

CONVERGENCE OF HARMONIC MAPS ON THE POINCARÉ DISK

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ABSTRACT. Let $\{f_n : \mathbb{D} \rightarrow \mathbb{D}\}$ be a sequence of locally quasiconformal harmonic maps on the unit disk \mathbb{D} with respect to the Poincaré metric. Suppose that the energy densities of f_n are uniformly bounded from below by a positive constant and locally uniformly bounded from above. Then there is a subsequence of $\{f_n\}$ that locally uniformly converges on \mathbb{D} , and the limit function is either a locally quasiconformal harmonic map of the Poincaré disk or a constant. Especially, if the limit function is not a constant, the subsequence can be chosen to satisfy some stronger conditions. As an application, it is proved that every point of the space $T_0(\mathbb{D})$, a subspace of the universal Teichmüller space, can be represented by a quasiconformal harmonic map that is an asymptotic hyperbolic isometry.

1. INTRODUCTION

Harmonic maps and their Hopf differentials on the Poincaré disk have been extensively studied (for example, see [1], [16]). Some relations between quasiconformal harmonic diffeomorphism and the universal Teichmüller space have also been found (cf. [15], [17]). Usually, we make the restriction that harmonic maps are quasiconformal or have finite total energy [2]. Our purpose in this paper is to study the normality of a sequence of harmonic maps on the Poincaré disk under weaker assumptions on the energy densities.

Let \mathbb{D} be the unit disk $\{|z| < 1\}$ with the Poincaré metric $\rho(z)|dz|^2$, where

$$\rho(z)|dz|^2 = \frac{4}{(1 - |z|^2)^2}|dz|^2, \quad \forall z \in \mathbb{D}.$$

In this paper, we only treat the (locally) quasiconformal harmonic maps of \mathbb{D} onto itself with respect to the Poincaré metric. We shall restrict ourselves to the case where a map $f : \mathbb{D} \rightarrow \mathbb{D}$ is called a harmonic map if it is a C^2 -homeomorphism of \mathbb{D} onto itself and its Hopf differential $\Phi = \phi_f(z)dz^2 = \rho(f(z))f_z\bar{f}_z dz^2$ is holomorphic in \mathbb{D} .

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Let $w = f(z)$ be a harmonic homeomorphism of \mathbb{D} onto itself. Then it satisfies the Euler-Lagrange equation

$$(1.1) \quad f_{z\bar{z}} + \partial_w(\log \rho) \circ f \cdot f_z f_{\bar{z}} = 0.$$

The ∂ and $\bar{\partial}$ energy densities are given by

$$|\partial w|^2 = \frac{\rho(w)}{\rho(z)} |w_z|^2, \quad |\bar{\partial} w|^2 = \frac{\rho(w)}{\rho(z)} |w_{\bar{z}}|^2.$$

Since the curvature of the Poincaré metric is -1 , the Euler-Lagrange equation gives the following equations [12]:

$$(1.2) \quad \Delta_\rho \log |\partial w|^2 = 2|\partial w|^2 - 2|\bar{\partial} w|^2 - 2,$$

$$(1.3) \quad \Delta_\rho \log |\bar{\partial} w|^2 = -2|\partial w|^2 + 2|\bar{\partial} w|^2 - 2,$$

where $\Delta_\rho = \frac{\Delta_0}{\rho}$ and Δ_0 is the Euclidean Laplacian.

Definition 1. A map $f : \mathbb{D} \rightarrow \mathbb{D}$ is called a *locally quasiconformal map* if for any subdomain E with $\bar{E} \subset \mathbb{D}$, f is quasiconformal in E .

A locally quasiconformal map f has locally L^2 -integrable formal generalized derivatives

$$(1.4) \quad \partial_z f = \frac{1}{2}(\partial_x f - i\partial_y f), \quad \partial_{\bar{z}} f = \frac{1}{2}(\partial_x f + i\partial_y f) \quad (z = x + iy)$$

and satisfies a Beltrami equation:

$$\partial_{\bar{z}} f = \mu(z)\partial_z f, \quad \|\mu\|_\infty \leq 1.$$

Here $\mu_f(z) = \mu(z) = \frac{\partial_{\bar{z}} f}{\partial_z f}$ is called the Beltrami coefficient or the complex dilatation of f . For any measurable subset S of \mathbb{D} we denote

$$\|\mu\|_S = \text{ess sup}_{z \in S} |\mu(z)|, \quad K_S[f] = \frac{1 + \|\mu\|_S}{1 - \|\mu\|_S}.$$

If $K[f] = K_{\mathbb{D}}[f] < \infty$, as usual, we say that f is a K -quasiconformal map of \mathbb{D} for any given $K \geq K[f]$.

