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QUASI-MINIMAL ABELIAN GROUPS

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Abstract. An abelian group G is said to be *quasi-minimal (purely quasi-minimal, directly quasi-minimal)* if it is isomorphic to all its subgroups (pure subgroups, direct summands, respectively) of the same cardinality as G. Obviously quasi-minimality implies pure quasi-minimality which in turn implies direct quasi-minimality, but we show that neither converse implication holds. We obtain a complete characterisation of quasi-minimal groups. In the purely quasi-minimal case, assuming GCH, a complete characterisation is also established. An independence result is proved for directly quasi-minimal groups.

§1. Introduction

The concept of quasi-minimality was first introduced in a topological context (see e.g. [6], [11] and [12]). Given a collection $\mathcal{C}$ of topological spaces a quasi-order, i.e., a reflexive and transitive but not necessarily anti-symmetric relation, "sub", is defined on $\mathcal{C}$ by: Y sub X if Y is homeomorphic to a subspace of X, for $X, Y \in \mathcal{C}$. Then a space $X \in \mathcal{C}$ is said to be *quasi-minimal* if Y sub X and $|Y| = |X|$ implies X is homeomorphic to Y. The present work investigates collections of abelian groups from a similar standpoint.

An abelian group G is said to be *quasi-minimal (purely quasi-minimal, directly quasi-minimal*, respectively) if G is isomorphic to each of its subgroups (pure subgroups, direct summands, respectively) of the same cardinality as G. It is possible to completely characterise quasi-minimal groups. Large classes of purely quasi-minimal groups can also be characterised; however, to achieve a complete characterisation we have assumed the general continuum hypothesis (GCH). It is not clear to us at this stage whether it is necessary to make such an additional set-theoretic assumption. We note that in the directly quasi-minimal case an example can be given to show that the direct quasi-minimality of a group may be undecidable in ZFC.

Our notation is standard and largely in accord with Fuchs [4] and [5], which contains all undefined group-theoretic terms used herein; an exception is that we write mappings on the right and write $A \sqsubset B$ to denote that A is a direct summand of B. The necessary set-theoretic background for this work is contained in [3]. In this paper the term "group" shall always denote an abelian group.

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§2. Quasi-minimal groups

Since a finite group is trivially quasi-minimal we concentrate on groups of cardinality κ where κ is a fixed but arbitrary infinite cardinal. Let Ab_κ denote the set of groups of cardinality κ. As mentioned above, a group $G \in Ab_\kappa$ is *quasi-minimal* if G is isomorphic to all its subgroups of cardinality κ. Since any group G can be written in the form $G = D \oplus R$ where D is divisible and R is reduced, we immediately get that a quasi-minimal group is either divisible or reduced. The following lemma gives a further reduction.

Lemma 2.1. *If G is quasi-minimal, then G is either torsion-free or a p-group.*

Proof. If $|tG| = |G|$, then $G \cong tG$, since G is quasi-minimal. Now, $tG = \bigoplus_{p\in\Pi} G_p$ where G_p is a p-group for all p. Choose p such that $G_p \neq 0$ and then $tG = G_p \oplus \bigoplus_{q\neq p} G_q$, so $|G| = |G_p|$ or $|G| = |\bigoplus_{q\neq p} G_q|$. If $|G| = |\bigoplus_{q\neq p} G_q|$, then $G \cong \bigoplus_{q\neq p} G_q$, a contradiction, since $G_p \neq 0$. Therefore $|G| = |G_p|$ and so $G \cong G_p$ is a p-group. Now suppose $|tG| < |G|$. Then $|G/tG| = |G| = \kappa$. If $\kappa = \aleph_0$, then tG is finite and so $tG \sqsubset G$, since tG is pure in G. Then $G = tG \oplus C$ where C is torsion-free and $|G| = |C| = \aleph_0$, and so $G \cong C$ (and $tG = 0$), i.e. G is torsion-free. If $\kappa > \aleph_0$, then $r(G/tG) = |G/tG|$; so we can choose κ linearly independent elements $\overline{g}_\alpha = g_\alpha + tG, \alpha < \kappa$. Let $C = \langle g_\alpha : \alpha < \kappa\rangle \leq G$. The group C is torsion-free since if $c = \sum_{\alpha<\kappa} k_\alpha g_\alpha \in tG$ where $k_\alpha = 0$ for almost all α, then $\sum_{\alpha<\kappa} k_\alpha g_\alpha + tG = tG$, so $\sum_{\alpha<\kappa} k_\alpha \overline{g}_\alpha = 0$ and hence $k_\alpha = 0$ for all α, i.e. $c = 0$. Now $|G| = |C|$ and so $G \cong C$, i.e. G is torsion-free. □

Quasi-minimal torsion-free groups and quasi-minimal p-groups are easily characterised.

Proposition 2.2. *If G is torsion-free of cardinality κ, then G is quasi-minimal if and only if G is free, of rank κ if $\kappa > \aleph_0$, and of rank 1 if $\kappa = \aleph_0$.*

Proof. Firstly, $\mathbb{Z}$ is quasi-minimal since the only non-zero subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z} \cong \mathbb{Z}$ where $n \in \mathbb{N}$. Also $\bigoplus_\kappa \mathbb{Z}$ is quasi-minimal for $\kappa > \aleph_0$ since if $H \leq \bigoplus_\kappa \mathbb{Z} \quad (\kappa > \aleph_0)$ with $|H| = \kappa$, then H is free of rank κ and so $H \cong G$. It remains to show that these are the only torsion-free quasi-minimal groups.

