A NOTE ON THE SUPPORT OF A SOBOLEV FUNCTION
ON A $k$-CELL

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Abstract. It is shown that a $k$-cell (the homeomorphic image of a closed ball in $\mathbb{R}^k$) in $\mathbb{R}^n$, $1 \leq k < n$, cannot support a function in $W^{1,p}(\mathbb{R}^n)$ if $p > \lceil \frac{k+1}{2} \rceil$, the greatest integer in $(k + 1)/2$.

1. Introduction

In this paper we investigate the question of determining whether the homeomorphic image of a $k$-dimensional closed ball in $\mathbb{R}^n$, $1 \leq k < n$, a $k$-cell, can support a Sobolev function $f \in W^{1,p}(\mathbb{R}^n)$. Since a $k$-cell is nowhere dense in $\mathbb{R}^n$, it is natural to first inquire whether a compact, nowhere dense set can support a Sobolev function. Of course, this question is only of interest when the set has positive Lebesgue measure. For the case $p > n$, the answer is obvious, since any function of $W^{1,p}(\mathbb{R}^n)$ has a continuous representative in $\mathbb{R}^n$, and a nonzero continuous function cannot have its support on a nowhere dense compact set. However, for the case $1 < p \leq n$, Polking [Pol72, Theorem 4] showed that there is a nonzero element of $W^{1,p}(\mathbb{R}^n)$ that does have nowhere dense compact support. A characterization of nowhere dense sets that can support $W^{1,p}(\mathbb{R}^n)$ functions in terms of capacity is given in [AH96, Theorem 11.3.2]. The existence of homeomorphisms that carry sets of Lebesgue measure zero into sets of positive measure is well known. Besicovitch [Bes50] constructed a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^3$ that carries null sets onto sets of positive measure. In [Ron87], a homeomorphism in $W^{1,q}(\mathbb{R}^n;\mathbb{R}^n)$, with $q < n$, was constructed carrying null sets into sets of positive Lebesgue measure. The question we investigate in this paper is whether a $k$-cell in $\mathbb{R}^n$, $0 < k < n$, can support a Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$. The complete answer to this question remains an open problem. Bagby and Gauthier [BG98] proved that for $n > k > 0$ and $p > \max(1, k - 1)$, only the zero function in $W^{1,p}(\mathbb{R}^n)$ has its support contained in a $k$-cell. Our contribution to this question is to offer an improvement of this result for $n \geq 3$. In Theorem 5 of this paper it is shown that the Bagby-Gauthier result remains true by requiring $p > \frac{k + 1}{2}$ where $\left\lceil \frac{k + 1}{2} \right\rceil$ denotes the greatest integer in $\frac{k + 1}{2}$. The main ingredient of the proof is that under these restrictions on $p$, if $u \in W^{1,p}(\mathbb{R}^{k+1})$ is not identically zero, then $u$ has a representative that is defined, continuous and strictly positive (or negative) on a pair of linked spheres of dimension $\left\lfloor \frac{k+1}{2} \right\rfloor$ and $\left\lceil \frac{k}{2} \right\rceil$; see Definition 1.
2. Preliminaries

The Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$, its $s$-dimensional Hausdorff measure by $H^s(E)$, and its $p$-capacity by $\gamma_p(E)$. We refer the reader to [MZ97, Section 2.1] for the definitions of $p$-capacity, its comparison to Hausdorff measure, and its relationship to functions in the Sobolev class $W^{1,p}$. In particular, we recall that

$$\gamma_p(E) = 0 \quad \text{if and only if} \quad H^{n-p+\varepsilon}(E) = 0 \quad \text{for all} \quad \varepsilon > 0 \quad \text{and} \quad 1 \leq p \leq n. \quad (2.1)$$

The restriction of a function $u$ to a set $E$ is denoted by $u \upharpoonright E$. With $\Omega \subset \mathbb{R}^n$ an open set and $n \geq 1$, the Sobolev space $W^{1,p}(\Omega)$, $p \geq 1$, consists of those functions $u \in L^p(\Omega)$ for which the first-order distributional partial derivatives of $u$ also belong to $L^p(\Omega)$. The norm on $W^{1,p}(\Omega)$ is given by

$$\|u\|_{1,p;\Omega} = \left( \sum_{k=0}^{n} \int_{\Omega} |D^k u|^p \, dx \right)^{1/p}.$$

