SYMMETRY OF EXTREMAL FUNCTIONS
FOR THE CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

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ABSTRACT. We study the symmetry property of extremal functions to a family of weighted Sobolev inequalities due to Caffarelli-Kohn-Nirenberg. By using the moving plane method, we prove that all non-radial extremal functions are axially symmetric with respect to a line passing through the origin.

1. INTRODUCTION

This paper is concerned with symmetry properties of extremal functions for the following weighted Sobolev inequalities due to Caffarelli, Kohn and Nirenberg ([3]): for all \( u \in C_0^\infty(\mathbb{R}^N) \),

\[
\left( \int_{\mathbb{R}^N} |x|^{-bp}|u|^p \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx
\]

where for \( N \geq 3 \):

\( -\infty < a < \frac{N-2}{2} \), \( a \leq b \leq a+1 \), and \( p = \frac{2N}{N-2 + 2(b-a)} \),

and for \( N = 2 \):

\( -\infty < a < 0 \), \( a < b \leq a+1 \), and \( p = \frac{2}{b-a} \).

Let \( D^{1,2}_a(\mathbb{R}^N) \) be the completion of \( C_0^\infty(\mathbb{R}^N) \), with respect to the inner product

\[
(u, v)_a = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v \, dx.
\]

Then inequalities (1) are extended to all \( u \in D^{1,2}_a(\mathbb{R}^N) \). Define

\[
S(a, b) = \inf_{u \in D^{1,2}_a(\mathbb{R}^N) \setminus \{0\}} E_{a,b}(u)
\]
to be the best embedding constants, where

\[ E_{a,b}(u) = \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} |x|^{-bp} |u|^p \, dx \right)^{\frac{2}{p}}}. \]

The extremal functions for \( S(a,b) \) are least-energy solutions of the Euler equation

\[ -\text{div}(|x|^{-2a} \nabla u) = |x|^{-bp} u^{p-1}, \quad u \geq 0, \quad \text{in } \mathbb{R}^N. \]

The best constant and the minimizers for the Sobolev inequality \( (a = b = 0) \) were given by Aubin \cite{1} and Talenti \cite{16}. In \cite{13}, Lieb considered the case \( a = 0, 0 < b < 1 \) and gave the best constants and explicit minimizers. In \cite{8}, Chou and Chu considered the \( a \)-nonnegative region and gave the best constants and explicit minimizers. The symmetry of the minimizers has also been studied in \cite{13} and \cite{8}.

In summary, for \( a \geq 0 \), all nonnegative solutions in \( \mathcal{D}^{1,2}_a(\mathbb{R}^N) \) for the corresponding Euler equation \( \text{(7)} \) are radial solutions (in the case \( a = b = 0 \), they are radial with respect to some point) and explicitly given \((1), (16), (13), (8)\). This was established in \cite{8}, using a generalization of the moving plane method \((e.g., (10), (11), (2))\).

Recently, Catrina and Wang \((4), (5)\) have discovered the symmetry-breaking phenomenon of the extremal functions for a sub-region of parameters when \( a < 0 \) occurs \((\text{see also (14) for a partial result})\). More precisely, they proved in \cite{4}, \cite{5} that there is a function \( h(a) \) defined for \( a \leq 0 \), satisfying \( h(0) = 0 \), \( a < h(a) \), and \( a + 1 - h(a) \) are strictly decreasing as \( a \to -\infty \), such that for any \((a,b)\) satisfying \( a < 0 \) and \( a < b < h(a) \), the extremal function \( S(a,b) \) is non-radial. In a recent preprint \cite{9} of Felli and Schneider, it is observed that this curve \( h(a) \) can be sharpened to the following:

\[ h(a) = 1 + a - \frac{N}{2} \left( 1 - \frac{N - 2 - 2a}{\sqrt{(N - 2 - 2a)^2 + 4(N - 1)}} \right). \]

A natural question is on the symmetry of these non-radial extremal functions. In \cite{6}, Catrina and Wang have proved that for \( c \in (0,1) \) fixed, for sufficiently large \( -a \), up to rotations and dilations, the extremal function to \( S(a,a+c) \) is unique and has \( \mathcal{O}(N-1) \) symmetry \((i.e., \text{the extremal function is axially symmetric with respect to a line passing through the origin})\).

In this paper, the main result is the following that gives the exact symmetry for all non-radial extremal functions.