If $\kappa = \aleph_0$, let $0 \neq g \in G$ and consider $\langle g\rangle$. Since G is torsion-free $|\langle g\rangle| = \aleph_0$ and so $G \cong \langle g\rangle \cong \mathbb{Z}$.

Now consider $\kappa > \aleph_0$. In this case $r(G) = \kappa$ so G has κ linearly independent elements $\{g_\alpha : \alpha < \kappa\}$. Now $\langle g_\alpha : \alpha < \kappa\rangle = \bigoplus_{\alpha<\kappa} \langle g_\alpha\rangle$ and $|\bigoplus_{\alpha<\kappa} \langle g_\alpha\rangle| = \kappa$. Therefore $G \cong \bigoplus_{\alpha<\kappa} \langle g_\alpha\rangle \cong \bigoplus_\kappa \mathbb{Z}$. □

Proposition 2.3. *If G is a p-group of cardinality $\kappa \geq \aleph_0$, then G is quasi-minimal if and only if $G \cong \mathbb{Z}(p^\infty)$ or $G \cong \bigoplus_\kappa \mathbb{Z}(p)$ for some prime p.*

Proof. Firstly, $\mathbb{Z}(p^\infty)$ is obviously quasi-minimal since its only subgroup of cardinality $\aleph_0$ is itself. Also $\bigoplus_\kappa \mathbb{Z}(p)$ is quasi-minimal since any subgroup is of the form $\bigoplus_\lambda \mathbb{Z}(p)$ and if the cardinalities are equal, then $\lambda = \kappa$. So consider a quasi-minimal

p-group G of cardinality $\kappa \geq \aleph_0$. Let B be a basic subgroup of $G, B = \bigoplus_{n\in\mathbb{N}} \bigoplus_{i\in I_n} \langle x_i \rangle$ with $o(x_i) = p^n$ for $i \in I_n$ where $I_n = \emptyset$ is allowed.

We first suppose $|G| > \aleph_0$. In this case we claim that $|G| = |G[p]|$ where $G[p]$ is the p-socle of G. If $|B| = |G|$, then $|B[p]| = |\bigoplus_{n\in\mathbb{N}} \bigoplus_{I_n} \langle p^{n-1}x_i \rangle| = \kappa$ and hence $|G[p]| \geq |B[p]| = \kappa = |G|$. If $|B| < |G|$, then $|G/B| = |G|$ and G/B is divisible, so $G/B = \bigoplus_{j\in J} \mathbb{Z}(p^\infty)$ where $|J| = \kappa$. There exists $C \leq G$, containing B, such that $C/B = \bigoplus_{j\in J} \mathbb{Z}(p)$. Then $C \cong B \oplus \bigoplus_{j\in J} \mathbb{Z}(p)$, since B is pure in C, and so $C[p] = B[p] \oplus \bigoplus_{j\in J} \mathbb{Z}(p)$. Therefore $|G| = |G/B| = \kappa = |C[p]| \leq |G[p]|$. Thus, in both cases, $|G[p]| = |G|$. Hence $G \cong G[p] \cong \bigoplus_{\kappa} \mathbb{Z}(p)$.

If $\kappa = \aleph_0$, then either $|B| = |G|$ or $|B|$ is finite. If $|B| = |G|$, then we get $G \cong G[p] \cong \bigoplus_{\aleph_0} \mathbb{Z}(p)$, as above. On the other hand, if B is finite, then $B \sqsubset G$, $G = B \oplus D$, say, with $D \cong G/B$, divisible and infinite. Therefore $G \cong D = \bigoplus_{I} \mathbb{Z}(p^\infty)$ for some index set I. If $|I| \geq 2$, then G contains a subgroup $H \cong \mathbb{Z}(p) \oplus \bigoplus_{J} \mathbb{Z}(p^\infty)$ where $I = J \cup \{i_0\}$, say. Now, $|H| = |G|$ so $G \cong H$, a contradiction, since H is not divisible. Thus $|I| = 1$ and $G \cong \mathbb{Z}(p^\infty)$. □

We summarise our characterisation of quasi-minimal groups in the following theorem:

Theorem 2.4. *A group G of cardinality κ is quasi-minimal if and only if*

(i) $\kappa < \aleph_0$*; or*

(ii) $(\kappa = \aleph_0)$ $\quad G = \mathbb{Z}, \quad \mathbb{Z}(p^\infty) \quad$ *or* $\quad \bigoplus_{\aleph_0} \mathbb{Z}(p)$*; or*

(iii) $(\kappa > \aleph_0)$ $\quad G = \bigoplus_{\kappa} \mathbb{Z} \quad$ *or* $\quad \bigoplus_{\kappa} \mathbb{Z}(p)$.

§3. Purely quasi-minimal groups

Next we consider the purely quasi-minimal groups in Ab_κ. Recall that a group $G \in Ab_\kappa$ is *purely quasi-minimal* if G is isomorphic to all its pure subgroups of cardinality κ. It is easily seen that we get the same reduction to either divisible or reduced groups as for the quasi-minimal groups. The divisible case is taken care of in the following lemma.