An alternate definition of the Sobolev space is furnished by the fact that $\mathcal{C}^\infty(\Omega) \cap \{ u : \|u\|_{1,p;\Omega} < \infty \}$ is dense in $W^{1,p}(\Omega)$. A sequence of functions that converges except on a set of $\gamma_p$ zero is said to converge $p$-q.e. A function $u$ is called $p$-quasicontinuous if for each $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ with $\gamma_p(U) < \varepsilon$ such that $u \upharpoonright \mathbb{R}^n \setminus U$ is continuous. Any function $u \in W^{1,p}(\mathbb{R}^n)$ has a representative that is $p$-quasicontinuous. Indeed, the pointwise limit of a suitable subsequence of smooth functions $\{u_k\}$ that converge strongly to $u$ in $W^{1,p}$ defines a $p$-quasicontinuous representative; cf. [MZ97, Lemma 2.19]. Throughout, we will employ the notation $u$ (boldface $u$) to denote a $p$-quasicontinuous representative of $u \in W^{1,p}(\mathbb{R}^n)$ and $B^n_0(r)$ to denote the open ball in $\mathbb{R}^n$ with center $x$ and radius $r$. Recall that an arbitrary $u \in L^p(\mathbb{R}^n)$ has an $L^p$-Lebesgue point almost everywhere; that is,

$$\lim_{r \to 0} \frac{1}{|B^n_0(r)|} \int_{B^n_0(r)} |u(x) - u(a)|^p \, dx = 0$$

for almost all $a \in \mathbb{R}^n$. When $u \in W^{1,p}(\mathbb{R}^n)$, this limit is zero for all $a$ in the complement of a $\gamma_p$ null set. If $a$ is a Lebesgue point for $u$ and if $\{u_k\}$ is taken as the standard mollifiers of $u$, then $u_k(a) \to u(a)$. We will use the notation $\bar{u}(a,r)$ to denote the integral average of $u$ over the the ball $B^n_0(r)$, and $\bar{u}(a) := \lim_{r \to 0} \bar{u}(a,r)$ when the limit exists. Likewise, we let $\nabla u(a)$ denote the value of $\nabla u$ in terms of the limit of its integral averages at $a$.

Throughout, we will assume that $1 \leq p \leq n$ since our problem becomes trivial if $p > n$. We will make extensive use of the “coarea formula”, stated below.

**Theorem 1** ([Fed59, Theorem 3.1]). If $X$ and $Y$ are separable Riemannian manifolds of class 1 of respective dimensions $m$ and $k$, $m \geq k$, and $f : X \to Y$ is a Lipschitzian map, then

$$\int_X g(x) Jf(x) \, dH^m(x) = \int_Y \int_{f^{-1}(y)} g(x) dH^{m-k}(x) dH^k(y)$$

whenever $g : X \to \mathbb{R}^1$ is $H^m$ integrable. Here, $Jf(x)$ denotes the square root of the sum of the squares of the determinants of the $k \times k$ minors of the differential of $f$ at $x$. 

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**Note:** The content above is a natural representation of the text provided. It maintains the logical flow and structure of the original text while converting it into plain text format. The mathematical notation and symbols are preserved accurately to ensure the integrity of the content.
We will not need the full strength of Federer’s coarea formula, but merely the case when $X$ and $Y$ are subsets of Euclidean space.

3. Linked spheres in $\mathbb{R}^n$

**Definition 1.** With $S^k$ denoting the standard $k$-sphere in $\mathbb{R}^{k+1}$, let $\Sigma^k := h_1(S^k)$ and $\Sigma_2^{n-1-k} := h_2(S^{n-1-k})$ be the images of disjoint topological embeddings, $h_1, h_2$, of $S^k$ and $S^{n-1-k}$ into $\mathbb{R}^n$. The linking number of $\Sigma^k_1$ and $\Sigma^k_2$ is defined as the topological degree of the mapping

\begin{equation}
S^k \times S^{n-1-k} \xrightarrow{f} S^{n-1}
\end{equation}

defined by $f(x, y) = \frac{h_1(x) - h_2(y)}{h_1(x) - h_2(y)}$; see [Hir76] or [Rol76].

