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D-SPACES AND FINITE UNIONS

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ABSTRACT. This article is a continuation of a recent paper by the author and R. Z. Buzyakova. New results are obtained in the direction of the next natural question: how complex can a space be that is the union of two (of a finite family) "nice" subspaces? Our approach is based on the notion of a D-space introduced by E. van Douwen and on a generalization of this notion, the notion of aD-space. It is proved that if a space X is the union of a finite family of subparacompact subspaces, then X is an aD-space. Under (CH), it follows that if a separable normal T_1-space X is the union of a finite number of subparacompact subspaces, then X is Lindelöf. It is also established that if a regular space X is the union of a finite family of subspaces with a point-countable base, then X is a D-space. Finally, a certain structure theorem for unions of finite families of spaces with a point-countable base is established, and numerous corollaries are derived from it. Also, many new open problems are formulated.

1. Some general facts and definitions

Addition theorems occupy an important place in general topology. One of the main questions considered in this paper is the following one: how complex can a space be if it is the union of two (of a finite family) "nice" subspaces? This might be especially important to know when we are constructing concrete spaces with a certain combination of properties. In [2] a necessary condition for a space to be the union of two metrizable subspaces was established. We continue this line of investigation in this article.

The Alexandroff compactification of any uncountable discrete space is the union of two metrizable (in fact, discrete) subspaces, while it is not first countable and, therefore, not metrizable. Another example of a non-metrizable compactum, which is the union of two metrizable subspaces, is the double circumference of Alexandroff and Urysohn. This space is first countable and, hence, Fréchet-Urysohn. In fact, E. Michael and M. E. Rudin established a curious fact: if a compact Hausdorff space X is the union of two metrizable compacta, then X is an Eberlein compactum [12], and therefore, X is Fréchet-Urysohn.

We define the extent $e(X)$ in a slightly unusual way (following [2]). A subset A of a space X is *discrete in X (locally finite in X)* if every point $x \in X$ has an open neighbourhood Ox containing not more than one element (only finitely many

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elements) of A. The *extent* $e(X)$ of a space X is the smallest infinite cardinal number τ such that $|A| \leq \tau$, for every subset A of X that is locally finite in X. This definition obviously coincides with the usual definition of the extent of X for all T_1-spaces.

Our approach to addition theorems is based on the notion of a D-space, introduced by E. van Douwen in [8], and on a generalization of it. A *neighbourhood assignment* on a topological space X is a mapping ϕ of X into the topology $\mathcal{T}$ of X such that $x \in \phi(x)$, for each $x \in X$. A space X is called *a D-space* if, for every neighbourhood assignment ϕ on X, there exists a subset A of X, locally finite in X, such that the family $\phi(A)$ covers X. A principal property of D-spaces is that the extent coincides with the Lindelöf number in such spaces. In particular, every countably compact D-space is compact, and every D-space with countable extent is Lindelöf. These facts make the notion of a D-space a useful tool in studying covering properties.

It is still an open problem (E. van Douwen, [8]) whether every regular Lindelöf space is a D-space. It is even unknown if every hereditarily Lindelöf regular T_1-space is a D-space. Van Douwen also asked whether there exists a subparacompact or metacompact space that is not a D-space. These questions are still open. Recall that a space X is said to be *subparacompact* if every open covering of X can be refined by a σ-discrete closed covering [4].

All metrizable spaces, and, more generally, all Moore spaces and semi-stratifiable spaces, are D-spaces [3]. A much more general result was recently obtained by R. Z. Buzyakova: every strong Σ-space is a D-space [6]. It follows from her theorem that all Tychonoff spaces with a countable network, all σ-spaces, all paracompact p-spaces, and all Lindelöf Σ-spaces (that is, all Tychonoff continuous images of Lindelöf p-spaces) are D-spaces. Recall that a *σ-space* is a space with a σ-discrete network.

On the other hand, there exists a Hausdorff, locally compact, locally countable, separable, first countable, submetrizable, σ-discrete, realcompact space with a G_δ diagonal that is not a D-space: the amazing space Γ constructed by H. H. Wicke and E. van Douwen in [9] has all these properties. Thus, there exists a locally compact σ-metrizable Tychonoff space with the diagonal G_δ that is not a D-space.

A space X is an *aD-space* [2] if for each closed subset F of X and each open covering γ of X there exist a subset A of F, locally finite in F, and a mapping ϕ of A into γ such that $a \in \phi(a)$, for each $a \in A$, and the family $\phi(A) = \{\phi(a) : a \in A\}$ covers F (any such mapping ϕ will be called a *pointer* (from A to γ)).