**Theorem 1.1.** Let \( N \geq 2 \). For all \((a,b)\) satisfying \( a < 0 \) and \( a < b < h(a) \), the extremal function \( u \) to \( S(a,b) \) has exact \( \mathcal{O}(N-1) \) symmetry. More precisely, up to a rotation, \( u(x) \) only depends on the radius \( r \) and the angle \( \theta_N \) between the positive \( x_N \)-axis and \( Ox \), and on each sphere \( \{ x \in \mathbb{R}^N \mid |x| = r \} \), \( u \) is strictly decreasing as the angle \( \theta_N \) increases, \(i.e., u = u(\theta_N, r) \) and \( \frac{\partial u}{\partial \theta_N}(\theta_N, r) < 0 \) for all \( r > 0 \) and all \( \theta_N \in (0,\pi) \).

Another natural question is what is the symmetry of the extremal functions for the parameters \((a,b)\) with \( h(a) \leq b < a+1 \) for \( a < 0 \). The following result offers a partial answer.
Theorem 1.2. Let $N \geq 3$, and $b_0 \in (0,1)$ be fixed. Then with $C > 0$ given, for all $(a,b)$ sufficiently close to $(0,b_0)$, any bound state solution of equation $u$ of (7) satisfying $\|u\|_a \leq C$ is radially symmetric with respect to the origin.

This theorem improves the result of [17], in which they obtained a result for the least-energy solutions. Our argument is different from that in [17].

In Section 2 we first recall a new formulation of the inequalities (1) from [4], [5]. For this alternative formulation some new ideas developed in [7], [15] for proving symmetry properties via the maximum principle can be applied to derive the desired results. Then we shall give the proofs of the main theorems.

2. Proofs of the main results

We start by recalling a reformulation of the inequalities (1) given in [4], [5]. The proofs of our main results will be based on this new formulation.

We shall use the notation $x = (\theta, t) \in C = S^{N-1} \times \mathbb{R}$, which is placed in $\mathbb{R}^{N+1}$ with the $t$-axis coinciding with the $x_{N+1}$-axis in $\mathbb{R}^{N+1}$. To $u$ a smooth function with compact support in $\mathbb{R}^N \setminus \{0\}$, we associate $v$ a smooth function with compact support on $C$, by the transformation

$$u(y) = |y|^{-\frac{N-2-2a}{2}} v \left( \frac{y}{|y|} - \ln |y| \right).$$

Here for $y \in \mathbb{R}^N \setminus \{0\}$, with $t = -\ln |y|$ and $\theta = \frac{y}{|y|}$ we have $(\theta, t) \in C$. It was proved that the mapping (9) is a Hilbert space isomorphism from $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$ to $H^1(C)$, where the inner product on $H^1(C)$ is

$$(v, w) = \int_C \nabla v \cdot \nabla w + \left( \frac{N-2-2a}{2} \right)^2 v w \ d\mu.$$

If there is no ambiguity, we still use $\| \cdot \|_a$ for the norm in $H^1(C)$ with respect to the above inner product.

Now we define an energy functional on $H^1(C)$ by

$$F_{a,b}(v) = \frac{\int_C |\nabla \theta v|^2 + v_t^2 + \left( \frac{N-2-2a}{2} \right)^2 v^2 \ d\mu}{\left( \int_C v^p \ d\mu \right)^{2/p}}.$$  

If $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^N)$ and $v \in H^1(C)$ are related through (9), then

$$E_{a,b}(u) = F_{a,b}(v).$$

Moreover, if $u$ is a solution of (7), then $v$ satisfies

$$-v_{tt} - \Delta \theta v + \left( \frac{N-2-2a}{2} \right)^2 v = v^{p-1}, \quad v > 0, \quad \text{on } C,$$

where $t = -\ln |y|$, and $\Delta \theta$ is the Laplace operator on the $N-1$ sphere. In summary we have the following from [4], [5].

Proposition 2.1. (i) The transformation (9) gives a Hilbert space isomorphism between $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$ and $H^1(C)$.

(ii) If $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^N)$ and $v \in H^1(C)$ are related through (9), then $E_{a,b}(u) = F_{a,b}(v)$. Therefore, for $S(a,b)$ as defined in (14), it follows that $S(a,b) = \inf_{H^1(C) \setminus \{0\}} F_{a,b}(v)$.
(iii) Solutions of (7) and (11) are in one-to-one correspondence, being related through [9].

We shall prove the following result regarding the new formulation, which implies Theorem 1.1. By the result in [5], without loss of generality, we may assume that any solution of (11) achieves its maximum at \( t = 0 \), and is even in \( t \) and monotonic decreasing for \( t > 0 \). For \( t \in \mathbb{R} \) we denote by \( O_t \) the point on the \( x_{N+1} \)-axis in \( \mathbb{R}^{N+1} \) with coordinate \( t \), i.e., \( O_t = (0, ..., 0, t) \).