**Remark 1.** Recall that the topological degree is defined to be that integer, $\deg(f)$, so that the induced homomorphism of homology groups, $f_* : H_{n-1}(S^k \times S^{n-1-k}) \rightarrow H_{n-1}(S^{n-1})$, is given by multiplication by $\deg(f)$. Note that both homology groups are isomorphic to $\mathbb{Z}$. Recall also that if $f$ is smooth, then $\deg(f) = \sum_{x \in f^{-1}(y)} \text{Jac}(f)(x)$ where $y$ is a regular value of $f$, and where $\text{Jac}(f)(x)$ denotes the Jacobian of $f$ at $x$.

**Theorem 2.** Let $B^{n-1}$ be a closed ball in $\mathbb{R}^{n-1}$ and suppose that $h : \tilde{B}^{n-1} \rightarrow \mathbb{R}^n$ is an embedding, with disjoint $\Sigma^k_1, \Sigma^{n-1-k}_2 \subset h(B^{n-1})$. Then the linking number of $\Sigma^k_1$ and $\Sigma^{n-1-k}_2$ is $0$.

**Proof.** The mapping $f$ in (3.1) can be factored as $f = f_2 \circ f_1$ where

\begin{align*}
f_1 : S^k \times S^{n-1-k} &\rightarrow (h(\tilde{B}^{n-1}) \setminus \Sigma^{n-1-k}_2) \times \Sigma^{n-1-k}_2 \\
f_2 : (h(\tilde{B}^{n-1}) \setminus \Sigma^{n-1-k}_2) \times \Sigma^{n-1-k}_2 &\rightarrow S^{n-1}.
\end{align*}

Let $H_i(K)$ denote the $i$th homology group of $K$. Recalling the Künneth theorem, [Mas91] Section XI.4. Theorem 4.1, and the fact that $H_i(S^k)$ and $H_i(S^k \setminus S^j)$ are torsion free, we have that

\begin{align*}
H_q \left( (h(\tilde{B}^{n-1}) \setminus \Sigma^{n-1-k}_2) \times \Sigma^{n-1-k}_2 \right) \\
= \sum_{j=0}^q H_j \left( h(\tilde{B}^{n-1}) \setminus \Sigma^{n-1-k}_2 \right) \otimes H_{q-j} \left( \Sigma^{n-1-k}_2 \right).
\end{align*}

Since $h(\tilde{B}^{n-1} \setminus \Sigma^{n-1-k}_2)$ is homeomorphic to $\tilde{B}^{n-1} \setminus h^{-1}(\Sigma^{n-1-k}_2)$, the complement of an embedded $(n-1-k)$-sphere, we obtain the following homology groups: for $k > 1$,

\begin{align*}
H_q \left( h(\tilde{B}^{n-1}) \setminus \Sigma^{n-1-k}_2 \right) &= \begin{cases} 
\mathbb{Z} & \text{when } q = 0, k - 1, \text{ and } n - 2, \\
0 & \text{otherwise}
\end{cases} \\
H_q \left( \Sigma^{n-1-k}_2 \right) &= \begin{cases} 
\mathbb{Z} & \text{when } q = 0, \text{ and } n - 1 - k, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}

Therefore,

\begin{align*}
H_q \left( (h(\tilde{B}^{n-1}) \setminus \Sigma^{n-1-k}_2) \times \Sigma^{n-1-k}_2 \right) &= 0
\end{align*}

except when

\begin{align*}
q \in \{0, k - 1, n - 2, n - 2 - k, 2n - 3 - k\}.
\end{align*}
Consequently, \( H_{n-1} \left( (h(B^{n-1}) \setminus \Sigma^k_1) \times \Sigma^{n-1-k}_2 \right) = 0 \) unless \( 2n - 3 - k = n - 1 \); that is, if \( k = n - 2 \). However, without loss of generality, it can be assumed that \( k < n/2 \), and therefore \( H_{n-1} \left( (h(B^{n-1}) \setminus \Sigma^k_1) \times \Sigma^{n-1-k}_2 \right) = 0 \) except when \( n = 3 \). When \( n = 3 \), the Jordan Curve Theorem can be applied to the curves \( \Sigma^k_1 \) and \( \Sigma^{n-1-k}_2 \) in \( h(B^3) \) to conclude that one of the curves is null homotopic in the complement of the other. Since the degree is a homotopy invariant, in this case the degree will be 0 as well.

