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ARC-ANALYTIC ROOTS OF ANALYTIC FUNCTIONS ARE LIPSCHITZ

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Abstract. Let g be an arc-analytic function (i.e., analytic on every analytic arc) and assume that for some integer r the function g^r is real analytic. We prove that g is locally Lipschitz; even C^1 if r is less than the multiplicity of g^r. We show that the result fails if g^r is only a C^k, arc-analytic function (even blow-analytic), $k \in \mathbb{N}$. We also give an example of a non-Lipschitz arc-analytic solution of a polynomial equation $P(x, y) = y^d + \sum_{i=1}^{d} a_i(x)y^{d-i}$, where a_i are real analytic functions.

1. Introduction

Let U be an open subset of $\mathbb{R}^n$. Following [13], we say that a map $f : U \to \mathbb{R}^k$ is *arc-analytic* if $f \circ \alpha$ is analytic for any analytic arc $\alpha : (-\varepsilon, \varepsilon) \to U$. In general, arc-analytic maps are very far from being analytic; in particular, there are arc-analytic functions that are not subanalytic [14], not continuous [4], and with a nondiscrete singular set [15]. Hence it is natural to consider only arc-analytic maps with subanalytic graphs. T.-C. Kuo (motivated by equisingularity problems) introduced in [10] the notion of *blow-analytic* functions, i.e., functions that become analytic after a composition with appropriate proper bimeromorphic maps (e.g., composition of blowing-up with smooth centres).

Clearly any blow-analytic mapping is arc-analytic and subanalytic. The converse holds in a slightly weaker form [3] (see also [18]).

Blow-analytic maps have been studied by several authors (see the survey [7]). It is known that in general subanalytic and arc-analytic functions are continuous [13], but not necessarily (locally) Lipschitz [7], [19].

On the other hand, it seems that the problem of taking r-th roots of smooth or analytic functions has been successfully studied only for $r = 2$. Glaeser [8] (see also Dieudonné [5]) proved that a nonnegative C^2 function in an open subset of $\mathbb{R}^n$, which vanishes to second order, has a positive square root of class C^1. This kind of result is no longer true for $r \geq 3$; see our remark 3.8. A more detailed study of the one-variable case can be found in [1].

The main result of this note is theorem 3.1. It states the following: if g is an arc-analytic function such that for some natural r the function $f = g^r$ is analytic,

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then g is locally Lipschitz. Moreover, if r is less than the multiplicity of f, then g is C^1.

By the curve selection lemma, the proof of Theorem 3.1 is reduced to the case of two variables. To study the two variables case we introduce (in section 2) a new tool, which we call the real tree model of a germ of a real analytic function in two variables. It takes into account not only the order of contact between the Newton-Puiseux roots of the function, but also the reality of some coefficients of Puiseux expansions of its roots. Combined with Theorem 2.1 (a result of Kuo and Lu [11], see also [12]), it enables us to control the orders of $f|\nabla f|^{1/r-1}$ on real analytic arcs.

In the end we give examples which prove that we cannot take $f = g^r$ only arc-analytic and C^k, $1 \leq k < r$. We give also a striking example of a non-Lipschitz arc-analytic solution of a polynomial equation $P(x,y) = y^d + \sum_{i=1}^d a_i(x)y^{d-i}$, where the a_i are real analytic functions (even polynomials).

Let us explain now how one can check in Theorem 3.1 the assumption that the function g is arc-analytic. In [16] we proved that if $g : U \to \mathbb{R}$, $0 \in U \subset \mathbb{R}^n$, is a continuous, bounded, subanalytic function that is analytic on all analytic arcs (germs) passing through the origin, then it is arc-analytic in a neighbourhood of the origin (arc-analyticity is an open property). This shows that when checking the arc-analyticity of g in a neighbourhood of the origin, it suffices to consider only the Fukui invariant of f [6], denoted by $A(f)$, which is the set of orders of $f \circ \gamma$, where $\gamma(t)$, $\gamma(0) = 0$, are arbitrary analytic arcs.

One has to test only that $A(f) \subset r\mathbb{N}$, and this can be effectively computed. On the other hand, these sets $A(f)$ have been recently described in [9].

