then*
(i) *if A or B is selfadjoint, then $\lambda \in \mathbb{R}$;*
(ii) *if either A or B is selfadjoint and the other is normal, then $\lambda \in \{-1, 1\}$; and*
(iii) *if both A and B are normal, then $|\lambda| = 1$.*

Proof. (i) Suppose that A is selfadjoint; then λA is normal. By the Fuglede-Putnam Theorem, we have $AB = \overline{\lambda} BA$. Hence $(\lambda - \overline{\lambda}) = 0$. This implies $\lambda \in \mathbb{R}$.

(ii) Suppose that A is normal and B is selfadjoint. By the Fuglede-Putnam Theorem, we have $A^* B = \overline{\lambda} BA^*$. Then $AA^* B = \overline{\lambda} ABA^* = |\lambda|^2 BAA^*$. From (i) and (iii) of Theorem 1.1, we have $\lambda \in \mathbb{R}$ and $|\lambda|^2 = 1$. Hence $\lambda \in \{-1, 1\}$.

(iii) From $AB = \lambda BA$, we have $B^* A^* = \overline{\lambda} A^* B^*$. By the Fuglede-Putnam Theorem, we get $A^* B = \overline{\lambda} BA^*$ and $BA^* = \lambda A^* B$. Hence $A^* B = |\lambda|^2 A^* B$. Since $AB \neq 0$ and A is normal, we can get $A^* B \neq 0$. Therefore $|\lambda| = 1$. $\square$

In fact, with similar deduction, we can generalize (i) of Theorem 2.1 in the following way:

Corollary 2.2. *Suppose $A, B, C \in B(H)$ and A, C are selfadjoint operators with $AB = \lambda BC \neq 0, \lambda \in \mathbb{C}$; then $\lambda \in \mathbb{R}$.*

The following result is a generalization of Theorem 1.2 in [1].

Theorem 2.3. *Let $A, B \in B(H)$. Then the following statements are equivalent.*

(i) *$ABB^* = B^*BA$ and $BAA^* = A^*AB$.*

(ii) *Both AB and BA are normal.*

(iii) *There exist unitary operators U and V in $B(H)$ such that $AB = UB^*A^*$ and $BA = VA^*B^*$.*

Proof. The equivalence of (i) and (ii) was proved by S. Gudder and G. Nagy in [8].

(ii)$\Rightarrow$ (iii) By the fact that an operator $N \in B(H)$ is normal if and only if there exists a unitary operator U such that $N = UN^*$, the proof is trivial.

(iii)$\Rightarrow$ (ii) Observe that $B^*A^* = ABU^*$, and so $AB = UABU^*$. Hence AB commutes with U and U^*, and similarly for B^*A^*. Thus we get $ABB^*A^* = UB^*A^*U^*AB = B^*A^*UU^*AB = B^*A^*AB$. Hence AB is normal. Similarly, from $BA = VA^*B^*$, we can get BA is normal. □

If both A and B are selfadjoint, then Theorem 2.3 becomes Proposition 4.1 and Proposition 4.2 in [1].

For given $0 \neq A \in B(H)$, let

$$S_A^n = \{B \in B(H) : \sigma(AB) \text{ has exactly } n \text{ distinct nonzero values}\}.$$

Then we have

Theorem 2.4. *For each $0 < n < \infty$ and $B \in S_A^n$, if $AB = \lambda BA$ for some $\lambda \in \mathbb{C}$, then $\lambda^n = 1$.*

Proof. Suppose $B \in S_A^n$ with $AB = \lambda BA$. Let $\sigma(AB) \setminus \{0\} = \{\alpha_1, \cdots, \alpha_n\}$. Then $\{\alpha_1, \cdots, \alpha_n\} = \{\lambda\alpha_1, \cdots, \lambda\alpha_n\}$, since $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$. Clearly, for each $\alpha_{i_1} \in \sigma(AB) \setminus \{0\}$, there exist $1 \leq p \leq n$ and $\alpha_{i_j} \in \sigma(AB) \setminus \{0\}, 1 < j \leq p$, with $\alpha_{i_k} \neq \alpha_{i_l}, 1 < k \neq l \leq p$, such that $\alpha_{i_1} = \lambda\alpha_{i_2} = \cdots = \lambda^{p-1}\alpha_{i_p} = \lambda^p\alpha_{i_1}$. If $\alpha_{i_1} = \lambda\alpha_{i_2} = \cdots = \lambda^{p-1}\alpha_{i_p} = \lambda^p\alpha_{i_1}$ and $\alpha_{j_1} = \lambda\alpha_{j_2} = \cdots = \lambda^{q-1}\alpha_{j_q} = \lambda^q\alpha_{j_1}$, for distinct i_1 and j_1, then $p = q$ and $\alpha_{i_l} \neq \alpha_{i_k}, 1 < l, k \leq p$. In fact, we assume that $p < q$. Then $\lambda^p = \lambda^q = 1$ and $\lambda^{q-p} = 1$; hence $\alpha_{j_1} = \alpha_{j_{q-p+1}}$. It is a contradiction. Then we have $\frac{n}{p} \in \mathbb{N}$. Hence $\lambda^n = 1$. □