Let $\{\gamma_n : \mathbb{D} \rightarrow \mathbb{C}\}$ be a sequence of maps on \mathbb{D} . If γ_n converge to some map $\gamma : \mathbb{D} \rightarrow \mathbb{C}$ uniformly on any compact subset of \mathbb{D} as $n \rightarrow \infty$, then we denote this by “ $\gamma_n \rightrightarrows \gamma$ (as $n \rightarrow \infty$)”.

The main results of this paper are the following two theorems:

Theorem 1. *Let $w_n = f_n : \mathbb{D} \rightarrow \mathbb{D}$ be a sequence of locally quasiconformal harmonic maps on the unit disk with respect to the Poincaré metric. Suppose the ∂ -energy densities $|\partial w_n|^2$ of f_n are uniformly bounded from below by a positive constant σ and locally uniformly bounded from above. Then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that locally uniformly converges on \mathbb{D} , and the limit function $w = f(z)$ of $\{f_{n_k}\}$ is either a locally quasiconformal harmonic map of the Poincaré disk or a constant. Especially, if $f(z)$ is not a constant, we can choose the subsequence $\{f_{n_k}\}$*

such that the following conditions also hold: as $n_k \rightarrow \infty$,

$$\begin{aligned}
 (1.5) \quad & |\partial w_{n_k}|^2 \rightrightarrows |\partial w|^2, \\
 (1.6) \quad & |\bar{\partial} w_{n_k}|^2 \rightrightarrows |\bar{\partial} w|^2, \\
 (1.7) \quad & \mu_{f_{n_k}}(z) \rightrightarrows \mu(z) = \mu_f(z), \\
 (1.8) \quad & \partial_z f_{n_k} \rightrightarrows \partial_z f, \\
 (1.9) \quad & \partial_{\bar{z}} f_{n_k} \rightrightarrows \partial_{\bar{z}} f, \\
 (1.10) \quad & \phi_{n_k} \rightrightarrows \phi = \phi_f.
 \end{aligned}$$

Theorem 1 tells us that certain boundedness of energy densities implies the normality of a sequence of harmonic maps as well as the normality of the corresponding sequence of Beltrami coefficients, and so on. Similar results have been obtained before under somewhat stronger assumptions. For example, Proposition 1.8 in [14] is achieved provided that the metrics $\rho(f_n)|\partial_z f_n|^2|dz|^2$ are complete on \mathbb{D} .

If the f_n are K -quasiconformal harmonic maps, we have the following corollary.

Corollary 1. *Let $w_n = f_n : \mathbb{D} \rightarrow \mathbb{D}$ be a sequence of K -quasiconformal harmonic maps on the Poincaré disk. Then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges locally uniformly on \mathbb{D} and the limit map $w = f(z)$ of the subsequence is either a K -quasiconformal harmonic map of the Poincaré disc or a constant. Moreover, if $f(z)$ is not a constant, then $\{f_{n_k}\}$ can be chosen such that (1.5)–(1.10) hold.*

In [9], the consequence “the limit map f is harmonic” of Corollary 1 plays a crucial role in the argument of the boundary value problem for harmonic maps related to the Schoen conjecture [11]. But there is a gap in his proof (see pp. 2041–2044 of [9]). Our Corollary 1 just helps bridge the gap.

Recall that the universal Teichmüller space $T(\mathbb{D})$ is defined as the collection of all normalized quasisymmetric homeomorphisms φ of $S^1 = \partial\mathbb{D}$ onto itself. A map $\varphi : S^1 \rightarrow S^1$ is said to be normalized quasisymmetric if it admits a quasiconformal extension to \mathbb{D} and fixes the points 1, -1 , and i .

Definition 2. A quasiconformal map $f : \mathbb{D} \rightarrow \mathbb{D}$ is called asymptotically conformal if for every $\varepsilon > 0$ there is a compact set E in \mathbb{D} such that f is $(1 + \varepsilon)$ -quasiconformal on $\mathbb{D} \setminus E$.

Let $T_0(\mathbb{D})$ be the set of all $\varphi \in T(\mathbb{D})$ that admit an asymptotically conformal extension to \mathbb{D} . $T_0(\mathbb{D})$ is a closed subspace of the universal Teichmüller space [6].