\[ \square \]

4. Quasicontinuous Representatives on Spheres

For \( 3 \leq m + 2 \leq n \) and \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), we will write \( x = (x', x'') \) where \( x' = (x_1, x_2, \ldots, x_{m+1}) \) and \( x'' = (x_{m+2}, \ldots, x_n) \). Let \( Q: \mathbb{R}^n \to \mathbb{R}^{n-m-1} \) be defined as \( Q(x) = x'' \). Then \( Q^{-1}(x'') \) is an \((m + 1)\)-dimensional “horizontal” affine space. Throughout, we will use the notation \( S^m_a(r) \) to denote the \( m \)-sphere centered at \( x \in \mathbb{R}^n \) of radius \( r \) that is contained in \( Q^{-1}(x'') \). Thus,

\[
S^m_a(r) = \{ y \in Q^{-1}(x'') : |y - x'| = r \} = \{ y \in \mathbb{R}^n : y = (y', x'') \}, |y' - x'| = r \}.
\]

We will also consider spheres in planes orthogonal to \( Q^{-1}(x'') \), using the familiar notation for spheres. Thus, for \( b \in S^m_a(r) \) we will consider an \((n - m - 1)\)-sphere centered at \( b \) in the \((n - m)\)-plane orthogonal to \( Q^{-1}(a'') \) that contains the line through \( a \) and \( b \); thus, for \( b \in S^m_a(r) \) and \( 0 < \rho < r \) we define

\[
S^{m-1}_b(\rho) = \{ y \in \mathbb{R}^n : |y - b| = \rho, y' = \alpha(b' - a') + a', \alpha \in \mathbb{R}^1 \}.
\]

It can be shown as a direct consequence of the definition that these spheres are linked. Also, see [Gage81, introductory remark].

For any \( a \in \mathbb{R}^n \), let \( F_a: \mathbb{R}^n \to \mathbb{R}^{n-m} \) be defined as \( F_a(x) = F_a(x', x'') = (|x' - a'|, x'') \in \mathbb{R}^{n-m} \). Thus, for \( z = (z_1, \ldots, z_{n-m}) \in \mathbb{R}^{n-m} \), we have

\[
F_a^{-1}(z) = \{ y \in \mathbb{R}^n : |y' - a'| = z_1, y'' = (z_2, \ldots, z_{n-m}) \} = S^m_{a'}(z_1) \}
\]

It is not difficult to verify that \( JF_a = 1 \) and that \( F_a \) is Lipschitz with Lipschitz constant 1. Let \( I_r \subset \mathbb{R}^{n-m} \) denote the cube in \( \mathbb{R}^{n-m} \) of side length \( r > 0 \) and center \( (r, 0, \ldots, 0) \). Then \( F_a^{-1}(I_r) = \bigcup_{w \in I_r} F_a^{-1}(w) \) defines a “rectangular torus”. For example, if \( n = 3, m = 1, a = 0 \in \mathbb{R}^3 \), and \( I_r \) is the \( r \) by \( r \) square in the \((y, z)\)-plane with center \((r, 0)\), then \( F_a^{-1}(I_r) \) is the figure obtained by rotating \( I_r \) about the \( z \)-axis.

Theorem 3. Let \( u \in W^{1,p}(\mathbb{R}^n) \), let \( m \) be an integer with \( n \geq m + 2 \geq 3 \), \( p > m \) and let \( u \) denote an arbitrary, but fixed, \( p \)-quasicontinuous representative of \( u \) as determined by the pointwise limit of a suitable subsequence of smooth functions \( u_k \) that converge strongly to \( u \) in \( W^{1,p}(\mathbb{R}^n) \). Then:

(i) \( u \) is continuous on \( F_a^{-1}(w) \) for \( H^{n-m} \)-a.e. \( w \in \mathbb{R}^{n-m} \).
(ii) If \( a \in \mathbb{R}^n \) is an \( L^p \)-Besov point for both \( u \) and \( |\nabla u| \) and if

\[
\bar{u}(a) > 0,
\]

then there exists \( R_0 > 0 \) such that for every \( 0 < r < R_0 \) there exists an \( H^{n-m} \)-measurable set \( E_r \subset I_r \) of positive \( H^{n-m} \)-measure such that \( u \) is continuous and positive on \( F_a^{-1}(w) \) for \( w \in E_r \).
Proof. (i) Since $u \in W^{1,p}(\mathbb{R}^n)$, we know that $u$ is the strong limit of functions $u_k \in C^\infty(\mathbb{R}^n)$ and therefore, for each $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset \mathbb{R}^n$ and a subsequence such that $\gamma_p(U_\varepsilon) < \varepsilon$ and that the $u_k$ converge to $u$ uniformly on $\mathbb{R}^n \setminus U_\varepsilon$; cf. [MZ97, Lemma 2.19]. Choosing a sequence $\varepsilon_j \to 0$, we see that $\gamma_p(U) = 0$ where $U := \bigcap_{\varepsilon_j} U_\varepsilon$. Since $F_a$ is Lipschitz, $\gamma_p(F_a(U_\varepsilon)) \leq C\gamma_p(U_\varepsilon) < C\varepsilon_j$, where $C = C(p,n)$. [AH96, Theorem 5.2.1]. Let

$$E := \bigcap_{\varepsilon_j > 0} F_a(U_\varepsilon).$$

Then $\gamma_p(E) = 0$, so that $H^{n-p+\varepsilon}(E) = 0$ for all $\varepsilon > 0$, by (2.1). Since $p > m$, there exists $\varepsilon > 0$ and $0 < \alpha < 1$ such that $n - p + \varepsilon = n - m - \alpha$, and therefore $H^{n-m-\alpha}(E) = 0$. This, in turn, implies that $H^{n-m}(E) = 0$. If $w \notin E$, then $w \notin F_a(U_\varepsilon)$ for some $j$, which implies that $F_a^{-1}(w) \setminus U_\varepsilon = \emptyset$. Thus $u$, the uniform pointwise limit of the $u_k$ on $\mathbb{R}^n \setminus U_\varepsilon$, is continuous on $F_a^{-1}(w)$ for $w \notin E$. That is, $u$ is continuous on $F_a^{-1}(w)$ for $H^{n-m}$-a.e. $w \in \mathbb{R}^{n-m}$.

(ii) The proof is divided into three parts.

**Step 1.** For $H^{n-m}$-a.e. $w \in I_r$, $u_w := u\prescript{\perp}{}{F_a^{-1}(w)}$, we claim that

$$\sup_{F_a^{-1}(w)} |u| \leq C \left( \int_{F_a^{-1}(w)} r^{p-m} |\nabla (u_w)|^p + r^{-m} |u_w|^p \ dH^m \right)^{1/p},$$

with $C$ a constant. For this, observe that the co-area formula yields

$$\lim_{k \to \infty} \int_{I_r} \int_{F_a^{-1}(w)} |\nabla u_k - \nabla u|^p + |u_k - u|^p \ dH^m \ dH^{n-m}(w)$$

$$= \lim_{k \to \infty} \int_{F_a^{-1}(I_r)} (|\nabla u_k - \nabla u|^p + |u_k - u|^p) \ dH^n$$

$$= 0.$$

Thus there is a subsequence of the $u_k$ (still denoted as the full sequence) such that for $H^{n-m}$-a.e. $w \in I_r$,

$$\lim_{k,l \to \infty} \int_{F_a^{-1}(w)} |\nabla u_k - \nabla u_l|^p + |u_k - u_l|^p \ dH^m = 0.$$

This subsequence converges strongly to some element of $W^{1,p}(F_a^{-1}(w))$, which we denote by

$$u\prescript{\perp}{}{F_a^{-1}(w)}.$$

Since $u_k \to u$ uniformly on $F_a^{-1}(w)$ for $w \notin E$, observe that $u\prescript{\perp}{}{F_a^{-1}(w)}$ is a continuous representative of $u\prescript{\perp}{}{F_a^{-1}(w)}$. To ease notation, we will write $u_w$ for $u\prescript{\perp}{}{F_a^{-1}(w)}$. For $g \in C^\infty(\mathbb{R}^n)$, it is well known that