The next result gives some restrictions that must be satisfied by a pair of operators commuting up to a scalar factor.

Theorem 2.5. *Let $A, B \in B(H)$ such that $AB = \lambda BA \neq 0$, for some $\lambda \in \mathbb{C}$. Then*

(i) *AB is bounded below if and only if both A and B are bounded below;*

(ii) *if A is normal and $R(B)$ is dense, then AB is not nilpotent.*

Proof. (i) If A and B are bounded below, clearly, we have that AB is bounded below. Conversely, if AB is bounded below, then B is bounded below. We also have that A is bounded below, since $AB = \lambda BA$.

(ii) If $AB = \lambda BA$ and A is normal, then $N(A)$ and $\overline{R(A)}$ are reducing subspaces of A and B. Thus A and B have forms $A = \text{diag}(A_1, 0)$ and $B = \text{diag}(B_1, B_2)$, with respect to $H = N(A)^\perp \oplus N(A)$. Since $AB = \text{diag}(A_1B_1, 0)$, we have that AB is nilpotent if and only if A_1B_1 is nilpotent. Hence, without loss of generality, we may assume that A is injective. Then $R(A)$ is dense, since A is normal and injective. If there exists $n \in \mathbb{N}$ such that $(AB)^n = 0$, then $(AB)^{n-1} = 0$, since $R(A)$ and $R(B)$ are dense. By the same deduction, we can get $AB = 0$. It is a contradiction. □

Next, we turn to the study of properties of pairs of operators commuting up to a scalar factor with one of the operators having finite rank.

As in the proof of Theorem 2.5, if $AB = \lambda BA$ and A is normal, then $N(A)$ and $\overline{R(A)}$ are reducing subspaces of A and B. Thus A and B have forms $A = \mathrm{diag}(A_1, 0)$ and $B = \mathrm{diag}(B_1, B_2)$, with respect to $H = N(A)^\perp \oplus N(A)$. Suppose that A has finite rank and $\{a_1, \cdots, a_m\}$ are the distinct nonzero eigenvalues of A. Then $A = \sum_{i=1}^m a_i P_i$, where P_i is the orthogonal projection from H onto $N(A - a_i I)$, for $1 \le i \le m$ and $P_i P_j = P_j P_i = 0, i \ne j$. Denote $H_i = P_i H$. Then A_1 and B_1 have operator matrix forms $A_1 = \mathrm{diag}(a_1 I, \cdots, a_m I)$ and $B_1 = (B_{ij})_{m\times m}$, with respect to the space decomposition $N(A)^\perp = \bigoplus_{i=1}^m H_i$, respectively. With the symbols above, we have the following three results.

Proposition 2.6. *Suppose that $A \in B(H)$ has finite rank and is normal. If $\sigma(A) \cap \sigma(\lambda A)$ has no nonzero element, then there does not exist nonzero $B \in B(H)$ such that $AB = \lambda BA \ne 0$. If $\sigma(A) \cap \sigma(\lambda A)$ has nonzero elements, then there exists $B \in B(H)$ such that $AB = \lambda BA \ne 0$ if and only if either $m = 1, \lambda = 1$ or $m > 1, \lambda \in \mathbb{C} \setminus \{0\}$.*