Let $q : S^1 \rightarrow S^1$ be a quasisymmetric map of the unit circle. The map q is said to be symmetric (see [6], [13]) if

$$(1.11) \quad \lim_{t \rightarrow 0} \frac{q(e^{i(t+t_0)}) - q(e^{it_0})}{q(e^{it_0}) - q(e^{i(t_0-t)})} = 1,$$

uniformly in t for all points $z_0 = e^{it_0} \in S^1$.

Strebel [13] and Fehlmann [5] showed respectively that a normalized quasisymmetric map $\varphi \in T_0(\mathbb{D})$ if and only if φ is a symmetric map of the unit circle.

Recently, Earle, Marković and Saric’s joint work [4] indicates that the barycentric extension (or Douady-Earle extension, see [3]) of every $\varphi \in T_0(\mathbb{D})$ is an asymptotic hyperbolic isometry. Following their idea, as an application of Corollary 1 we claim

that the quasiconformal harmonic extension of $\varphi \in T_0(\mathbb{D})$ is also an asymptotic hyperbolic isometry. To be precise, we have

Theorem 2. *Every $\varphi \in T_0(\mathbb{D})$ admits a unique quasiconformal harmonic extension to \mathbb{D} that is an asymptotic hyperbolic isometry.*

2. SOME LEMMAS

We have restricted ourselves to the case that a harmonic map should be a C^2 -diffeomorphism. So for a harmonic map w , $|\partial w|^2 > 0$ and $J(w) > 0$, and hence the function

$$u = \frac{1}{2} \log |\partial w|^2$$

is well defined in \mathbb{D} . The following Bochner equation can be deduced from equation (1.2):

$$(2.1) \quad \Delta_\rho u = e^{2u} - \|\Phi\|^2 e^{-2u} - 1,$$

where

$$(2.2) \quad \|\Phi\| = \rho^{-1}(z)|\phi(z)|, \quad \phi(z) = \rho(w)w_z\bar{w}_z.$$

We include two main results from [16] here as Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *Let $w = f(z)$ be a harmonic map of \mathbb{D} onto itself. Then $w = f(z)$ is a quasiconformal map if and only if the Bers norm of its Hopf differential $\phi(z) = \rho(w)w_z\bar{w}_z = \rho(w)w_z\bar{w}_z$ is bounded, i.e.,*

$$(2.3) \quad \|\phi\|_B = \sup_{z \in \mathbb{D}} |\rho^{-1}(z)\phi(z)| < \infty.$$

Lemma 2.2. *Let $w = f(z)$ be a quasiconformal harmonic map of \mathbb{D} onto itself. Then we have*

$$(2.4) \quad 1 \leq |\partial w|^2 = \frac{\rho(f(z))}{\rho(z)} |f_z(z)|^2 \leq \frac{1}{2} (1 + \sqrt{1 + 4\|\phi\|_B^2}), \quad \forall z \in \mathbb{D},$$

where ϕ is the Hopf differential of $w = f(z)$.

From Lemma 2.2, we can easily deduce the following corollary:

Corollary 2.1. *Let $w = f(z)$ be a quasiconformal harmonic map of \mathbb{D} onto itself. Then*

$$(2.5) \quad \|\phi\|_B \leq \frac{K[f] - 1}{2}$$

and

$$(2.6) \quad 1 \leq |\partial w|^2 \leq \frac{K[f] + 1}{2}.$$

Proof. Since $\mu_f(z) = \frac{\rho^{-1}(z)\overline{\phi(z)}}{|\partial w|^2}$, we have

$$(2.7) \quad \|\mu_f\|_\infty \geq \frac{2\|\phi\|_B}{1 + \sqrt{1 + 4\|\phi\|_B^2}} \geq \frac{2\|\phi\|_B}{2 + 2\|\phi\|_B} = \frac{\|\phi\|_B}{1 + \|\phi\|_B}.$$

Thus, (2.5) and (2.6) follow from (2.4) and (2.7). □

To prove our main results, we have to estimate the gradient of the energy density of a harmonic map. The estimate is very important in the proof of Theorem 1.

Lemma 2.3. *Suppose w is a harmonic diffeomorphism on \mathbb{D} with $|\partial w|^2 > \sigma$, where σ is a positive constant. Let $\Phi = \phi dz^2$ be the Hopf differential of w . Set*

$$\mathbb{D}_r = \{|z| < r < 1\}$$

and

$$m_r = \sup_{z \in \mathbb{D}_{\frac{1+r}{2}}} |\partial w|^2.$$

Then

$$(2.8) \quad \sup_{z \in \mathbb{D}_r} |\nabla(\log |\partial w|^2)| \leq C m_r,$$

for some constant C depending only on r , where ∇ is the gradient in the Poincaré disk \mathbb{D} .