Lemma 3.1. *For a divisible purely quasi-minimal group G of cardinality κ one of the following is true:*

(i) $G \cong \mathbb{Q} \quad (\kappa = \aleph_0)$,

(ii) $G \cong \mathbb{Z}(p^\infty)$*, for some p* $\quad (\kappa = \aleph_0)$,

(iii) $G \cong \bigoplus_{\kappa} \mathbb{Q} \quad (\kappa > \aleph_0)$,

(iv) $G \cong \bigoplus_{\kappa} \mathbb{Z}(p^\infty)$*, for some p* $\quad (\kappa > \aleph_0)$.

Proof. We first note that $\mathbb{Q}, \mathbb{Z}(p^\infty), \bigoplus_{\kappa} Q, \bigoplus_{\kappa} \mathbb{Z}(p^\infty)$ are indeed purely quasi-minimal since a pure subgroup of a divisible group is again divisible.

Let G be divisible and purely quasi-minimal. Then $G = \bigoplus_{I} \mathbb{Q} \oplus \bigoplus_{p\in\Pi} \bigoplus_{I_p} \mathbb{Z}(p^\infty)$, for some index sets I, I_p. If $|G| = \aleph_0$, then it is immediate that G must be isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$, for some p. Now suppose $|G| = \kappa > \aleph_0$. Then either $|I| = \kappa$ or $|\bigoplus_{p\in\Pi}\bigoplus_{I_p}\mathbb{Z}(p^\infty)| = \kappa$. If $|I| = \kappa$ we have $G \cong \bigoplus_\kappa \mathbb{Q}$. On the other hand, if $|I| < \kappa$, then $\kappa = \sup_p |I_p|$. If $|I_p| = \kappa$ for some p, then $G \cong \bigoplus_{I_p}\mathbb{Z}(p^\infty)$. If $|I_p| < \kappa$ for all p, then there exist infinitely many $I_p \neq \emptyset$. Consider $H = \bigoplus_{I_p(p\neq q)}\mathbb{Z}(p^\infty)$ where $I_q \neq \emptyset$. Then $|H| = |G|$, so $H \cong G$, a contradiction, since H has no q-component. We conclude that if $|I| < \kappa$, then $G \cong \bigoplus_{I_p}\mathbb{Z}(p^\infty)$ for some p. □

Turning to the reduced case we have a further reduction to homocyclic or torsion-free groups:

Theorem 3.2. *If a reduced group G is purely quasi-minimal, then G is either a homocyclic p-group, i.e. $G = \bigoplus_I \mathbb{Z}(p^n)$ for some index set I and some $n \in \mathbb{N}$, or G is torsion-free.*

Proof. To begin we show that a homocyclic group is purely quasi-minimal. Let $G = \bigoplus_{i<\kappa} y_i\mathbb{Z}(p^n)$ for some n and some infinite cardinal κ, and let $H \leq_* G$ with $|H| = |G|$. The group H is again a direct sum of cyclics, $H = \bigoplus_I \langle x_i\rangle$, say, with $|I| = \kappa$ since $|H| = \kappa$. Suppose $o(x_i) = p^m$ for some $m < n$ and some $i \in I$. We have $x_i = \sum_{j<\kappa} k_j y_j$ where $k_j = 0$ for almost all j. Since $p^m x_i = 0$ it follows that $0 = \sum_{j<\kappa} p^m k_j y_j$ and so $p^m k_j y_j = 0$ for all j. Therefore p^n divides $p^m k_j$ for all j, so p divides k_j for all j and hence $x_i = p\sum_{j<\kappa} r_j y_j$ where $k_j = pr_j$ for all j. We get that p divides x_i in G and so p divides x_i in H since H is pure in G, a contradiction, x_i being a generator of $\langle x_i\rangle$ in H. We conclude that $o(x_i) = p^n$ for all $i \in I$ and so $H \cong G$.

Now let G be a reduced purely quasi-minimal group. If $tG = 0$, then G is torsion-free, so suppose $tG \neq 0$. Choose p such that $G_p \neq 0$ and G_p is not p-divisible; such exists since G is reduced and $tG \neq 0$. Let B be a p-basic subgroup of G; clearly $B \neq 0$. We claim that B cannot be torsion-free. Since G_p is not divisible there exists some $x \in G_p[p]$ with finite p-height (see [4, 20(c)]), and as G/B is p-divisible we have $x = p^n g_n + b_n$, for each n, where $g_n \in G$ and $b_n \in B$. Now, $px = 0$ implies $p^{n+1}g_n + pb_n = 0$, so $p^{n+1}g_n = -pb_n \in p^{n+1}G \cap B = p^{n+1}B$ and hence $pb_n = p^{n+1}b_n'$ where $b_n' \in B$. If B is torsion-free, then we get $b_n = p^n b_n'$ and so $x = p^n(g_n + b_n')$, i.e., p^n divides x for all n, a contradiction.

Therefore $B = B_0 \oplus \bigoplus_{n=1}^\infty B_n$ where B_0 is free or trivial and not all $B_n = 0$. Let $k > 0$ be the smallest integer with $B_k \neq 0$. Then $G = B_k \oplus (B^* + p^k G)$ where $B^* = B_0 \oplus \bigoplus_{nk} B_n$. If $|G| = |B_k|$, then $G \cong B_k$ and so $G \cong \bigoplus_\kappa \mathbb{Z}(p^k)$ and hence G is homocyclic. Otherwise $|G| = |B^* + p^k G|$ and hence $G \cong B^* + p^k G = H$, say. We claim that B^* is a p-basic subgroup of H.