Clearly, every closed subspace of an aD-space is an aD-space. It was proved in [3] that every subparacompact space is an aD-space. Below we use the next two easy-to-prove statements.

Lemma 1.1. *If $X = Y \cup Z$, where Y is an aD-space (D-space) and Y is closed in X, and every subspace of Z closed in X is an aD-space (a D-space), then X is also an aD-space (a D-space, respectively).*

Proposition 1.2 ([2]). *If $X = Y \cup Z$, where Y and Z are aD-spaces (D-spaces) and Y is closed in X, then X is also an aD-space (a D-space, respectively).*

Notice the following crucial property of aD-spaces [2]: every aD-space of countable extent is Lindelöf. Therefore, we have the next statement, which allows us to identify many non-aD-spaces:

Proposition 1.3. *Every pseudocompact aD-space of countable extent is compact.*

A Tychonoff space that is the union of a countable collection of metrizable subspaces can have rather complex structure and an unusual combination of properties: it suffices to refer to the σ-discrete space Γ of van Douwen and Wicke. It is natural to ask whether the union of a finite collection of metrizable spaces can be as complicated as the space Γ. In that direction, it was shown in [2] that if a regular space X is the union of finitely many metrizable subspaces, then X is a D-space. The proof of this statement was based on the following result of Buzyakova [2]: every space with a point-countable base is a D-space. Since every D-space of countable extent is Lindelöf, it follows that every regular space of countable extent, if it is the union of a finite family of metrizable spaces, is Lindelöf [2]. Note that the space Γ is a Tychonoff σ-metrizable space of countable extent that is not a D-space. However, it was shown in [2] that if a regular space X is the union of a countable family γ of dense metrizable subspaces, then X is a D-space. Also, if a space X is the union of a countable family of open metrizable subspaces, then X is a D-space [2]. Both results follow from the principal theorem in [2] that every space with a point-countable base is a D-space. The next theorem on finite unions is a new result.

Theorem 1.4. *If a space X is the union of a finite collection $\{X_i : i = 1, ..., k\}$ of subparacompact subpaces, then X is an aD-space.*

We will prove this statement by induction. To make induction possible, we need the following lemma:

Lemma 1.5. *Suppose that X is a space, Y a subspace of X such that every closed subspace F of X contained in $X \setminus Y$ is an aD-space, and γ is an open covering of X. Then, for every locally finite in Y family η of subsets of Y refining γ, there exist a locally finite in X subset A and a pointer $f : A \to \gamma$ such that $\bigcup \eta \subset \bigcup f(A)$.*

Proof. Let F be the set of all elements of X at which the family η is not locally finite. Clearly, F is a closed subset of X disjoint from Y. Therefore, F is an aD-space, and there exist a locally finite in F subset B of F and a pointer $g : B \to \gamma$ such that F is covered by the subfamily $g(B)$ of γ. Put $U = \bigcup g(B)$, $H = (\bigcup \eta) \setminus U$, and $\xi = \{P \setminus U : P \in \eta, P \neq \emptyset\}$. Clearly, the family ξ is locally finite in X and $\bigcup \xi = H$. For each $P \in \xi$ fix a $c_P \in P$ and $V_P \in \gamma$ such that $P \subset V_P$ (recall that η refines γ). Put $C = \{c_P : P \in \xi\}$, $A = B \cup C$, $f(c_P) = V_P$, for each $c_P \in C$, and $f(b) = g(b)$, for each $b \in B$. Then f is a pointer from A to γ and, clearly, $f(A) \supset (g(B) \cup f(C)) \supset U \cup (\bigcup \eta) \supset \bigcup \eta$, and the set A is locally finite in X. $\square$

Now we are going to present a proof of Theorem 1.4.

Proof. We argue by induction. If $k = 0$, then $X = \emptyset$ and the statement is obviously true. Assume now that the statement holds for $k = n$, for some $n \in \omega$, and let us show that it is also true for $k = n + 1$.

Let γ be an open covering of X. For each $i = 1, ..., n + 1$, fix a covering $\mathcal{P}_i = \bigcup\{\eta_{ij} : j \in \omega\}$ of X_i satisfying the following conditions:

1) $\mathcal{P}_i$ refines γ;

2) each element of $\mathcal{P}_i$ is a closed subset of the space X_i;

3) the family η_{ij} is discrete in X_i, for each $i = 1, ..., n + 1$ and $j \in \omega$.

If a subset F of $X \setminus X_i$ is closed in X, then F is the union of not more than n subparacompact subspaces, since $F \cap X_j$ is closed in X_j, for each $j = 1, ..., n + 1$.