2. Definitions and notation

2.1. **Newton-Puiseux factorisation of analytic functions.** For later use we need to recall the following facts, which can be found in [12].

Let us consider a germ of a holomorphic function $f(x,y)$ in a neighbourhood of the origin in $\mathbb{C}^2$. Replacing, if necessary, x by $cy + x$ with c generic, and applying the Preparation Theorem, we can write

$$f(x,y) = u(x,y) \cdot F(x,y), \tag{2.1}$$

where u is a unit and $F(x,y) = y^d + \sum_{i=1}^d a_i(x)y^{d-i}$, with a_i real analytic functions, is the associated Weierstrass polynomial.

Then the Newton-Puiseux factorisation is of the form

$$f(x,y) = u(x,y) \cdot \prod_{i=1}^{d} [y - \beta_i(x)], \tag{2.2}$$

where the β_i are fractional power series with orders $O(\beta_i) \geq 1$.

By a fractional (convergent) power series we mean a series of the form

$$\lambda : \; y = \lambda(x) := c_1 x^{n_1/N} + c_2 x^{n_2/N} + \cdots, \quad c_i \in \mathbb{C}, c_1 \neq 0, \tag{2.3}$$

where $N \leq n_1 < n_2 < \cdots$ are positive integers, having no common divisor, such that $\lambda(t^N)$ has positive radius of convergence. We call $O(\lambda) = n_1/N$ the order of λ. By convention the order of $\lambda \equiv 0$ is $+\infty$. We will identify λ with the analytic arc $\lambda :\; x = t^N$, $y = c_1 t^{n_1} + c_2 t^{n_2} + \cdots$, $|t|$ small, which is not tangent to the y-axis (since $n_1/N \geq 1$).

Two fractional power series $\lambda_1(x), \lambda_2(x)$ are *congruent modulo* $q \in \mathbb{Q}^+$ if their difference is of the form

$$\lambda_1(x) - \lambda_2(x) = cy^q + \cdots, \quad c \in \mathbb{C}. \tag{2.4}$$

In this case we write $\lambda_1 \equiv \lambda_2 \mod q$.

We say that $\beta(x)$ is a *(Newton-Puiseux) root modulo* q of $f = 0$ if there exists $\beta_i(x)$ such that

$$f(x, \beta_i(x)) \equiv 0 \quad \text{and} \quad \beta \equiv \beta_i \mod q. \tag{2.5}$$

If in the Newton-Puiseux factorisation (2.2) there are exactly m roots β_i, $\beta_i \equiv \beta$ mod q, we say that β *is a modulo* q *root of* $f = 0$ *of multiplicity* m.

We recall the following theorem.

Theorem 2.1 (Compare Lemma (3.3), [11])**.** *If* $\beta(x)$ *is a modulo* q *root of* $f = 0$ *of multiplicity* m*, then it is a modulo* q *root of* $\partial f/\partial y = 0$ *of multiplicity* $m - 1$.

2.2. **Real part of tree model.** Let us describe the tree model associated to an analytic function $f(x, y)$. This is completely determined by the Newton-Puiseux expansions of the roots β_i of f, as in 2.2—more precisely, by the way the roots are congruent.

For instance the polynomial $f(x, y) = (x^2 + y^4)(x^2 - y^6)$, has the tree shown in Figure 1.