By definition, B^* is a direct sum of cyclics. Also B^* is pure in B which is pure in G and so B^* is pure in G and hence in H. Finally $B^* = B \cap H$ since if $b \in B \cap H$, then $b = b^* + p^k g$ where $b^* \in B^*$ and $g \in G$, and so $b - b^* \in p^k G \cap B = p^k B \leq B^*$, which means that $B \cap H \leq B^*$. The converse inclusion is obvious. Therefore $H/B^* = H/(B \cap H) \cong (H+B)/B = G/B$ is p-divisible.

Since $H \cong G$ we get $B^* \cong B$, a contradiction, since $B_k \neq 0$. Hence $|G| = |B_k|$ and G is a homocyclic group. □

It remains to consider the reduced torsion-free case. Before the characterisation can be established we need some general results on reduced torsion-free groups. First recall the definition of high subgroups.

Definition 3.3. Let A be a subgroup of a torsion-free group G. A subgroup K of G is an A-*high subgroup* of G if $A \cap K = 0$, and if $K' \supseteq K$ such that $A \cap K' = 0$, then $K' = K$, i.e. K is maximal with respect to the property $A \cap K = 0$.

The following result is well known but we include the simple proof for completeness.

Lemma 3.4. *Let A be a subgroup of a torsion-free group G and let K be an A-high subgroup of G. Then $K \leq_* G$ and $G/(A \oplus K)$ is torsion.*

Proof. Suppose that $mg \in K$ for some $g \in G \backslash K$ and some $m \in \mathbb{Z}$. Then $\langle K, g\rangle \cap A \neq 0$, so there exists some non-zero $c \in A$ with $c = k + ng$ for some $k \in K$ and $n \in \mathbb{Z}$. Therefore $mc = mk + mng \in A \cap K = 0$ and so $c = 0$, since A is torsion-free, a contradiction. Hence $mg \notin K$ if $g \notin K$ and so $K \leq_* G$.

For the second part consider $g \in G \setminus A \oplus K$. Then $g \in G \setminus K$ and hence $\langle K, g\rangle \cap A \neq 0$. Therefore $k + ng = c$ for some $k \in K, c \in A$ and $n \in \mathbb{Z}$. We have $ng = c - k$ and so $n(g + A \oplus K) = 0$ and hence $G/(A \oplus K)$ is torsion. □

Lemma 3.5. *If G is a reduced torsion-free purely quasi-minimal group, then G is t-homogeneous for some type t.*

Proof. Let $t \in T(G)$, the typeset of G. Then there exists some $g' \neq 0$ in G such that $t(g') = t$. Consider $G(t) = \{g \in G : t(g) \geq t\} \neq 0$. It is well known that $G(t) \leq_* G$ and we claim that $|G(t)| = |G|$.

Let K be a $G(t)$-high subgroup of G. Lemma 3.4 now tells us that $K \leq_* G$ and $G/(K \oplus G(t))$ is torsion. Therefore $G = (K \oplus G(t))_*$ and hence $|G| = |K \oplus G(t)| \cdot \aleph_0 = |K| \cdot |G(t)| \cdot \aleph_0 = |K|$ or $|G(t)|$. If $|G| = |K|$, then $G \cong K$, since $K \leq_* G$, and we get $G(t) \cong K(t) \leq G(t) \cap K = 0$, a contradiction, since $G(t) \neq 0$. Therefore $|G| = |G(t)|$ and hence $G \cong G(t)$ whenever $G(t) \neq 0$. Now, if $t, s \in T(G)$, then there exist $a, b \in G$ such that $t(a) = t$ and $t(b) = s$ where the types are with respect to G. Since $G \cong G(t)$ and $G \cong G(s)$ we get $G(t) \overset{\phi}{\cong} G(s)$. Now, $t(a) = t(a\phi)$ and $t(b) = t(b\phi^{-1})$ imply that $t = s$ and therefore G is t-homogeneous. □

Following Griffith [7] we make the following definition:

Definition 3.6. A linearly independent subset S of a torsion-free group G is *quasi-pure independent* if $\bigoplus_{x \in S} \langle x\rangle_*$ is a pure subgroup of G where $\langle x\rangle_* = \langle x\rangle$ whenever $\langle x\rangle_*$ is cyclic.

Note that every torsion-free group has quasi-pure independent subsets and Zorn's Lemma implies that any quasi-pure independent set is contained in a maximal one.

Next we state some results concerning quasi-pure independent subsets of a torsion-free group G.

Lemma 3.7. *Let G be any torsion-free group. Then*

(i) *If T, S are two infinite maximal quasi-pure independent sets of G, then $|T| = |S|$.*
(ii) *If S a maximal quasi-pure independent subset of G, then $|G| \leq (|S|+1)^{\aleph_0}$.*

Proof. See [7, Corollary 125 and Theorem 126]. □

Definition 3.8. A subgroup H of a torsion-free group G is *pure essential* in G if $H \leq_* G$ and if $A \leq G$ with $A \cap H = 0$ and $A \oplus H \leq_* G$, then $A = 0$, in other words, $G/(A \oplus H)$ is not torsion-free for any such non-zero $A \leq G$.