It follows by the inductive assumption that F is an aD-space. Therefore, Lemma 1.5 is applicable: for each $i = 1, ..., n+1$ there exist a locally finite in X set $D_{i,1}$ and a pointer $f_{i,1} : D_{i,1} \to \gamma$ such that $\bigcup \eta_{i,1}$ is covered by $f_{i,1}(D_{i,1})$. Note that we can define the sets $D_{i,1}$ and pointers $f_{i,1}$ one after another in such a way that $D_{j,1}$ will lie outside of $f_{i,1}(D_{i,1})$ whenever $i < j$. Put $D_1 = \bigcup\{D_{i,1} : i = 1, ..., n+1\}$, and define a pointer $f_1 : D_1 \to \gamma$ by the requirement that the restriction of f_1 to $D_{i,1}$ coincides with $f_{i,1}$, $1 = 1, ..., n+1$. Put $U_1 = \bigcup f_1(D_1)$. Clearly, U_1 is an open subset of X.

Let us assume that we have already defined an open subset U_j of X, for some positive integer j. Obviously, we can apply Lemma 1.5 to the space $X \setminus U_j$ and obtain, for each $i = 1, ..., n+1$, a locally finite in $X \setminus U_j$ set $D_{i,(j+1)}$ and a pointer $f_{i,(j+1)} : D_{i,(j+1)} \to \gamma$ such that $\bigcup \eta_{i,(j+1)} \setminus U_j$ is covered by $f_{i,(j+1)}(D_{i,(j+1)})$. Note that we can define the sets $D_{i,(j+1)}$ and pointers $f_{i,(j+1)}$ one after another in such a way that $D_{m,(j+1)}$ will lie outside of $f_{l,(j+1)}(D_{l,(j+1)})$ whenever $l < m$.

Put $D_{j+1} = \bigcup\{D_{i,(j+1)} : i = 1, ..., n+1\}$, and define a pointer $f_{j+1} : D_{j+1} \to \gamma$ by the requirement that the restriction of f_{j+1} to $D_{i,(j+1)}$ coincides with $f_{i,(j+1)}$, $1 = 1, ..., n+1$. Put $U_{j+1} = \bigcup f_{j+1}(D_{j+1}) \cup U_j$. Clearly, U_{j+1} is an open subset of X.

The family $\{U_j : j \in \omega\}$ of open subsets of X obtained in this way covers X, since all elements of $\mathcal{P}$ are covered by it. Since $D_j \subset U_j \subset U_{j+1}$ and $D_{j+1} \cap U_j = \emptyset$, the set $D = \bigcup\{D_j : j \in \omega\}$ is locally finite in X. Thus, the pointer $f : D \to \gamma$ defined by the requirement that the restriction of f to D_i coincides with f_i is a pointer we are looking for. □

Problem 1.6. Suppose that a space X is the union of a finite collection $\{X_i : i = 1, ..., k\}$ of Moore spaces. Is then X a D-space?

Problem 1.7. Suppose that $X = Y \cup Z$, where Y and Z are D-spaces. Is X a D-space? Is X an aD-space?

Problem 1.8. Suppose that $X = Y \cup Z$, where Y and Z are D-spaces, and X is countably compact. Is X compact?

Problem 1.9. Suppose X is a countably compact space that is the union of a countable family of D-spaces. Is X compact?

A partial answer to the last question was given in [1], where it was shown that if a linearly ordered countably compact space X is the union of a countable family of aD-spaces, then X is compact.

Theorem 1.10 (CH)**.** *If a separable normal T_1-space X is the union of a finite number of subparacompact subspaces, then X is Lindelöf.*

Proof. It follows from Theorem 1.4 that X is an aD-space. On the other hand, from a lemma of F. B. Jones [10] and CH it follows that the extent of X is countable. Therefore, X is Lindelöf [2]. □

Even the following corollary seems to be new.

Corollary 1.11 (CH)**.** *If a separable normal T_1-space X is the union of a finite number of metrizable subspaces, then X is Lindelöf.*

Of course, one cannot just drop (CH) in the above two statements, since it is consistent that there exists a normal Mrowka space that is not Lindelöf (see [4], [10]).

It was observed in [1] that every countably metacompact σ-metrizable (σ-paracompact) space of countable extent is Lindelöf. However, the answer to the next question remains unknown:

Problem 1.12. Is every countably metacompact σ-metrizable space a D-space? An aD-space?