Proof. If $A = 0$, it is trivial. If $A \ne 0$, then we have $AB = \lambda BA \ne 0$ if and only if $A_1B_1 = \lambda B_1A_1 \ne 0$, since $AB = \mathrm{diag}(A_1B_1, 0)$. Hence, without loss of generality, we may assume that A is injective. Also, $AB = \lambda BA$ if and only if $a_iB_{ij} = \lambda a_j B_{ij}$, for $1 \le i, j \le m$. If $\sigma(A) \cap \sigma(\lambda A)$ has no nonzero element, then $a_i \ne \lambda a_j$, for i, j. Hence there does not exist B_{ij} such that $a_iB_{ij} = \lambda a_j B_{ij} \ne 0$, for i, j. Clearly, if $m = 1$ and $\lambda = 1$, then we have $AB = \lambda BA \ne 0$, for each $0 \ne B \in B(H)$. If $m > 1$ and $\sigma(A) \cap \sigma(\lambda A)$ has nonzero elements, then there exists $1 \le i, j \le m$ such that $a_i = \lambda a_j$. Hence for each B with $B_{ij} \ne 0$ and $B_{st} = 0, s \ne i, t \ne j$, we have $AB = \lambda BA \ne 0$. On the other hand, if $AB = \lambda BA \ne 0$ and $m = 1$, then we have $\lambda = 1$, since $\sigma(A) \cap \sigma(\lambda A)$ has nonzero element a_1. □

If, with respect to a suitable decomposition of the space, B has a form $B = \bigoplus_{i=1}^s J_k^{(i)} \bigoplus_{i=s+1}^{s+t} M_{j_i}^{(i)} (\oplus 0) \oplus B_2$, then we call it a standard form, where the $k \times k$ block operator matrix $J_k^{(i)}$ and the $j_i \times j_i$ block operator matrix $M_{j_i}^{(i)}$ are defined by

$$J_k^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & & 0 \\ & 0 & \ddots & \\ & & \ddots & B_{(k-1)k}^{(i)} \\ B_{k1}^{(i)} & & & 0 \end{pmatrix}, \quad M_{j_i}^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & & 0 \\ & 0 & \ddots & \\ & & \ddots & B_{(j_i-1)j_i}^{(i)} \\ 0 & & & 0 \end{pmatrix},$$

$s, t \in \mathbb{N} \cup \{0\}, k, j_i \in \mathbb{N}$ and $B_{(\cdot,\cdot)}^{(\cdot)}$ are exactly the nonzero entries of $(B_{ij})_{m\times m}$. In this case, if there exist $a^{(i)} \in \sigma(A) \setminus \{0\}, 1 \le i \le s+t$ with $a^{(p)} \ne a^{(l)}$ for $1 \le p \ne l \le s+t$, such that A has a form $A = \bigoplus_{i=1}^s I_k^{(i)} \bigoplus_{i=s+1}^{s+t} N_{j_i}^{(i)} \oplus A'$, $\lambda^k = 1$ and $j_i \le k$ for $s+1 \le i \le s+t$, where A' is a diagonal operator and the $k \times k$ block operator matrix $I_k^{(i)}$ and the $j_i \times j_i$ block operator matrix $N_{j_i}^{(i)}$ are defined by

$$I_k^{(i)} = \begin{pmatrix} a^{(i)}I & & & \\ & \lambda a^{(i)}I & & \\ & & \ddots & \\ & & & \lambda^{k-1}a^{(i)}I \end{pmatrix}, \quad N_{j_i}^{(i)} = \begin{pmatrix} a^{(i)}I & & & \\ & \lambda a^{(i)}I & & \\ & & \ddots & \\ & & & \lambda^{j_i-1}a^{(i)}I \end{pmatrix},$$

respectively, then we say A and B are compatible.

Theorem 2.7. *Suppose that $0 \neq A \in B(H)$ has finite rank and is normal. Then $AB = \lambda BA$ if and only if one of the following conditions holds:*

(i) *there exists a $B_{ii} \neq 0$, $1 \leq i \leq m$, $\lambda = 1$ and $B = \operatorname{diag}(B_{11}, \cdots, B_{mm}, B_2)$;*

(ii) *$B_{ii} = 0$, for $1 \leq i \leq m$, and A and B are compatible.*

Proof. It is easy to show that (i) and (ii) imply $AB = \lambda BA$, respectively.