The following result, which we shall exploit to obtain our characterisation of purely quasi-minimal groups, is due to Griffith.

Theorem 3.9. *Every torsion-free group G has a completely decomposable pure essential subgroup C such that $|G| \leq |C|^{\aleph_0}$.*

Proof. See [7, Theorem 129]. □

We are now ready to establish the characterisation of reduced torsion-free purely quasi-minimal groups under the assumption of the Generalised Continuum Hypothesis (GCH). Recall that GCH states that the successor of any infinite cardinal κ is 2^κ. Note that we do not need GCH in the countable case.

Theorem 3.10 (GCH)**.** *A torsion-free reduced group G of cardinality κ is purely quasi-minimal if and only if either $G \cong R$ ($\kappa = \aleph_0$) or $G \cong \bigoplus_\kappa R$ ($\kappa > \aleph_0$), for some rank 1 group R.*

Proof. Firstly, we show that R and $\bigoplus_\kappa R$ are purely quasi-minimal. If $0 \neq H \leq_* R$, then R/H is torsion-free, which is impossible since R has rank 1. Hence the only non-zero pure subgroup of R is R itself and so R is purely quasi-minimal. If $H \leq_* \bigoplus_\kappa R$, then H is also homogeneous completely decomposable of the same type as R. Therefore $H = \bigoplus_I R$ and if $|H| = \kappa > \aleph_0$, then $|I| = \kappa$ and so $H \cong \bigoplus_\kappa R$.

Now let G be a torsion-free reduced purely quasi-minimal group. Lemma 3.5 tells us that G is t-homogeneous for some type t. By Theorem 3.9 there exists a pure essential completely decomposable subgroup C of G such that $|G| \leq |C|^{\aleph_0}$. Let $C = \bigoplus_I R$ where R is a rank 1 group whose type must be t since $R \leq_* C \leq_* G$.

If $|G| = \aleph_0$, then $|R| = |C| = |G| = \aleph_0$ and hence $G \cong R$. So consider $|G| = \kappa > \aleph_0$. If $|C| = |G| = \kappa$, then $G \cong C$ and we are finished. We wish to prove that $|C| < |G|$ is impossible. Let us assume that $|C| < |G|$ to obtain a contradiction. First note that $|C| < |G|$ implies $2^{|C|} \leq |G|$, assuming GCH, and $|G| \leq |C|^{\aleph_0} \leq (2^{|C|})^{\aleph_0} = 2^{|C|}$, so $|G| = 2^{|C|}$. Now consider the short exact sequence $0 \longrightarrow C \stackrel{i}{\longrightarrow} G \stackrel{\pi}{\longrightarrow} G/C \longrightarrow 0$ where i is inclusion and π is canonical projection. The induced sequence $0 \longrightarrow \mathrm{Hom}(G/C, G) \longrightarrow \mathrm{Hom}(G, G) \longrightarrow \mathrm{Hom}(C, G)$ is exact. We claim that $\mathrm{Hom}(G/C, G) = 0$.

Let $g + C \in G/C$. Then $t_{G/C}(g + C) \geq t_G(g) = t$ since homomorphisms do not decrease types. If $t_{G/C}(g + C) = t$, then $R \cong \langle g + C\rangle_*$. Denote $\langle g + C\rangle_*$ by B/C, a pure subgroup of G/C. Then $C \leq_* B$ and B/C is homogeneous completely decomposable of type t and every element of $B \setminus C$ is of type t since $B \leq_* G$, so $C \sqsubset B$. Therefore $B = C \oplus R_1$ where $R_1 \cong R$, and $B \leq_* G$, but this contradicts the fact that C is pure essential in G. We conclude that $t_{G/C}(g + C) > t$ for all $g \in G$. Hence $\mathrm{Hom}(G/C, G) = 0$, again since homomorphisms do not decrease types. Therefore $0 \longrightarrow \mathrm{Hom}(G, G) \longrightarrow \mathrm{Hom}(C, G)$ is exact, so $\mathrm{Hom}(G, G)$ is isomorphic to a subgroup of $\mathrm{Hom}(C, G)$ and hence $|\mathrm{Hom}(G, G)| \leq |\mathrm{Hom}(C, G)| \leq |G|^{|C|} = (2^{|C|})^{|C|} = 2^{|C|} = |G|$. But $|G| > \aleph_0$ means that $r(G) = |G|$ and so there exists a maximal linearly independent set X in G of cardinality κ. Then this set X contains 2^κ different linearly independent subsets $\{S\}$ of G of cardinality κ (see [9, p. 43]). Each of these subsets S generates a pure subgroup $\langle S \rangle_*$ of G. Furthermore, if $S_1 \neq S_2$, then $\langle S_1 \rangle_* \neq \langle S_2 \rangle_*$ since otherwise, for any $s \in S_1 \setminus S_2$, we have $s \in \langle S_2 \rangle_*$ and so there exist non-zero integers $n, n_1, \ldots, n_k$, for some k, such that $ns = n_1 x_1 + \cdots + n_k x_k$ with $x_1, \ldots, x_k \in S_2$; but this contradicts the fact that $S_1 \cup S_2$ is contained in the linearly independent subset X of G.