Note that even the union of two discrete subspaces need not be countably metacompact, as is witnessed by a space constructed by J. Chaber in [7], Example 2.4 (see also [14]). Recall that a subset P of a space X is said to be *locally closed* if P is open in its closure (that is, if P can be represented as the intersection of an open subset of X with a closed subset of X). For example, every discrete subspace of a space X is locally closed in X.

Lemma 1.13. *If a space Y is the union of a finite collection of locally closed D-spaces, then Y is a D-space.*

Proof. We argue by induction. Let $Y = P_1 \cup ... \cup P_{n+1}$ where each P_i is locally closed in X and a D-space. Assume that the lemma holds whenever the number of summands does not exceed n. For $i = 1, ..., n+1$ put $F_i = \overline{P_i} \setminus P_i$. Then F_i is closed in Y and $P_i \cap F_i = \emptyset$. Therefore, F_i is the union of $\leq n$ locally closed D-spaces $P_j \cap F_i$. By the inductive assumption, it follows that each F_i is a D-space. Clearly, $\overline{P_i} = P_i \cup F_i$. Since F_i is closed in $\overline{P_i}$, and both F_i and P_i are D-spaces, it follows that $\overline{P_i}$ is a D-space, for each $i = 1, ..., n + 1$. Hence, Y is also a D-space, as the union of a finite number of closed D-spaces. □

The next theorem partially generalizes one of the main results in [2], stating that every space with a point-countable base is a D-space.

Theorem 1.14. *Suppose that $X = X_1 \cup ... \cup X_k$, where X is a regular space and each X_i is a space with a point-countable base. Then X is a D-space.*

Proof. We again argue by induction. Assume that the statement is true for not more than $k-1$ summands. Put $H_{i,j} = \overline{\overline{X_i} \cap X_j}$, $W_{i,j} = \overline{X_i} \setminus H_{i,j}$, for $i, j = 1, ..., k$, and $F = \bigcap\{H_{i,j} : i, j = 1, ..., k\}$.

Let $i \neq j$. Then, obviously, $W_{i,j} \cap X_j = \emptyset$. Therefore, $W_{i,j}$ is the union of $\leq k-1$ spaces with a point-countable base, and the inductive assumption implies that $W_{i,j}$ is a D-space. Clearly, $W_{i,j}$ is locally closed in X. The space $V_i = X \setminus \overline{X_i}$ is also locally closed in X, for each $i = 1, ..., k$. Since the sets V_i and X_i are disjoint, the space V_i is the union of $\leq k-1$ spaces with a point-countable base, and the inductive assumption implies that V_i is a D-space. It follows from Lemma 1.13 that the subspace $E = (\bigcup\{W_{i,j} : i, j = 1, ..., k\}) \cup (\bigcup\{V_i : i = 1, ..., k\})$ of X is a D-space (note that $W_{i,j}$ is empty whenever $i = j$). Take any $x \in X \setminus E$. Then $x \in \overline{X_i}$ and $x \notin W_{i,j}$, for all $i, j = 1, ..., k$. It follows that $x \in H_{i,j}$, for all $i, j = 1, ..., k$. Hence, $x \in F$. Thus, $X = E \cup F$.

Let us show that the space F has a point-countable base. Fix a point-countable base $\mathcal{B}_i$ in each space X_i, $i = 1, ..., k$. For each $V \in \mathcal{B}_i$, let $\phi(V)$ be the largest open subset of $\overline{X_i}$ such that $\phi(V) \cap X_i = V$. Put $\mathcal{G}_i = \{\phi(V) : V \in \mathcal{B}_i\}$, $\mathcal{G} = \bigcup\{\mathcal{G}_i : i = 1, ..., k\}$, and $\mathcal{S} = \{W \cap F : W \in \mathcal{G}\}$. From regularity of X it easily follows that $\mathcal{S}$ is a base of the space F. *Claim*: The family $\mathcal{S}$ is point-countable. Clearly, to prove the claim it is enough to show that, for any $j = 1, ..., k$ and any $x \in F$, the family $\mathcal{G}_j$ is countable at x; that is, only countably many elements of $\mathcal{G}_j$ contain x. Obviously, $x \in X_i$, for some $i = 1, ..., k$. Then $x \in F \subset H_{i,j} = \overline{\overline{X_i} \cap X_j}$. However, the space $\overline{X_i}$ is first countable at the point x, since $x \in X_i$ and the space X is regular. It follows that the tightness of the space $\overline{X_i}$ at the point x is countable, and there exists a countable subset $M \subset \overline{X_i} \cap X_j$ such that $x \in \overline{M}$. Let γ_x be the family of all elements of $\mathcal{G}_j$ containing x. Clearly, $x \in \overline{X_j}$, and each $W \in \gamma_x$ is an open neighbourhood of x in $\overline{X_j}$. Since $M \subset X_j$ and $x \in \overline{M}$, every $W \in \gamma_x$ contains at least one point of M. However, M is countable, and the family $\mathcal{G}_j$ is point-countable at each point of M, since $M \subset X_j$ and $\mathcal{B}_j$ is a point-countable base of the space X_j. Therefore, γ_x is also countable. It follows that $\mathcal{S}$ is a point-countable base of F. Hence, F is a D-space [2]. Since F is closed in X, and $X = E \cup F$, where E is also a D-space, we conclude that X is a D-space. □