Conversely, from $AB = \lambda BA$, we have $a_i B_{ij} = \lambda a_j B_{ij}$. If there exists a $B_{ii} \neq 0$, for some $1 \leq i \leq m$, then $\lambda = 1$ and $AB = BA$. Since $(A - a_i I)B = B(A - a_i I)$, by the Fuglede-Putnam Theorem, H_i is a reducing subspace of B. Then $B_{ij} = 0, i \neq j$. Hence $B = \operatorname{diag}(B_{11}, \cdots, B_{mm})$. If $B_{ii} = 0$ for $1 \leq i \leq m$, then $AB = \lambda BA$ implies that $a_i B_{ij} = \lambda a_j B_{ij}$. Since $a_i \neq a_j, i \neq j$, we obtain that there is at most one nonzero entry in each row and column of $(B_{ij})_{m\times m}$. It is easy to see that with respect to a suitable decomposition of the space, B has a form $B = \bigoplus_{i=1}^{s} J_{k_i}^{(i)} \bigoplus_{i=s+1}^{s+t} M_{j_i}^{(i)} \oplus B_2$, where

$$J_{k_i}^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & & 0 \\ & 0 & \ddots & \\ & & \ddots & B_{(k_i-1)k_i}^{(i)} \\ B_{k_i 1}^{(i)} & & & 0 \end{pmatrix}, \quad M_{j_i}^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & & 0 \\ & 0 & \ddots & \\ & & \ddots & B_{(j_i-1)j_i}^{(i)} \\ 0 & & & 0 \end{pmatrix},$$

$s, t \in \mathbb{N} \cup \{0\}, k_i, j_i \in \mathbb{N}$ and $B_{(\cdot,\cdot)}^{(\cdot)}$ are exactly the nonzero entries of $(B_{ij})_{m\times m}$. Also, simple computation, we can get that there exists $a^{(i)} \in \sigma(A)\backslash\{0\}, 1 \leq i \leq s+t$ with $a^{(p)} \neq a^{(l)}$ for $1 \leq p \neq l \leq s+t$ such that A has a form $A = \bigoplus_{i=1}^{s} I_{k_i}^{(i)} \bigoplus_{i=s+1}^{s+t} N_{j_i}^{(i)} \oplus A'$, with respect to the same decomposition of the space, where A' is a diagonal operator and the $k_i \times k_i$ block operator matrix $I_{k_i}^{(i)}$ and the $j_i \times j_i$ block operator matrix $N_{j_i}^{(i)}$ are defined by

$$I_{k_i}^{(i)} = \begin{pmatrix} a^{(i)}I & & & \\ & \lambda a^{(i)}I & & \\ & & \ddots & \\ & & & \lambda^{k_i-1}a^{(i)}I \end{pmatrix}, \quad N_{j_i}^{(i)} = \begin{pmatrix} a^{(i)}I & & & \\ & \lambda a^{(i)}I & & \\ & & \ddots & \\ & & & \lambda^{j_i-1}a^{(i)}I \end{pmatrix}.$$

We will show that if $s > 1$, then $k_1 = \cdots = k_s$. We assume that $k_i < k_j, 1 \leq i, j \leq s$; then $\lambda^{k_i} = 1$ and $\lambda^{k_j} = 1$. Hence $\lambda^{k_j - k_i} = 1$. Therefore $a^{(i)} = \lambda^{k_j-k_i} a^{(i)}$. It is a contradiction with $a_i \neq a_j, i \neq j$. Hence $k_1 = \cdots = k_s$. Let $k = k_i$; then, clearly we have $\lambda^k = 1$ and the $j_i \leq k$, for $s + 1 \leq i \leq s + t$. That is to say, A and B are compatible.

□

Remark. If A is compact and normal, a result similar to Theorem 2.7 can be obtained by the same deduction. Its representation is more complex, and so it is omitted.

For given $0 \neq A \in B(H)$, let $S_A^\lambda = \{B \in B(H) : AB = \lambda BA\}$. Then S_A^λ is a closed subspace of $B(H)$. If $\lambda^n = 1$, for some $n \in \mathbb{N}$, then for each $B \in S_A^\lambda$, we have $B^{n+1} \in S_A^\lambda$. In fact, $AB^{n+1} = \lambda^{n+1}B^{n+1}A = \lambda B^{n+1}A$. If H is finite dimensional, by Theorem 4.4.6 in [10], we have $S_A^\lambda \neq \{0\}$ if and only if $\sigma(A) \cap \sigma(\lambda A) \neq \phi$.

Proposition 2.8. *Suppose that H is finite dimensional and $0 \neq A \in B(H)$ is normal. Then*

$$\dim S_A^\lambda = (\dim N(A))^2 + \sum_{1 \le i,j \le m} \dim H_i \dim H_j \psi(i,j),$$

where $\psi(i,j) = \begin{cases} 1 & a_i = \lambda a_j \\ 0 & a_i \neq \lambda a_j \end{cases}$.