**Theorem 2.2.** Let \( N \geq 2 \). Let \( v \) be a least-energy solution of (11) such that \( v \) is non-radial (i.e., \( v \) depends on \( \theta \)), and let \( P_0 \) be a maximum point of \( v \). Then for each fixed \( t \), \( v(\theta, t) \) is axially symmetric with respect to the line passing through \( O_t \) and \( P_0 \). Moreover, by assuming that \( P_0 \) is located on the positive \( x_N \)-axis and denoting by \( \theta_N \) the angle between the vector from \( O_t \) to \( P_0 + O_t \) and the vector from \( O_t \) to \( x \) with \( x = (\theta, t) \), we have that \( v \) depends only on \( t \) and \( \theta_N \), and \( \frac{\partial v}{\partial \theta_N}(\theta_N, t) < 0 \) for all \( \theta_N \in (0, \pi) \) and all \( t \in \mathbb{R} \).

We shall follow the ideas developed recently in [7], [15], which are along the line of research of using the moving plane method for symmetry of positive solutions (10), (11).

**Proof.** For simplicity, we write \( \lambda_0 = \left( \frac{N-2}{2} - p \right)^2 \). Let \( T \) be any hyperplane in \( \mathbb{R}^{N+1} \) that passes through the \( x_{N+1} \)-axis. Let \( C^+ \) be one of the half cylinders of \( C \setminus T \). For \( x \in C^+, x^* \) denotes the reflection point of \( x \) with respect to the plane \( T \).

We claim that one of the following assertions holds: (a) \( v(x) = v(x^*) \) for all \( x \in C^+ \); (b) \( v(x) > v(x^*) \) for all \( x \in C^+ \); (c) \( v(x) < v(x^*) \) for all \( x \in C^+ \).

To this end, we first prove that either \( v(x) \geq v(x^*) \) for all \( x \in C^+ \), or \( v(x) \leq v(x^*) \) for all \( x \in C^+ \). Suppose this is not true. Then the following two sets are both nonempty:

\[ D_+ = \{ x \in C^+ | v(x) > v(x^*) \}, \quad D_- = \{ x \in C^+ | v(x) < v(x^*) \}. \]

Define \( w(x) = v(x) - v(x^*) \) for \( x \in C^+ \). Then \( w \) satisfies

\[
\begin{cases}
-\Delta w + \lambda_0 w = c(x)w, & x \in C^+, \\
w = 0, & x \in \partial C^+
\end{cases}
\]

where \( c(x) = (p - 1) \int_0^1 (sv(x) + (1 - s)v(x^*))^{p-2}ds \). Denote \( D_+ = \{ x^* | x \in D_+ \} \) and define \( u \) on \( C \) as follows:

\[
u(x) = \begin{cases}w(x), & x \in D_+, \\
dw(x^*), & x \in D_-^*, \\
0, & \text{otherwise.}
\end{cases}
\]

Here the constant \( d > 0 \) is chosen such that

\[ \int_C u(x) \phi_1(x) = 0 \]

with \( \phi_1 \) being the first eigenfunction of the following eigenvalue problem:

\[
\begin{cases}
-\Delta \phi + \lambda_0 \phi - (p - 1)v^{p-2}\phi = \mu \phi, & x \in C, \\
\phi = 0, & x \in \partial C,
\end{cases}
\]
which is well-defined due to the decay property of \( v \) as \( |t| \to \infty \). Let \( \mu_2 \) be the second eigenvalue of \([14]\). Since \( v \) is a least-energy solution of \([11]\) we have \( \mu_2 \geq 0 \). On the other hand, we may check that \( u \) is not identically zero and

\[
-\Delta u + \lambda \alpha u - (p-1)v^{p-2}u = \begin{cases} 
\leq 0, & x \in D_+, \\
\geq 0, & x \in D_+, \\
= 0, & otherwise.
\end{cases}
\]

Following these we have a contradiction as follows:

\[
0 > \int_{\mathcal{C}} u(-\Delta u + \lambda \alpha u - (p-1)v^{p-2}u)dx = \int_{\mathcal{C}} (|\nabla u|^2 + \lambda \alpha u^2 - (p-1)v^{p-2}u)dx \geq 0.
\]

Thus we have proved that either \( w > 0 \) or \( w \leq 0 \) for \( x \in \mathcal{C}^+ \). By the strong maximum principle, we have either \( w > 0 \), or \( w < 0 \), or \( w = 0 \) for all \( x \in \mathcal{C}^+ \), corresponding to the three alternatives in the claim. This proves the claim.