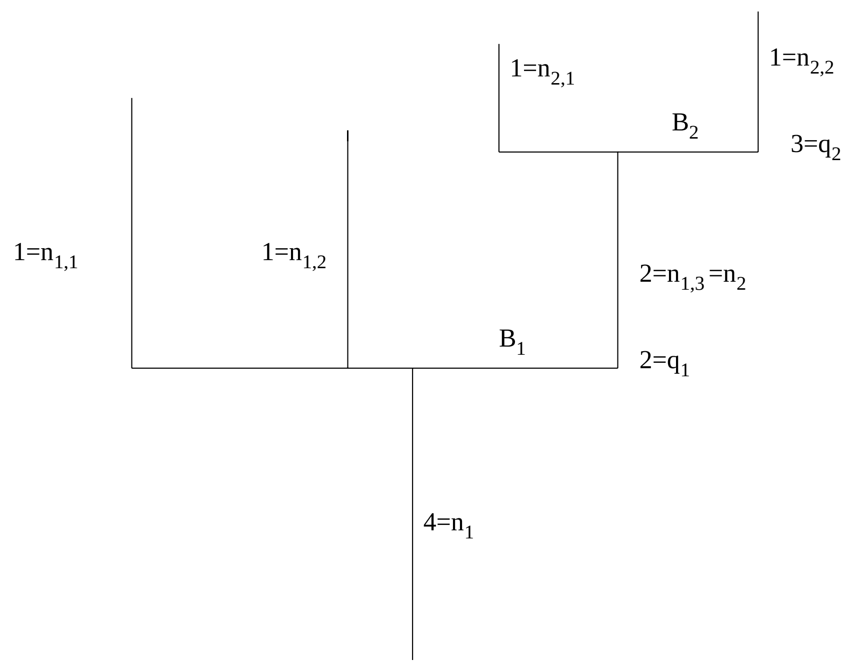

FIGURE 1.

Informally, the tree model is obtained as follows. In Figure 2, below, the bottom vertical consists of all the truncated roots (truncated at q_1, the lowest order of terms for which at least two roots differ). In this way the first bar B_1 appears (if not, we are in the trivial case when all the roots coincide and the tree is just one vertical). In particular, there are roots growing on B_1 that differ at this height (i.e., their "contact" is strictly less than q_1). From this height onward the roots growing on the bar B_1 group into different bunches of roots, depending on their contacts—if their contact is greater than q_1 they determine the same new bunch, otherwise they belong to different new bunches. The procedure repeats for every newly obtained bunch, giving rise to a tree model. The horizontal and vertical dashed lines (in Figure 2) represent possible extra bars and bunches of roots respectively.

This process can be formally described as follows. We assume that f is written in the form (2.2).

To each $q \in \mathbb{Q}^+$ we associate a partition R_q of the roots β_i, $i = 1, \ldots, d$, of f, defined by

$$R_q = \{A_{I_q}, I_q \in P_q\},$$

where $P_q = \{I_q\}$ is a partition of the set $\{1, 2, ..., d\}$, defined as follows: $i, j \in I_q$ if and only if $\beta_i(y), \beta_j(y)$ are *congruent modulo* $q \in \mathbb{Q}^+$. Accordingly we define $A_{I_q} = \{\beta_i \mid i \in I_q\}$.

Each A_{I_q} is called a *q-bunch*, and its cardinal is by definition called the *multiplicity of* A_{I_q} and will be denoted by $m(A_{I_q})$. Note that for all q, the multiplicities satisfy $1 \leq m(A_{I_q}) \leq d$ and $\sum_{I_q \in P_q} m(A_{I_q}) = d$.

For $q' \leq q$ we can easily see that the partition R_q refines the partition $R_{q'}$.

Clearly there are finitely many rational numbers $q_1 < \cdots < q_n$ such that

- $\mathbb{Q}^+ = [0, q_1] \cup (q_1, q_2] \cup ... \cup (q_{n-1}, q_n] \cup (q_n, \infty)$,
- R_q is constant on each interval $[q_j, q_{j+1})$, $j = 0, 1, \ldots, n$, and
- for any $j = 1, \ldots, n-1$ the partition $R_{q_{j+1}}$ is strictly finer than the partition R_{q_j}.

We say that a q-bunch A_{I_q} *succeeds* a q'-bunch $A_{I_{q'}}$ (or that $A_{I_{q'}}$ *precedes* A_{I_q}), if, by definition, A_{I_q} is strictly included in $A_{I_{q'}}$ (then necessarily $q > q'$).

The rational numbers $q_j, j = 1, ..., n$, are called the *heights* of the tree. At each height $q_j, j = 1, ..., n$, we draw one or more supporting bars $B_{I_{q_j}}$. Namely, a bar $B_{I_{q_j}}$ appears only if $A_{I_{q_j}}$, at that level, splits into at least two successors. More precisely, if $A_{I_{q_j}}$ splits into k_{q_j} successors, the corresponding bar $B_{I_{q_j}}$ will indicate exactly k_{q_j} points from which new bunches will grow.