Proof. Suppose that A is normal. Then $AB = \lambda BA$ if and only if $A_1B_1 = \lambda B_1A_1$. Hence $\dim S_A^\lambda = \dim S_{A_1}^\lambda + (\dim N(A))^2$. Next, we assume that A is injective. We will show that $\dim S_A^\lambda = \sum_{1\le i,j\le m} \dim H_i \dim H_j\psi(i,j)$. In this case, $AB = \lambda BA$ if and only if $a_iB_{ij} = \lambda a_jB_{ij}$, for $1 \le i,j \le m$. If $a_i \neq \lambda a_j$, *i.e.* $\psi(i,j) = 0$, for some $1 \le i,j \le m$, then $AB = \lambda BA$ implies B_{ij} must be zero. If $a_i = \lambda a_j$, *i.e.* $\psi(i,j) = 1$, for some $1 \le i,j \le m$, then for each $B_{ij} \in M_{\dim H_i \times \dim H_j}$, we have $a_iB_{ij} = \lambda a_jB_{ij}$. Thus $\dim S_A^\lambda = \sum_{1\le i,j\le m} \dim H_i \dim H_j\psi(i,j)$. □

3. Sequential quantum measure

We first prove a lemma.

Lemma 3.1. *Suppose that A is an injective positive operator and $B \in B(H)$ with dense range. If $A \circ B \in P(H)$, then $AB = BA = I$.*

Proof. Since $\overline{R(A)}$ is dense in H, we have that $R(A^{\frac{1}{2}})$ is dense. That is, $\overline{R(A^{\frac{1}{2}})} = H$. Hence $\overline{R(BA^{\frac{1}{2}})} = H$. This implies that $\overline{R(A^{\frac{1}{2}}BA^{\frac{1}{2}})} = H$. By the assumption $A \circ B \in P(H)$, we have $A^{\frac{1}{2}}BA^{\frac{1}{2}} = I$. Therefore $R(A^{\frac{1}{2}}) = H$. This shows that A is invertible. Hence $B = A^{-1}$, i.e., $AB = BA = I$. □

If A is positive and $B \in B(H)$, then A and B have operator matrices $A = \mathrm{diag}(A_1, 0)$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with respect to the space decomposition $H = \overline{R(A)} \oplus N(A)$, respectively. Since

$$\begin{pmatrix} {A_1}^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} {A_1}^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} {A_1}^{\frac{1}{2}}B_{11}{A_1}^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix},$$

$A \circ B \in P(H)$ is equivalent to $A_1 \circ B_{11} \in P(\overline{R(A)})$. With the symbols above we have

Theorem 3.2. *Suppose that $A \in B(H)$ is positive and that $B \in B(H)$ is selfadjoint. Then $N(A)$ is an invariant subspace of B, $N(B|_{\overline{R(A)}})$ is an invariant subspace of A, and $A \circ B \in P(H)$ if and only if $AB = BA$ and $A|_{\overline{R(B|_{\overline{R(A)}})}} \circ B|_{\overline{R(B|_{\overline{R(A)}})}} \in P(\overline{R(B|_{\overline{R(A)}})})$.*

Proof. Suppose that $A \in B(H)$ is positive, $B \in B(H)$ is selfadjoint, $N(A)$ is an invariant subspace of B, $N(B|_{\overline{R(A)}})$ is an invariant subspace of A and $A \circ B \in P(H)$. Then we have $A = \mathrm{diag}(A_1, 0)$, $B = \mathrm{diag}(B_{11}, B_{22})$ and $A_1 \circ B_{11} \in$

$P(\overline{R(A)})$, where $A_1 = A|_{\overline{R(A)}}$, $B_{11} = B|_{\overline{R(A)}}$ and $B_{22} = B|_{N(A)}$. Hence $B_{11}A_1B_{11} = B_{11}$. If we consider the space decomposition $\overline{R(A)} = N(B_{11}) \oplus \overline{R(B_{11})}$, we have $A_1 = \mathrm{diag}(A_{11}, A_{22})$ and $B_{11} = \mathrm{diag}(0, B'_{11})$. Hence $B_{11}A_1|_{\overline{R(B_{11})}} = I_{\overline{R(B_{11})}}$ and $B'_{11}A_{22}B'_{11} = B'_{11}$. Then we have $A_{22} \circ B'_{11} \in P(\overline{R(B_{11})})$. By Lemma 3.1, we have $A_{22}B'_{11} = I_{\overline{R(B_{11})}}$. Hence $A_1B_{11} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22}B'_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = B_{11}A_1$. Therefore $AB = \mathrm{diag}(0, I, 0) = BA$ and $A|_{\overline{R(B|_{\overline{R(A)}})}} \circ B|_{\overline{R(B|_{\overline{R(A)}})}} \in P(\overline{R(B|_{\overline{R(A)}})})$.