Now, if K_1 and K_2 are two such pure subgroups of G, then $G \cong K_1$ and $G \cong K_2$, since G is purely quasi-minimal. If $\phi_1 : G \longrightarrow K_1$ and $\phi_2 : G \longrightarrow K_2$ are isomorphisms, then $K_1 \neq K_2$ implies $\phi_1 \neq \phi_2$ and thus there exist at least 2^κ different endomorphisms of G. Therefore $2^{|G|} \leq |\mathrm{End}(G)| \leq |G|$ which is obviously a contradiction. Hence we can deduce that $|C| < |G|$ is impossible and so $|C| = |G|$ and G is homogeneous completely decomposable. □

Note that in Theorem 3.10, if $|G| \leq \aleph_\omega$, then it is enough to assume the continuum hypothesis (CH), i.e. $2^{\aleph_0} = \aleph_1$, as the following argument shows. The Hausdorff Formula (see [8, Theorem 1.6.12]) tells us that $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}$ for all $\alpha, \beta \in Ord$ with $\beta < \alpha + 1$; so $\aleph_1^{\aleph_0} = \aleph_1 \cdot \aleph_0^{\aleph_0} = \aleph_1 \cdot 2^{\aleph_0} = \aleph_1$, using CH, and a simple induction argument now gives $\aleph_n^{\aleph_0} = \aleph_n$ for all $n > 1$. Therefore, in Theorem 3.10, if:

(i) $|G| = \aleph_1$ and $|C| = \aleph_0$, then CH gives us that $|G| = 2^{|C|}$ and we get a contradiction as in the proof of the theorem;

(ii) $|G| = \aleph_\alpha$ where $1 < \alpha \leq \omega$ and $|C| = \aleph_n, n < \alpha$, then $|C|^{\aleph_0} = \aleph_n^{\aleph_0} = \aleph_n$ (CH) $< \aleph_\alpha = |G|$, a contradiction to $|G| \leq |C|^{\aleph_0}$.

As in the quasi-minimal case, we summarise what we have established concerning the purely quasi-minimal groups in a theorem:

Theorem 3.11. *A group $G \in Ab_\kappa$ is purely quasi-minimal if and only if*

(i) $\kappa < \aleph_0$*: or*
(ii) $(\kappa = \aleph_0)$ $\quad G = R, \quad \mathbb{Z}(p^\infty)$ *or* $\bigoplus_{\aleph_0} \mathbb{Z}(p^k)$*; or*
(iii) $(\kappa > \aleph_0)$ $\quad G = \bigoplus_\kappa R$ *(GCH),* $\bigoplus_\kappa \mathbb{Z}(p^\infty)$ *or* $\bigoplus_\kappa \mathbb{Z}(p^k)$*, where R is a rank 1 group, p is any prime and k is any positive integer.*

Note. The authors are not convinced that an additional set-theoretic hypothesis is necessary in Theorem 3.10 but see no obvious way of avoiding the use of GCH. Even the assumption that the type R involved is of idempotent type does not give any significant simplification since, as observed by Lutz Strüngman, a homogeneous torsion-free group G of type R is purely quasi-minimal if and only if its subgroup $G_R = \{g \in G : \chi_G(g) \geq \chi_R(1)\}$ is purely quasi-minimal. This latter group is, of course, homogeneous of idempotent type.

§4. Directly quasi-minimal groups

The final type of quasi-minimal group G we consider is where we require that G be isomorphic only to all its direct summands of the same cardinality as itself. In this case G is called directly quasi-minimal. Again the usual reduction to either divisible or reduced groups is true. We can characterise the divisible directly quasi-minimal groups as in Lemma 3.1.

Lemma 4.1. *For a divisible directly quasi-minimal group G one of the following is true:*

(i) $G \cong \mathbb{Q} \quad (\kappa = \aleph_0)$,
(ii) $G \cong \mathbb{Z}(p^\infty)$, *for some p* $\quad (\kappa = \aleph_0)$,
(iii) $G \cong \bigoplus_\kappa \mathbb{Q} \quad (\kappa > \aleph_0)$,
(iv) $G \cong \bigoplus_\kappa \mathbb{Z}(p^\infty)$, *for some p* $\quad (\kappa > \aleph_0)$.

Proof. The arguments are similar to those in the purely quasi-minimal case. □

Theorem 3.2 is also true in the directly quasi-minimal case.

Theorem 4.2. *If a reduced group G is directly quasi-minimal, then G is either a homocyclic p-group or G is torsion-free.*

Proof. Apply the same arguments as before. □

It remains to consider the torsion-free reduced case. Every indecomposable torsion-free reduced group is trivially directly quasi-minimal. Such groups exist in abundance: Shelah [13] has shown that, for each infinite cardinal κ, there exist 2^κ non-isomorphic indecomposable groups of cardinality κ. Recall that a decomposable group G is superdecomposable if G has no indecomposable direct summands. The existence of countable superdecomposable groups was first established by Corner [1]. For a countable decomposable directly quasi-minimal group we have the following lemma.

Lemma 4.3. *If G is a countable torsion-free directly quasi-minimal group, then either G is indecomposable or*

(i) $G \cong \bigoplus_n G$, *for all $n \in \mathbb{N}$ and so G must have infinite rank;*
(ii) $G^* = Hom(G, \mathbb{Z}) = 0$;
(iii) *G is superdecomposable.*

Proof. (i) Assuming G is decomposable we have $G = A \oplus B$ where $|A| = |B| = |G| = \aleph_0$. Therefore $G \cong A$ and $G \cong B$ and so $G \cong G \oplus G$. A straightforward induction now completes the proof.