$$\sup_{S^m_\infty(1)} |g| \leq C \left( \int_{S^m_\infty(1)} |\nabla g|^p + |g|^p \ dH^m \right)^{1/p},$$

with $C = C(m,p)$, and by a simple scaling argument that

$$\sup_{S^m_\infty(r)} |g| \leq C \left( \int_{S^m_\infty(r)} r^{p-m} |\nabla g|^p + r^{-m} |g|^p \ dH^m \right)^{1/p}.$$
Since $u_k \n F_a^{-1}(w)$ converges uniformly to $u \n F_a^{-1}(w)$ and strongly to $u_w$ in the sense of \eqref{4.13}, applying \eqref{4.14} with $g$ replaced by $u_k$ yields \eqref{4.13}.

**Step 2.** We will show that there exist a constant $C_2 > 0$ and an $H^{n-m}$-measurable set $E_r \subset I_r$ of positive $H^{n-m}$-measure such that

\begin{equation}
\int_{E_r} |\nabla u|^p + \left| \frac{u - \bar{u}(a,r)}{r} \right|^p \, dH^n \leq C_2 r^n \text{ for each } w \in E_r.
\end{equation}

From the hypotheses that $a$ is an $L^p$-Lebesgue point for both $u$ and $|\nabla u|$ and that $\bar{u}(a) > 0$, it follows that there exist positive numbers $R$ and $\kappa$ such that for $r \in (0, R)$ we have

\begin{equation}
\bar{u}(a, r) > \kappa > 0
\end{equation}

and

\begin{equation}
\int_{B^n_2(r)} |\nabla u|^p \, dH^n \leq \left( |\nabla u(a)|^p + 1 \right) H^n(B^n_2(r)).
\end{equation}

Using Poincaré’s inequality and \eqref{4.9}, there exists $C_1 = C_1(n, p)$ such that

\begin{align*}
\int_{B^n_2(r)} |u - \bar{u}(a, r)|^p \, dH^n &\leq C_1 \int_{B^n_2(r)} |\nabla u|^p r^p \, dH^n \\
&\leq C_1 \alpha_n \left( |\nabla u(a)|^p + 1 \right) r^{n+p}
\end{align*}

where $\alpha_n$ is the volume of the unit ball in $\mathbb{R}^n$, and consequently,

\begin{equation}
\int_{B^n_2(r)} \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \, dH^n \leq C_1 \alpha_n \left( |\nabla u(a)|^p + 1 \right) r^n \text{ for } r \in (0, R).
\end{equation}

Employing the co-area formula, \eqref{4.10} and \eqref{4.14}, we have for all $r \in (0, R)$,

\begin{align*}
\int_{I_r} \int_{F_a^{-1}(w)} |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \, dH^m(t) \, dH^{n-m}(w) &= \int_{F_a^{-1}(I_r)} |JF_a| \left( |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \right) \, dH^n \\
&\leq \int_{B^n_2(r + r \sqrt{n/2})} \left( |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \right) \, dH^n \\
&\leq \alpha_n \left( 1 + \frac{\sqrt{n}}{2} \right) \left( C_1 + 1 \right) \left( |\nabla u(a)|^p + 1 \right) r^n.
\end{align*}

That is, setting $C_2 = \alpha_n \left( 1 + \sqrt{n/2} \right) \left( C_1 + 1 \right) \left( |\nabla u(a)|^p + 1 \right)$, we have

\begin{equation*}
\int_{I_r} \int_{F_a^{-1}(w)} |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \, dH^m \, dH^{n-m}(w) \leq C_2 r^n.
\end{equation*}

Let $G(w)$ denote the inner integral in this expression, so that we have

\begin{equation*}
\int_{I_r} G(w) \, dH^{n-m}(w) \leq C_2 r^n,
\end{equation*}

which establishes \eqref{4.7}.
Step 3. Finally, we will establish (ii) of our theorem. Since $E_r \subset I_r$, notice that for $w \in E_r$, $F_a^{-1}(w)$ is an $m$-sphere whose radius, $w_1 := \rho$, has the property that $r/2 \leq \rho \leq 3r/2$. Thus, using (4.3) and (4.7), we obtain