On the other hand, suppose that $AB = BA$ and $A|_{\overline{R(B|_{\overline{R(A)}})}} \circ B|_{\overline{R(B|_{\overline{R(A)}})}} \in P(\overline{R(B|_{\overline{R(A)}})})$. With respect to $H = \overline{R(A)} \oplus N(A)$, A and B have operator matrices $A = \mathrm{diag}(A_1, 0)$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}$, respectively. By the assumption, we have $AB = \begin{pmatrix} A_1B_{11} & A_1B_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_{11}A_1 & 0 \\ B_{12}^*A_1 & 0 \end{pmatrix} = BA$. Hence $B_{12} = 0$ and $A_1B_{11} = B_{11}A_1$. Therefore $N(A)$ is an invariant subspace of B. With respect to $\overline{R(A)} = N(B_{11}) \oplus \overline{R(B_{11})}$, A_1 and B_{11} have operator matrices $A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$ and $B_{11} = \mathrm{diag}(0, B'_{11})$, respectively. From $A_1B_{11} = B_{11}A_1$ we have $A_{12}B'_{11} = 0$. Hence $A_{12} = 0$ and $A \circ B = \mathrm{diag}(0, A_{22}^{\frac{1}{2}}B'_{11}A_{22}^{\frac{1}{2}}, 0)$. Therefore $N(B|_{\overline{R(A)}})$ is an invariant subspace of A and $A \circ B \in P(H)$, since $A|_{\overline{R(B|_{\overline{R(A)}})}} \circ B|_{\overline{R(B|_{\overline{R(A)}})}} \in P(\overline{R(B|_{\overline{R(A)}})})$. $\square$

If $A, B \in \varepsilon(H)$, then we can easily get that the part of the necessity condition of Theorem 3.2 is Theorem 1.3 in [9]. The virtue of our proof is that we can know the structure of A, B and AB clearly. One can use this method to prove Corollary 2.4, Theorem 2.5 and Theorem 2.6 in [9].

The following theorem is an answer to the question raised by S. Gudder and G. Nagy in [9], which is also a generalization of Theorem 2.6 (c) in [9]. This question was also independently answered by A. Gheondea and S. Gudder in [7]. Though our proof is essentially the same as A. Gheondea and S. Gudder's, maybe our presentation of it is better and simpler. We still retain the proof here.

Theorem 3.3. *If $A, B \in \varepsilon(H)$, then we have $A \circ B \geq B$ if and only if $AB = BA = B$.*

Proof. It is clear that $AB = BA = B$ implies $A \circ B \geq B$. Conversely, if

$$A \circ B = \begin{pmatrix} A_1^{\frac{1}{2}}B_{11}A_1^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix},$$

then $B_{12} = B_{22} = 0$. Hence $B = \mathrm{diag}(B_{11}, 0)$ and $A_1^{\frac{1}{2}}B_{11}A_1^{\frac{1}{2}} \geq B_{11}$. This implies that $A_1^{\frac{n}{2}}B_{11}A_1^{\frac{n}{2}} \geq B_{11}$, for $n \in \mathbb{N}$. Suppose that $A_1 = \int_0^1 \lambda dE\lambda$ is the spectral calculus of A_1. Let $H_1 = E(1)$ and $H_2 = E([0, 1))$. Then for each $x \in H_2$, we have $A_1^{\frac{n}{2}}x \to 0$ as $n \to \infty$. This implies $B_{11}x = 0$, for $x \in H_2$. Hence $A = \mathrm{diag}(I, A'_1, 0)$ and $B = \mathrm{diag}(B'_{11}, 0, 0)$, with respect to $H = H_1 \oplus H_2 \oplus N(A)$, where $A'_1 = A|_{H_2}$

and $B'_{11} = B|_{H_1}$. Therefore,

$$AB = \begin{pmatrix} B'_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = BA = B.$$

□

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