For a given q-bunch A_{I_q}, $q \in (q_{i-1}, q_i]$ as above, we associate the following:

(1) $b(I_q) = \sup\{q' | A_{I_q} \text{succeeds}\, A_{I_{q'}}\}$, *the q-basic degree of the bunch* A_{I_q}, where clearly $b(I_q) \leq q_{i-1}$ and $b(I_q) \in \{q_j \mid j = 0, ..., n\}$;
(2) $\beta(I_q)$, *the representative of the bunch* A_{I_q}, obtained from any $\beta_i(y) \in A_{I_q}$ by deleting all the terms of degree greater than or equal to q_i;
(3) $\beta_b(I_q)$, *the q-basic representative of the bunch* A_{I_q} obtained from $\beta(I_q)$ by deleting all the terms of degree greater than $b(I_q)$.

Definition 2.2. A q-bunch A_{I_q} is called real if its representative $\beta(I_q)$ is real. A q-bunch A_{I_q} is called almost real if its q-basic representative $\beta_b(I_q)$ is real. In particular, if a q-bunch is real, then it is almost real as well.

Accordingly, if $A_{I_{q_j}}$, $j \in \{1, ..., n\}$, is real and the bar $B_{I_{q_j}}$ appears, then we call it a real supported bar.

If we consider a real Puiseux arc $\alpha(t) = (x(t), y(t))$, $x = t$, $y = \sum_{i \geq N} \alpha_i t^{i/N}$, then, from the tree model of f, we will describe the orders of f and $\text{grad}(f)$ along this arc. We are interested in the case when f is real. Let us assume that our arc

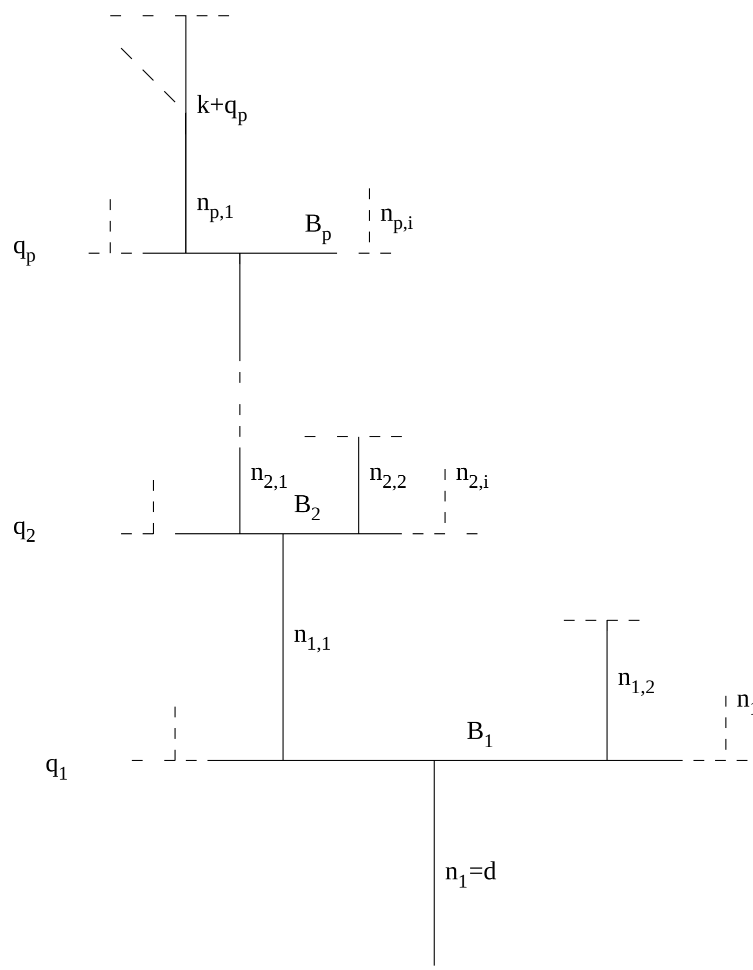

FIGURE 2.

has contact with the tree of f as in Figure 2, B_p being the last bar in contact with our curve. Figure 2 emphasizes the part of the tree with maximal contact with our analytic arc α. In the picture the oblique dashed line represents the possible remaining part (after α leaves the tree at that level) of our analytic curve. For the simplicity of our picture, we use the simpler notation B_p instead of B_{I_q} (and accordingly for their multiplicities).