Next, let \( P_0 \) be a maximum point of \( v \). Without loss of generality, we may assume that \( P_0 = (0, \ldots, 0, 1, 0) \in \mathbb{R}^{N+1} \), i.e., \( P_0 \) is on the \( x_N \)-axis. Let \( T_0 \) be the hyperplane in \( \mathbb{R}^{N+1} \) that has \( x_N \) as its normal direction. Let \( \mathcal{C}^+ = \{ x \in \mathcal{C} \mid x_N > 0 \} \) and for \( x \in \mathcal{C}^+ \), let \( x^- \) be the reflection point of \( x \) with respect to \( T_0 \). Then from the first part of the proof we have \( v(x) > v(x^-) \) for all \( x \in \mathcal{C}^+ \). Since otherwise, \( v(P_0) = v(P_0^-) = \max \mathcal{C} v \) and we may produce a contradiction as follows. Let \( T \) be any hyperplane in \( \mathbb{R}^{N+1} \) that contains the \( x_{N+1} \)-axis such that \( P_0 \notin T \), and let \( \mathcal{C} \) be the half cylinder of \( \mathcal{C} \setminus T \) such that \( P_0 \in \mathcal{C} \). For \( x \in \mathcal{C} \), let \( x^* \) denote the reflection point of \( x \) with respect to \( T \). Then we have \( v(P_0) \geq v(P_0^-) \) and \( v((P_0^-)^*) \leq v(P_0^-) \). By the proof in the first part, we obtain \( v(x) = v(x^*) \) for all \( x \in \mathcal{C} \). Since \( T \) is arbitrary, we assert that \( v \) must be independent of its \( \theta \) component, which is a contradiction with the assumption.

Finally, choose any two-dimensional plane in \( \mathbb{R}^N \) that contains the \( x_N \)-axis. For simplicity we assume this is generated by the \( x_1 \) and \( x_N \) directions. Let \( R_{c_N} \) be the ray from the origin on this two-dimensional plane that has an angle \( \omega \) with the positive \( x_1 \) direction for \( \omega \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) with \( \omega = \frac{\pi}{2} \) corresponding to the positive \( x_N \)-axis. Let \( \nu_\omega \) be the normal vector of the ray \( R_{c_N} \) in this two-dimensional plane with \( \nu_\omega = c_N \) (the \( N \)-th axis direction) and let \( T_\omega \) be the \( N \)-dimensional hyperplane in \( \mathbb{R}^{N+1} \) that has \( \nu_\omega \) as its normal vector. Let \( \mathcal{C}_\omega \) be the half cylinder of \( \mathcal{C} \setminus T_\omega \) that contains \( P_0 \) for \( \omega \in [0, \frac{\pi}{2}) \), and for \( x \in \mathcal{C}_\omega \), let \( x_\omega \) be the reflection point of \( x \) with respect to \( T_\omega \). Define \( u_\omega(x) = v(x) - v(x_\omega) \) for \( x \in \mathcal{C}_\omega \). Then we have

\[
\begin{cases} 
-\Delta u_\omega + \lambda \alpha u_\omega - c_\omega(x)u_\omega = 0, & x \in \mathcal{C}_\omega, \\
u_\omega = 0, & x \in \partial \mathcal{C}_\omega.
\end{cases}
\]

where \( c_\omega(x) = (p-1) \int_0^1 (sv(x) + (1-s)v(x_\omega))^p - 2ds \). For \( \omega = 0 \), we have \( u_0(x) > 0 \) for all \( x \in C_0 \). Let

\[
\omega_0 = \sup \{ \omega \mid u_\omega(x) \geq 0, \forall x \in \mathcal{C}_\omega, \forall 0 \leq \omega' \leq \omega \leq \frac{\pi}{2} \}.
\]