Corollary 1.15. *If a regular space X of countable extent is the union of a finite collection of subspaces with a point-countable base, then X is Lindelöf.*

Recall that the *spread* of a space X is countable if every discrete subspace of X is countable.

Corollary 1.16. *If a pseudocompact space X of countable spread is the union of a finite collection of subspaces with a point-countable base, then X is a hereditarily Lindelöf compactum.*

Proof. Every subspace Y of X is a space of countable extent and has a point-countable base. Therefore, by Corollary 1.15, X is hereditarily Lindelöf. It follows that X is compact. □

A closer look at the proof of Theorem 1.14 leads us to the next general statement, interesting in itself:

Theorem 1.17. *If a regular space X is the union of a finite family of subspaces with a point-countable base, then X is the union of a finite collection of locally closed subspaces with a point-countable base.*

Proof. We argue by induction, using the construction from the proof of Theorem 1.14. To make the induction possible, it suffices to notice that every locally closed subset Z of a locally closed subspace Y of a space X is locally closed in X. Note that the number of summands in the second representation of X can be much larger than in the first one. □

The next lemma is obvious.

Lemma 1.18. *If A is a locally closed subset of a regular space X, and A is not nowhere dense in X, then A contains a nonempty open subset of X.*

Theorem 1.19. *If a regular space X is the union of a finite family of subspaces with a point-countable base, then X has a dense open subspace with a point-countable base.*

Proof. By Theorem 1.17, X is the union of a finite collection γ of locally closed subspaces with a point-countable base. The same is true for every nonempty open subset U of X. Since no space is the union of a finite family of nowhere dense subspaces, it follows that, for at least one $Y \in \gamma$, $Y \cap U$ is not nowhere dense in U. From Lemma 1.18 it follows that there exists a nonempty open subset V of U such that $V \subset Y \cap U$. Clearly, V is open in X and V has a point-countable base. Now a standard application of Zorn's Lemma provides us with a disjoint family η of nonempty open subsets of X such that $\bigcup \eta$ is dense in X and each element of η has a point countable base. Clearly, $Z = \bigcup \eta$ is a dense open subspace of X with a point-countable base. $\square$

Corollary 1.20. *If X is a regular countably compact T_1-space that is the union of a finite collection of subspaces with a point-countable base, then X is the union of a finite family of locally compact metrizable subspaces (which are locally closed in X).*

Proof. By Theorem 1.17, X can be represented as the union of a finite family η of locally closed subspaces with a point-countable base. Observe that each $Y \in \eta$ is open in $\overline{Y}$. Every countably compact T_1-space with a point-countable base is metrizable and compact, by A. Mischenko's theorem [10]. Also, every locally separable, locally metrizable Hausdorff space with a point-countable base is metrizable, by a theorem of P. Alexandroff and P. Urysohn [10]. It follows that every element of η is a locally compact metrizable space. $\square$

Notice that, under the assumptions in Corollary 1.20, the space X is compact. Indeed, it follows from the results in [15] that if a countably compact space X is the union of a countable collection of subspaces with a point-countable base, then X is compact. See also [13].

Corollary 1.21. *If X is a regular Lindelöf p-space that is the union of a finite collection of subspaces with a point-countable base, then X is the union of a finite family of locally separable metrizable subspaces (which are locally closed in X).*

Proof. The proof is practically the same as the proof of Corollary 1.20; it is enough to observe that every Lindelöf p-space with a point-countable base is separable metrizable [10]. $\square$

After the above arguments, the next statement becomes obvious. It generalizes an interesting result of M. Ismail and A. Szymanski from [11], where they studied locally compact Hausdorff spaces that are unions of finite families of metrizable subspaces.

Corollary 1.22. *If a Lindelöf p-space X is the union of a finite family of subspaces with a point-countable base, then X has an open dense metrizable subspace.*

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