Remark 2.3. Here the rational number k can be zero, and also B_p can be the last bar in that bunch.

3. Main result and examples

Now we state the main result of our paper:

Theorem 3.1. *Let g be an arc-analytic function, and assume that for a natural number r, $g^r = f$ is analytic. Then g is locally Lipschitz. Moreover, if at any point of $f = 0$ the multiplicity of f is greater than r, then g is of class C^1.*

It is useful to consider the notion of arc-analytic even for complex-valued functions, where we understand that they are analytic on real analytic arcs. We cannot avoid arc-analytic solutions in the sense above. Indeed, we have the following type of examples, which appear in our context:

Example 3.2. Consider $P(z,x,y) = z^4 - x^8 - y^8$ as a polynomial in z. It has the obvious roots

$$z_1 = \sqrt[4]{x^8+y^8}, \quad z_2 = -\sqrt[4]{x^8+y^8}, \quad z_3 = i\sqrt[4]{x^8+y^8}, \quad z_4 = -i\sqrt[4]{x^8+y^8}.$$

In fact our theorem 3.1 can be stated in a more general way using Fukui's invariant. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a germ of an analytic function. For any germ of an analytic arc $\alpha : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$, the composition $f \circ \alpha(t)$ is a convergent power series in t. We denote by $o_\alpha(f)$ its order at $t = 0$. We call the set of integers

$$A_{\mathbb{R}}(f) = \{o_\alpha(f);\, \alpha : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0) \text{ analytic}\}$$

the *real Fukui invariant of the germ* f. In the same way, we define the *complex Fukui invariant of the germ* f as the set of integers

$$A_{\mathbb{C}}(f) = \{o_\alpha(f);\, \alpha : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0) \text{ holomorphic}\},$$

where we also denote by f the complexification of the germ f. Clearly

$$A_{\mathbb{R}}(f) \subset A_{\mathbb{C}}(f).$$

Now Theorem 3.1 follows from the following.

Theorem 3.3. *Let $f : U \to \mathbb{R}$ be an analytic function defined in a neighbourhood U of $0 \in \mathbb{R}^n$. Assume that $A_{\mathbb{R}}(f) \subset r\mathbb{N}$, i.e., that r divides the order of $f \circ \alpha(t)$ for all real analytic arcs α. Then $|f|^{\frac{1}{r}}$ is Lipschitz in a neighbourhood of* 0 *or, equivalently,*

$$o_\alpha(|\operatorname{grad} f|) \geq \frac{r-1}{r} o_\alpha(f), \quad \textit{for any analytic arc } \alpha.$$

Moreover, if r is smaller than the multiplicity of f, then $|f|^{\frac{1}{r}}$ is of class C^1 in a neighbourhood of 0.

We offer the following easy example.

Example 3.4. If $f = x^2 + y^2$, then clearly 2 divides the order of $f \circ \alpha(t)$ for all analytic arcs α. We have that $\sqrt{x^2+y^2}$ is Lipschitz, despite the fact that it is not arc-analytic.

Remark 3.5. Our theorem 3.3 has a purely real character. Let us take for instance $f = x^2 + y^2$; then $A_{\mathbb{R}}(f) = 2\mathbb{N}^+$ and $A_{\mathbb{C}}(f) = \{n \in \mathbb{N}; n \geq 2)$. If we consider $g = (x^2+y^2)^{1/2}$ as a multivalued function in a neighbourhood of $0 \in \mathbb{C}^2$, then g is not Lipschitz at points of the complex lines $x^2 + y^2 = 0$. On the other hand, if we assume that $A_{\mathbb{C}}(f) \subset r\mathbb{N}$, then it is easy to check that f must be an r-th power of a holomorphic function; so clearly $f^{\frac{1}{r}}$ is Lipschitz.

In the rest of this section we discuss some examples related to our theorems 3.1 and 3.3. One can ask a more general question; are the arc-analytic roots of polynomials,

$$P(x,y) = \sum_{i=0}^{d} a_i(x) y^{d-i},$$

with a_i real analytic functions in n variables, necessarily locally Lipschitz?

Let us note first an example which shows that if a polynomial P is not monic, then an arc-analytic root of P is not necessarily locally Lipschitz.