$$
\sup_{F_a^{-1}(w)} \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \leq C \int_{F_a^{-1}(w)} \left( \frac{\rho^{p-m} \left| \nabla \left( \frac{u_w - \bar{u}(a, r)}{r} \right) \right|^p + \rho^{-m} \left| \frac{u_w - \bar{u}(a, r)}{r} \right|^p}{r} \right) dH^m
$$

$$
\leq C \frac{3}{2} \rho^{-m} \int_{F_a^{-1}(w)} \left( \nabla (u_w)^p + \frac{u_w - \bar{u}(a, r)}{r} \right) dH^m
$$

$$
\leq C \left( \frac{3}{2} \right)^p 2^{m-2} \int_{F_a^{-1}(w)} \left( \nabla (u_w)^p + \frac{u_w - \bar{u}(a, r)}{r} \right) dH^m
$$

$$
\leq C_2 C \left( \frac{3}{2} \right)^p 2^{m}.
$$

With $K := \left( C_2 C \left( \frac{3}{2} \right)^p 2^{m} \right)^p$, we have $\sup_{F_a^{-1}(w)} |u - \bar{u}(a, r)| \leq Kr$ for $w \in E_r$. This, along with (4.8), implies there exists $R_0 > 0$ such that $u > 0$ on $F_a^{-1}(w)$ for $w \in E_r$, $0 < r < R_0$.

Theorem 4. Let $n \geq 3$, $n > m$ and $p > m > n - 1 - m \geq 1$. If $u$ is a non-zero element of $W^{1,p}(\mathbb{R}^n)$, then $u$ has a pair of linked spheres of dimensions $m$ and $n - 1 - m$ in its support.

Proof. If $u \in W^{1,p}(\mathbb{R}^n)$ is not identically zero, then there exists a $a \in \mathbb{R}^n$ such that $a$ is an $L^p$-Lebesgue point for $u$ and $\nabla u$. We will assume without loss of generality, that $\bar{u}(a) > 0$. Applying Theorem 3 we obtain $r_0 > 0$ and a Borel set $E_{r_0} \subset I_{r_0}$ of positive $H^{n-m}$-measure such that for $w \in E_{r_0} \subset \mathbb{R}^{n-m}$, $u$ is continuous and positive everywhere on $F_a^{-1}(w) = S_{(a', w_{n-m})}^m(w_1)$. With a slight abuse of the notation introduced at the beginning of Section 4, we let $w'' := (w_2, \ldots, w_{n-m})$ so that we now have

$$
F_a^{-1}(w) = S_{(a', w'')}^m(w_1).
$$

Let $W_a := \bigcup_{w \in E_{r_0}} F_a^{-1}(w)$. Since $E_{r_0}$ is $H^n$-measurable and $JF_a = 1$, we can appeal to the co-area formula to conclude that

$$
H^n(W_a) = \int_{E_{r_0}} H^m(F_a^{-1}(w)) \ dH^{n-m}(w) > 0.
$$

Note that $u$ is defined and is positive at all points of $W_a$. For suitable $w \in W_a$, we will construct an $(n - m - 1)$-sphere that will link with $S_{(a', w'')}^m(w_1)$ and that will lie in a “radial” $(n - m)$-plane emanating from $(a', w'')$ orthogonal to $Q^{-1}(w'')$. For this purpose define

$$
P : \mathbb{R}^n \setminus \bar{B}_{\theta}(a', w'')(r_0/2) \to S_{(a', w'')}^m(1) \quad \text{by} \quad P(x) = \left( a' + \frac{x' - a'}{|x' - a'|} w'' \right).
$$

Observe that $P$ is locally Lipschitz and that $P^{-1}(\theta)$ is independent of $w$ for $\theta \in S_{(a', w'')}^m(1)$. Proceeding as in the proof of Theorem 3 Step 1, with $F_a$ replaced by $P$, an application of the co-area formula yields that $u \bigcup P^{-1}(\theta) \in W^{1,p}(P^{-1}(\theta))$ for $H^m$-a.e. $\theta \in S_{(a', w'')}^m(1)$ and that $u \bigcup P^{-1}(\theta)$ is a $p$-quasicontinuous representative for $u \bigcup P^{-1}(\theta)$; see (4.3) and (4.7). Since $H^n(W_a) > 0$, the co-area formula also
implies that $H^{n-m}(W_a \cap P^{-1}(\theta)) > 0$ for $H^m$-a.e $\theta \in S^m_{(a',w')} (1)$. Thus, for such $\theta$, there exists