(ii) Again we assume G is decomposable and note that Stein's Theorem (see [4, Corollary 19.3]) tells us that $G = N \oplus F$ where F is free and N has no free quotient groups (or equivalently, $N^* = 0$). If $F \neq 0$, then $G \cong F$ (and $N = 0$), so $G = \mathbb{Z}$ since G is directly quasi-minimal, a contradiction to the decomposability of G. Therefore $F = 0$ and $G \cong N$ and hence $G^* = 0$.

(iii) If A is a direct summand of G, then, as in (i), $G \cong A$ and so A is decomposable. □

Note that (iii) implies (ii) in Lemma 4.3, since if ϕ is a non-zero homomorphism from G to $\mathbb{Z}$, then $G/\text{Ker}\phi \cong \text{Im}\phi \cong \mathbb{Z}$ and so $G \cong \text{Ker}\phi \oplus \mathbb{Z}$.

Properties (i) and (ii) of Lemma 4.3 are not sufficient to characterise the countable decomposable directly quasi-minimal torsion-free groups as the following example shows:

Corner [2] (see also [5, Theorem 91.6]) has given an example of a countable group G with countable endomorphism ring where $G \cong \bigoplus_n G$ for all n but $G \ncong \bigoplus_{\aleph_0} G$, and it is a standard exercise to extend this to show the existence of a family of $2^{\aleph_0}$ groups G_j with $G_j \cong \bigoplus_n G_j$ for all j but $\mathrm{Hom}(G_j, G_i) = 0$ if $j \neq i$. Stein's Theorem tells us that for each j, $G_j = N_j \oplus F_j$ where N_j and F_j are as in Lemma 4.3. Therefore $G_j^* \cong N_j^* \oplus F_j^* = F_j^*$. Since $G_j^* \cong G_j^* \oplus G_j^*$ we get that $F_j^* \cong F_j^* \oplus F_j^*$ and so either $F_j^* = 0$ or F_j^* has infinite rank. Hence the same must be true for F_j, since F_j is free. Now, if F_j has infinite rank, then $|F_j^*| = 2^{\aleph_0}$ and so $|G_j^*| = 2^{\aleph_0}$. But $G_j^* \leq \mathrm{End}(G_j)$, since G_j is torsion-free, so G_j^* is countable. We conclude that $F_j^* = G_j^* = 0$. Now, if we take two such groups $G_1 \ncong G_2$, then $G_1 \oplus G_2$ clearly satisfies properties (i) and (ii) but is obviously not directly quasi-minimal.

In a similar way we can show that superdecomposability is not sufficient for direct quasi-minimality:

Corner [1] (see also [5, Theorem 91.5]) has given another example of a countable superdecomposable group with the property that every non-zero idempotent ε of the group has a corresponding non-zero idempotent ζ such that $\zeta = \zeta\varepsilon = \varepsilon\zeta \neq \varepsilon$. Now consider two such groups A and B with $\mathrm{Hom}(A, B) = \mathrm{Hom}(B, A) = 0$ and set $G = A \oplus B$. We show that G is superdecomposable. First of all $E(G) \cong \begin{pmatrix} E(A) & 0 \\ 0 & E(B) \end{pmatrix}$ where $E(G), E(A), E(B)$ denote the endomorphism rings of G, A and B, respectively. So, if $X \neq 0$ is some summand of G and ε is projection onto X, along some complementary summand, then ε is an idempotent in $E(G)$ and so $\varepsilon = \begin{pmatrix} \varepsilon_a & 0 \\ 0 & \varepsilon_b \end{pmatrix}$, for some idempotents $\varepsilon_a \in E(A)$ and $\varepsilon_b \in E(B)$. Now there exist idempotents $\zeta_a \in E(A)$ and $\zeta_b \in E(B)$ such that $\zeta_a = \zeta_a\varepsilon_a = \varepsilon_a\zeta_a \neq \varepsilon_a$ and $\zeta_b = \zeta_b\varepsilon_b = \varepsilon_b\zeta_b \neq \varepsilon_b$. Then, setting $\zeta = \begin{pmatrix} \zeta_a & 0 \\ 0 & \zeta_b \end{pmatrix}$, we get that $\zeta = \zeta\varepsilon = \varepsilon\zeta \neq \varepsilon$. We have $G\zeta = G\zeta\varepsilon \subseteq G\varepsilon = X$ and $G\zeta$ is a non-zero summand of G, so $G\zeta$ is a non-zero summand of X. In fact, it can easily be verified that $X = G\zeta \oplus G(\varepsilon - \zeta)$. Therefore G is a countable superdecomposable group but G is obviously not directly quasi-minimal.

It is an open question whether there exist groups satisfying (i) and (iii) which are not directly quasi-minimal.

Turning to the uncountable case, every purely quasi-minimal group is, of course, directly quasi-minimal. The following lemma gives an example of an uncountable decomposable torsion-free reduced directly quasi-minimal group which is not purely quasi-minimal.