Example 3.6. $(x^2 + y^4)z - x(x^2 + 2y^4) = 0.$

Also, if a solution is not arc-analytic, then it need not be Lipschitz, as the following example shows.

Example 3.7. $g^3 = x^2y^2 + x^{100} + y^{100}.$

Remark 3.8. Note that the above example shows that Glaeser's result (mentioned in the introduction), cannot be extended to $r = 3$. Indeed, our example is nonnegative, even analytic, and has the vanishing order 3, but its cubic root is no longer Lipschitz.

4. Proof of Theorem 3.1

First we present a reduction to the case where the coefficients are analytic functions of two variables.

Assume that $g(x)$ is an arc-analytic solution of the monic polynomial,

$$P(x, y) = y^r - f(x),$$

where f is a real analytic function in n variables, and that g is not Lipschitz. Note that g has semi-analytic graph, and then by the curve selection lemma (cf., for instance, [2], [17]) it follows that one partial derivative of g is unbounded along an analytic arc at the origin, say $\alpha(t)$. Assume that $(\partial g/\partial x_1)(\alpha(t))$ is unbounded. If we consider the analytic transformation $\gamma(t, s) = \alpha(t) + se_1$, $e_1 = (1, 0, \ldots, 0)$, we observe that $P(\gamma(t, s))$ becomes a monic polynomial with coefficients analytic functions in two variables, and has $h(s, t) = g(\gamma(t, s))$ as an arc-analytic solution. Moreover, its partial derivative with respect to s at $(t, 0)$ is nothing but $(\partial g/\partial x_1)(\alpha(t))$. This, indeed, shows that we can assume that f is an analytic function in 2 variables.

From now on, we assume that f is of the form (2.1) considered in section 2,

$$f(x, y) = u(x, y) \cdot F(x, y),$$

where u is a unit and $F(x, y) = y^d + \sum_{i=1}^{d} a_i(x)y^{d-i}$, with a_i real analytic functions, is the associated Weierstrass polynomial.

The proof of our theorem is a corollary of the following lemma.

Lemma 4.1. *If r divides the order of $f \circ \alpha(t)$, for all analytic arcs α, then r divides the multiplicities of all almost real q-bunches in the tree of f (and therefore the multiplicities of all its real q-bunches).*

Proof. We are going to prove the lemma by choosing convenient curves so that we can cover the multiplicities of all the almost real q-bunches in the tree of f (and therefore the multiplicities of all its real q-bunches). It will be enough to prove that r divides $n_{p,1}$ in Figure 2, or divides $n_p = n_{p,1} + \cdots + n_{p,i} + \ldots$, when $k = 0$. Note that in the second case we must have the equality $n_p = n_{p-1,i}$ for some i. Let α be an analytic arc; by abuse of notation we assume that it has the form $(x, \alpha(x))$, where $\alpha(x)$ is a Puiseux series. Let us denote by β a root of f that has maximal contact with α. In our case, $\alpha(x) = \beta_{k+q_p}(x) + h.o.t.$, where β_{k+q_p} is the root β, $(k + q_p)$-truncated. Note that, afterwards, α has no more contact with the tree.

Case 1. Assume that $k > 0$ and let us write β_{k+q_p} with the last two consecutive terms:

$$\beta_{k+q_p}(x) = \cdots + ax^s + bx^{k+q_p},$$

where $q_p \le s < k + q_p$; note that a and b are real (but not necessarily nonzero). We consider now the family of curves $\tau_{l,n}(t) = (x(t), y(t))$ with

$$x(t) = t^n, \quad y(t) = \beta_{k+q_p}(x) - bx^{k+q_p} + dx^s t^l,$$

where $d \neq 0$ is real. The exponents l and n are integers; moreover, n is large enough so that $sn + l < (k + q_p)n$. We can also choose n as a multiple of all denominators of exponents of β_{k+q_p}. So we may assume that $\tau_{l,n}$ is analytic.