\begin{equation} w \in W_a \cap P^{-1}(\theta) \tag{4.11} \end{equation}

such that $w$ is a Lebesgue point for both $u \ll P^{-1}(\theta)$ and $\nabla (u \ll P^{-1}(\theta))$. Since $H^{n-m}(W_a \cap P^{-1}(\theta)) > 0$ and $u \ll P^{-1}(\theta) > 0$ on $W_a \cap P^{-1}(\theta)$, it follows that we can also require $w$ to have been chosen so that

\begin{equation} u \ll P^{-1}(\theta)(w) > 0. \tag{4.12} \end{equation}

With $w$ determined by (4.11) and (4.12), it follows that $u \ll P^{-1}(\theta)$ satisfies the hypotheses of Theorem 3 (ii) with the ambient space $\mathbb{R}^n$ replaced by $P^{-1}(\theta)$ and with $F_n$ replaced by $D : P^{-1}(\theta) \to \mathbb{R}^1$, defined by $D(x) = |x - (a',w')|$. Theorem 3 (ii) provides a number $0 < \bar{r} < w_1/2$ and a set $A \subset (\bar{r}/2,3\bar{r}/2)$ of positive $H^1$-measure such that $u \ll P^{-1}(\theta)$ is defined and positive on each $D^{-1}(\rho)$, $\rho \in A$. Thus, we have that $u > 0$ on the $(n-m-1)$-sphere $D^{-1}(\rho)$ and $u > 0$ on the $m$-sphere $S^m_{(a',w')} (\bar{r})$. These spheres are linked, since they are similar to the linked spheres (4.1) and (4.2).

\section*{Theorem 5}

Let $h : \tilde{B}^k \to \mathbb{R}^n$ be an embedding of the closed ball $\tilde{B}^k \subset \mathbb{R}^{k+1}$ where $1 \leq k < n$ and $n \geq 3$. If $u \in W^{1,p}(\mathbb{R}^n)$, $p > \frac{k+1}{2}$ and $\text{spt } u \subset h(\tilde{B}^k)$, then $u \equiv 0$.

\begin{proof}

First, assume $k$ is even, $k + 1 < n$, and by contradiction, suppose that $H^n(\text{spt } u) > 0$. Writing $x \in \mathbb{R}^n$ as $x = (x', x'')$ where $x' \in \mathbb{R}^{k+1}$, recall that $Q : \mathbb{R}^n \to \mathbb{R}^{n-k-1}$ is defined as $Q(x) := x''$. Then we have $H^{k+1}(Q^{-1}(x'') \cap \text{spt } u) > 0$ for all $x''$ in a set $E$ of positive $H^{n-k-1}$-measure and, as in (4.5), $u$ is a nonzero element of $W^{1,p}(Q^{-1}(x''))$ for $H^{n-k-1}$-a.e. $x'' \in E$. Redefine $E$ to include only such $x''$. For $x'' \in E$, we employ Theorem 3 with $\mathbb{R}^n$ replaced by the $(k+1)$-dimensional affine space $Q^{-1}(x'')$ and $m$ replaced by $k/2$ to conclude that $u \in W^{1,p}(Q^{-1}(x''))$ contains a pair of linked spheres, both of dimension $k/2$, in its support. Call these spheres $S_1$ and $S_2$. With $h$ as in the statement of our theorem, let $H := h^{-1} \ll (Q^{-1}(x'') \cap h(\tilde{B}^k))$; so $H$ is a homeomorphism of $(Q^{-1}(x'') \cap h(\tilde{B}^k))$ into $\mathbb{R}^k$. Since $S_1$ and $S_2$ are linked spheres in $(Q^{-1}(x'') \cap h(\tilde{B}^k))$ and since $H$ is a homeomorphism, it follows from Definition 11 that $H(S_1)$ and $H(S_2)$ are linked in $\mathbb{R}^k$, which contradicts Theorem 2.

The above proof is easily modified and simpler for the case $k + 1 = n$. A similar argument holds when $k$ is odd.
\end{proof}

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