Lemma 4.4. *The Baer-Specker group* $\prod_{\aleph_0} \mathbb{Z}$ *is directly quasi-minimal but not purely quasi-minimal.*

Proof. Let $P = \prod_{\aleph_0} \mathbb{Z}$. First note that P is not purely quasi-minimal since it contains a pure free subgroup of rank $2^{\aleph_0}$ (see [5, Corollary 97.4]). If $P = A \oplus B$, then both A and B are products of countably many copies of $\mathbb{Z}$ (see [3, IX, Theorem 1.4]) and either $|A| = |P| = 2^{\aleph_0}$ or $|B| = |P| = 2^{\aleph_0}$ or both. Suppose that $A = \prod_I \mathbb{Z}$ and

$|A| = |P|$. Since $|I|$ is countable and $|A| = |P|$ we must have $|I| = \aleph_0$ and hence $A \cong P$. □

However, if we consider $G = \prod_{\kappa} \mathbb{Z}$ where $\kappa > \aleph_0$, then the direct quasi-minimality of G may be undecidable in ZFC, as is shown by our final proposition.

Proposition 4.5. *Let $\kappa > \aleph_0$, κ not ω-measurable, and let $G = \prod_{\kappa} \mathbb{Z}$. Then:*

(i) *assuming* GCH*, G is directly quasi-minimal;*
(ii) *assuming* MA+¬ CH*, then, for all $\aleph_0 < \kappa < 2^{\aleph_0}$, G is not directly quasi-minimal.*

Proof. (i) Suppose $G = A \oplus B$ with $|A| = |G| = 2^{\kappa}$, say. Then $A = \prod_{I} \mathbb{Z}$ with $|I| = \lambda$, for some $\lambda \leq \kappa$, (see [3, IX, Theorem 1.4]) and so $2^{\lambda} = 2^{\kappa}$. Assuming GCH we get $\lambda = \kappa$ and so $A \cong G$ and hence G is directly quasi-minimal.

(ii) Now assume that MA+¬ CH holds. Let $G = \prod_{\kappa} \mathbb{Z}$ where $\aleph_0 < \kappa < 2^{\aleph_0}$, and we can assume that $2^{\aleph_0} = 2^{\kappa}$, since this is a consequence of MA+¬ CH (see [3, p. 177]). Then $G = \prod_{\aleph_0} \mathbb{Z} \oplus B$, say, with $|\prod_{\aleph_0} \mathbb{Z}| = |G|$ but $\prod_{\aleph_0} \mathbb{Z} \ncong G$, since $\bigoplus_{\aleph_0} \mathbb{Z} = (\prod_{\aleph_0} \mathbb{Z})^* \ncong G^* \cong \bigoplus_{\kappa} \mathbb{Z}$ (see [3, III, Corollary 3.7]) and so G is not directly quasi-minimal. □

Since both GCH and MA+¬ CH can be shown to be consistent with ZFC (see [14]) we can deduce that the direct quasi-minimality of e.g. $G = \prod_{\aleph_1} \mathbb{Z}$ is not decidable in ZFC. We note that the full strength of MA+¬ CH is not required: we simply need any model consistent with ZFC in which $2^{\kappa} = 2^{\aleph_0}$ holds for cardinals $\aleph_0 < \kappa < 2^{\aleph_0}$. Such a model may be obtained from any model of ZFC in which $2^{\aleph_0}$ is regular, by using Easton forcing with a constant index function equal to $2^{\aleph_0}$. Details of Easton forcing may be found in Kunen's book [10].

References

1. A.L.S. Corner, *Every countable reduced torsion-free ring is an endomorphism ring*, Proc. London Math. Soc. **13** (1963) 687–710. MR **27:**3704
2. A.L.S. Corner, *On a conjecture of Pierce concerning direct decompositions of abelian groups*, Proc. Colloq. Abelian Groups, Budapest (1964), 43–48. MR **30:**148
3. P. C. Eklof and A. H. Mekler, *Almost Free Modules*, North-Holland Mathematical Library **46** (1990). MR **92e:**20001
4. L. Fuchs, *Infinite Abelian groups I*, Academic Press, New York (1970). MR **41:**333
5. L. Fuchs, *Infinite Abelian groups II*, Academic Press, New York (1973). MR **50:**2362
6. J. Ginsburg and B. Sands, *Minimal infinite topological spaces*, Amer. Math. Monthly **86** (1979), 574–576. MR **80e:**54007
7. P. Griffith, *Infinite Abelian Groups*, Chicago Lectures in Mathematics, Chicago and London (1970).
8. M. Holz, K. Steffens, and E. Weitz, *Introduction to Cardinal Arithmetic*, Birkhäuser Advanced Texts, Birkhäuser-Verlag, Basel-Boston-Berlin (1999). MR **2000h:**03001
9. T. Jech, *Set Theory*, Academic Press (1978). MR **80a:**03062
10. K. Kunen, *Set Theory*, North-Holland (1980). MR **82f:**03001
11. J. Matier and T.B.M. McMaster, *Iteration of the "anti" operation in ordered topological spaces and in other categorical contexts*, Boll. Un. Mat. Ital. **7** (1993), 369–378. MR **94g:**18002
12. P. T. Matthews and T.B.M. McMaster, *A viewpoint on minimality in topology*, Irish Math. Soc. Bull. **32** (1994), 7–16.

13. S. Shelah, *Infinite abelian groups, Whitehead problem and some constructions*, Israel J. Math. **18** (1974), 243–256. MR **50:**9582
14. R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. **94** (1971), 201–245. MR **45:**3212

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