The contact of $\tau_{l,n}$ with the tree is equal to the order $f \circ \tau_{l,n}$, which we denote by $o_{l,n}(f)$. Clearly, by 2.2 and the choice of l, n we have

$$o_{l,n}(f) = n_{p,1}(sn + l) + c(\alpha),$$

and $c(\alpha)$ does not depend on l. Recall that $l < n(k + q_p - s)$. So if we choose n large enough, we will have two analytic curves, $\tau_{l,n}$ and $\tau_{l+1,n}$. By the assumption of the lemma the orders $o_{l,n}(f)$ and $o_{l+1,n}(f)$ are divisible by r. Hence r divides $n_{p,1} = o_{l+1,n}(f) - o_{l,n}(f)$. This proves the lemma when $k > 0$.

Case 2. Assume that $k = 0$; so the bar B_p is real supported. Let us take a rational h_p such that $1 \le h_p < q_p$ and there are no more greater powers in β below B_p (this is always possible when $q_p > 1$). Consider the family of curves $\tau_{l,n}(t) = (x(t), y(t))$ with

$$x(t) = t^n, \quad y(t) = \beta_{h_p}(x) + ax^{h_p} t^l,$$

where $a \neq 0$ is real. The exponents l and n are integers; moreover, n is large enough so that $nh_p + l < nq_p$, and as before we may assume that $\tau_{l,n}$ is analytic.

Clearly the order of $f \circ \tau_{l,n}$ is equal to $n_p(nh_p + l) + c(\alpha)$, where $c(\alpha)$ does not depend on l. But this order is divisible by r. So we can conclude, as in case 1, that n_p is divisible by r. If $q_p = 1$, we may take the generic line $y = \tau(x) = ax$. □

Now the proof of our main theorem 3.1 goes like this. By the curve section lemma, it is enough to prove that $|g_y| = |f_y|\,|f|^{1/r-1}$ is bounded on any real analytic arc α. So it suffices to show that $o_\alpha(f_y)$, the order of f_y on α, and $o_\alpha(f)$, the order of f on α, satisfy the following inequality:

$$o_\alpha(f_y) \ge \frac{r-1}{r} o_\alpha(f). \tag{4.1}$$

We study the contact of α with the tree of f . If $k > 0$, then we have

$$o_\alpha(f) = n_{p,1}(k + q_p) + (n_p - n_{p,1})q_p + (n_{p-1} - n_p)q_{p-1} + \ldots.$$

On the other hand, it follows from Theorem 2.1 that the order of the derivative f_y along α is

$$o_\alpha(f_y) \ge (n_{p,1} - 1)(k + q_p) + (n_p - n_{p,1})q_p + (n_{p-1} - n_p)q_{p-1} + \ldots.$$

We know from Lemma 4.1 that r divides $n_{p,1}$. So

$$\frac{n_{p,1} - 1}{n_{p,1}} \ge \frac{r-1}{r},$$

which proves inequality (4.1) (actually it suffices only to prove that $r \le n_{p,1}$).

When α leaves the tree through B_p it follows that B_p is real supported, and therefore r divides n_p. Replacing $n_{p,1}$ by n_p in the above calculations will give us the result.

A careful look at our proof will give the second claim as well. Actually we can see that with the second hypothesis we have that $o_\alpha(f_y) > \frac{r-1}{r} o_\alpha(f)$.

Remark 4.2. For curves tangent to the y-axis, using the fact that f is *mini-regular* in y, one can check easily the inequality $o_\alpha(|\mathrm{grad}\, f|) \geq \frac{r-1}{r} o_\alpha(f)$. On the other hand, if α is not tangent to the y-axis, we clearly have $o_\alpha(f_x) \geq \frac{r-1}{r} o_\alpha(f)$. This explains why we are interested only in the derivative with respect to y.

Remark 4.3. Note that this theorem is no longer true if we ask f to be only blow-analytic and Lipschitz. Indeed, we give the following example.

Example 4.4.

$$f = y^k \frac{x^2 + y^4}{x^2 + 2y^4} \quad \text{and} \quad g = y \sqrt[k]{\frac{x^2 + y^4}{x^2 + 2y^4}}, \qquad k \geq 2.$$

One can see that f is blow-analytic, and even of class $C^{k-1/2}$, $g^k = f$, but g is only blow-analytic, not Lipschitz.