We want to prove \( \omega_0 = \frac{\pi}{2} \). If this is not true, we produce a contradiction as follows. From the definition of \( \omega_0 \) and the proofs in the first part we have for \( 0 \leq \rho \leq \omega_0 \), \( u_\omega(x) > 0 \) for \( x \in \mathcal{C}_\omega, \frac{\partial u_\omega}{\partial \nu_\omega}(x) > 0 \) for \( x \in \mathcal{T}_\omega \), and \( u_\omega(x) = 0 \) for all \( x \in \mathcal{C}_\omega \). Since \( P_0^{\omega_0} \) is also a global maximum point of \( v \), we have \( P_0^{\omega_0} \in \mathcal{T}_{\omega_1} \) for some \( \omega_1 \in (0, \omega_0) \) and \( \nabla v(P_0^{\omega_0}) = 0 \). We have a contradiction with \( 2 \frac{\partial u}{\partial \nu_\omega}(P_0^{\omega_0}) = \frac{\partial u}{\partial \nu_\omega}(P_0^{\omega_1}) > 0 \).
Therefore we obtain $u_\pm(x) \geq 0$ in $\mathbb{C}_\pm^n$. Using the same argument we may obtain also $u_-\pm(x) \geq 0$ in $\mathbb{C}_-\pm^n$. This leads to $u_\pm(x) = 0$ for all $x \in \mathbb{C}_\pm^n$. Now it follows that for all $\omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $2 \frac{\partial v}{\partial \omega}(x) = \frac{\partial u}{\partial \omega}(x) > 0$ for all $x \in T_\omega$. Note that for $t$ fixed, the angle $\theta_N$ between the vector from $O_t$ to $P_b + O_t$ and the vector from $O_t$ to the point $x$ is given by $\theta_N = \frac{\pi}{2} - \omega$; so we get $\frac{\partial u}{\partial \theta_N}(x) < 0$. This completes the proof of Theorem $2.2$. \hfill \Box

**Remark.** The method above was used in [7] for more general nonlinearity $f(v)$ than $v^{p-1}$. This also applies to our equation (11). In fact, the same result of the problem (see [17], [18]). Thus if we confine ourselves to the subspace of even functions in $t$, the radial solutions are isolated. Without loss of generality, we assume that a maximum point of $v$ is achieved at $x = 0$; otherwise, due to the translation invariance in $t$, we conclude that the radial solutions are isolated. By the result of [3] that any (even) solution is monotonically decreasing to zero as $|t| \to \infty$, we have that for $(a, b)$ near $(0, b_0)$, all solutions uniformly tend to zero as $|t| \to \infty$. Let $C > 0$ be fixed. Now if for a sequence $(a_n, b_n) \to (0, b_0)$ with $a_n < 0$ (in equation (11) this corresponds to $p_n \to p_0 = \frac{2N}{N-2+2q_0} \in (2, 2')$, there are non-radial solutions $v_n$ for these parameters satisfying $\|v_n\|_{a_n} \leq C$, then we must have for a subsequence, $\max_{\mathcal{C}} v_n = M_n \to \infty$. Otherwise, due to the energy bound and using the concentration-compactness arguments we would get a sequence of non-radial solutions converging to the even (nontrivial) radial solution at $(0, b_0)$, which is a contradiction with the fact that even radial solutions are isolated. Without loss of generality, we assume that a maximum point of $v_n$ is at $P = (1, 0, \ldots, 0) \in \mathcal{C} \subset \mathbb{R}^{N+1}$ and for given $0 < r < 1$ we define a map from $B_r(P) := \{x \in \mathcal{C} \mid (x_1 - 1)^2 + x_2^2 + \ldots + x_{N+1}^2 < r^2\}$ onto $B_r(0) \subset \mathbb{R}^N$ by $\psi(x) = (x_2, \ldots, x_N, x_{N+1}) \in \mathbb{R}^N$. It is easy to see that $\psi$ has an inverse map and the Jacobians of both $\psi$ and $\psi^{-1}$ are $1$ at $P$ and $0$ respectively. Define

$$w(x) = M_n^{-1}v_n(\psi^{-1}(M_n^{\frac{p_0-2}{2}} x)), \text{ for } x \in \mathbb{R}^N, \text{ with } |x| < M_n^{\frac{p_0-2}{2}} r.$$ 

Then $0 \leq w_n \leq 1$ and $w_n(0) = 1$ and $w_n$ satisfies an elliptic equation of the form

$$- \sum_{i,j=2}^{N+1} a_{i,j,n}(x) \frac{\partial w_n}{\partial x_i \partial x_j} + \sum_{i=2}^{N+1} b_{i,n}(x) \frac{\partial w_n}{\partial x_i} + c_n w_n = w_n^{p_n-1}.$$

Using the fact that the Jacobians of $\psi$ and $\psi^{-1}$ both tend to $1$ at $P$ and $0$ respectively, we have $a_{i,j,n}(x) \to Id_{N \times N}$, $b_{i,n}(x) \to 0$, $c_n(x) \to 0$, as $n \to \infty$, all uniformly in a bounded set of $x$. By elliptic theory we have that $w_n$ converges to $w$ in $C^2_{loc}(\mathbb{R}^N)$ and $w$ is a positive solution of $-\Delta w = w^{p_0-1}$ in $\mathbb{R}^N$. However,
this is a contradiction since it is well known that such a solution does not exist for $p_0 \in (2, 2^*)$. This completes the proof of Theorem 1.2.

The question on the symmetry of extremal functions for $h(a) \leq b < a + 1$, in general, remains open.

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