Remark 4.5. The proof of our theorem shows that actually we have a bit more. Indeed, as we pointed out before, to obtain $o_\alpha(f_y) \geq \frac{r-1}{r} o_\alpha(f)$, it suffices only to have $r \leq n_{p,1}$ (where the curve α leaves the tree; in other words, for all terminal almost real bunches). This could be useful if one is interested in the above inequality only for some special curves.

Finally, we point out that our result cannot be extended, in this degree of generality, to even polynomials that are solvable by radicals.

Remark 4.6. Our theorem 3.1 cannot be generalised to more than one radical. There are irreducible polynomials solvable by radicals and which have arc-analytic solutions that are not Lipschitz. We give the following striking example.

Example 4.7.

$$f = \sqrt[4]{x^2 + y^8 - \sqrt{x^4 + y^{20}}},$$

This function is arc-analytic but not Lipschitz!

References

1. D. Alekseevsky, A. Kriegl, P. W. Michor, and M. Losil, *Choosing roots of polynomials smoothly*, Israel Journal of Mathematics, **105** (1998), 203-233. MR **2000c:**58017
2. E. Bierstone and P. D. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math., **67** (1988), 5-42. MR **89k:**32011
3. E. Bierstone and P. D. Milman, *Arc-analytic functions*, Invent. Math., **101** (1990), 411-424. MR **92a:**32011
4. E. Bierstone, P. D. Milman, and A. Parusiński, *A function which is arc-analytic but not continuous*, Proc. Amer. Math. Soc., **113** (1991), 419-423. MR **91m:**32008
5. J. Dieudonné, *Sur un théorème de Glaeser*, J. Analyse Math., Vol. 23 (1970), pp. 85–88. MR **42:**4678
6. T. Fukui, *Seeking invariants for blow-analytic equivalence*, Compositio Math., **105** (1997), 95-108. MR **98c:**32009
7. T. Fukui, S. Koike and T.-C. Kuo, *Blow-analytic equisingularities, properties, problems and progress*, in "Real analytic and algebraic singularities", Pitman Research Notes in Mathematics Series, **381**, Longman, Harlow, 1998, pp. 8–29. MR **99a:**32051

8. G. Glaeser, *Racine carrée d'une fonction différentiable*, Annales de l'Institut Fourier (Grenoble), Vol. 13 (1963), fasc. 2, pp. 203–210. MR **29:**1294
9. S. Izumi, S. Koike and T.-C. Kuo, *Computations and Stability of the Fukui Invariant*, Compositio Mathematica **130(1)** (2002), 49-73. MR **2003a:**32050
10. T.-C. Kuo, *On classification of real singularities*, Invent. Math., **82** (1985), 257–262. MR **87d:**58025
11. T.-C. Kuo and Y.C. Lu, *On analytic function germs of two complex variables*, Topology, **16** (1977), 299-310. MR **57:**704
12. T.-C. Kuo and A. Parusiński, *Newton Polygon Relative to an Arc*, in Real and Complex Singularities (São Carlos, 1998), Chapman & Hall Res. Notes Math., **412**, Boca Raton, FL, 2000, pp. 76–93. MR **2000j:**32043
13. K. Kurdyka, *Ensembles semi-algébriques symétriques par arcs*, Math. Ann., **282** (1988), 445–462. MR **89j:**14015
14. K. Kurdyka, *A counterexample to subanalyticty of an arc-analytic function*, Ann. Polon. Math. **55** (1991), 241-243. MR **92j:**32019
15. K. Kurdyka, *An arc-analytic function with nondiscrete singular set*, Ann. Polon. Math. **59,** 1 (1994), 251-254. MR **95g:**32009
16. K. Kurdyka and L. Paunescu, *Arc-analyticity is an open property for subanalytic functions*, preprint, Univ.-Savoie, 2002.
17. S. Łojasiewicz, *Ensembles semi-analytiques*, preprint, Inst. Hautes Études Sci., 1965.
18. A. Parusiński, *Subanalytic functions*, Trans. Amer. Math. Soc. **344**, 2 (1994), 583–595. MR **94k:**32006
19. L. Paunescu, *An example of blow-analytic homeomorphism* in "Real analytic and algebraic singularities", Pitman Research Notes in Mathematics Series, **381**, Longman, Harlow, 1998, pp. 62–63. MR **98i:**32001

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