Proof. Let $u = \frac{1}{2} \log |\partial w|^2$. Since w is a locally quasiconformal diffeomorphism, $|\partial w|^2 > \sigma$ everywhere and hence u is well-defined and smooth in \mathbb{D} . By (2.1) we have

$$(2.9) \quad \sup_{z \in \mathbb{D}_r} \rho^{-1}(z) |\Delta_0 u| \leq C_1 m_r,$$

where C_1 is a constant depending only on r . Let $z_0 = x_0 + iy_0 \in \mathbb{D}_r$. Set $\delta = \frac{1-|z_0|}{4}$ and $B_\delta(z_0) = \{z : |z - z_0| < \delta\}$. Then $B_{2\delta}(z_0) = \{z : |z - z_0| < 2\delta\}$ is contained in $\mathbb{D}_{\frac{1+r}{2}}$. Applying the estimate on p. 70 in [7] to $B_\delta(z_0)$ and $B_{2\delta}(z_0)$, we see that there is an absolute constant C_2 such that

$$(2.10) \quad (1 - |z_0|^2) |\nabla_0 u|(z_0) \leq C_2 (\sup_{B_{2\delta}} |u| + m_r),$$

where ∇_0 is the usual Euclidean gradient. Using the fact that $\log(1 + t) \leq t$ for $t \geq 0$, we have

$$\sup_{B_{2\delta}} |u| \leq \frac{1}{2} \left(\log \frac{m_r}{\sigma} + |\log \sigma| \right) \leq \frac{m_r}{\sigma} + \frac{1}{2} |\log \sigma| \leq C_3 m_r,$$

where C_3 depends only on r . Thus the lemma follows. □

We also need the following classical results on *good approximation* to a quasiconformal map.

Lemma 2.4 ([8]). *Let $w_n, n = 1, 2, \dots$, be a sequence of K -quasiconformal maps of G that converges to a quasiconformal map w with complex dilatation μ , uniformly on compact subsets of G . If the complex dilatations $\mu_n(z)$ of w_n tend to limit $\mu_\infty(z)$ almost everywhere in G , then w_n is a good approximation to w , i.e., $\mu_\infty(z) = \mu(z)$ almost everywhere in G . Moreover, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\partial_z w_{n_k} \rightarrow \partial_z w, \partial_{\bar{z}} w_{n_k} \rightarrow \partial_{\bar{z}} w$ (as $n_k \rightarrow \infty$) almost everywhere in G .*

Finally, we introduce

Definition 3. Let $\mu \in L^\infty(\mathbb{D})$ be a Beltrami coefficient of a quasiconformal map of \mathbb{D} . We say that μ vanishes at infinity on \mathbb{D} if and only if for every $\varepsilon > 0$ there is a compact set E in \mathbb{D} such that $\|\mu\|_{\mathbb{D} \setminus E}$ is less than ε .

Let $d_{\mathbb{D}}(z, w) = 2 \tanh^{-1} \left| \frac{z-w}{1-\bar{w}z} \right|$ denote the hyperbolic distance of $z, w \in \mathbb{D}$. For any $a \in \mathbb{D}$ and $r > 0$, let $\bar{B}(a, r)$ be the closed hyperbolic ball of radius r and center a .

Let $\{p_n\}$ be a sequence in \mathbb{D} . We say that p_n approaches infinity in \mathbb{D} and write $p_n \rightarrow \infty$ if and only if $\lim_{n \rightarrow \infty} d_{\mathbb{D}}(p, p_n) = \infty$ for some, hence all, $p \in \mathbb{D}$. With these definitions the following lemma was proved in [4].

Lemma 2.5. *Let μ be a Beltrami coefficient, and let r be a positive real number. Then μ vanishes at infinity on \mathbb{D} if and only if $\lim_{n \rightarrow \infty} \|\mu\|_{\bar{B}(z_n, r)} = 0$ for every sequence $\{z_n\}$ in \mathbb{D} such that $z_n \rightarrow \infty$.*

3. PROOF OF THEOREM 1

In the proof, we shall apply the diagonal trick to choose a convergent subsequence from some sequence many times. For simplicity and without loss of generality, unless otherwise stated, we still use the original sequence to represent the subsequence after passing to a subsequence if necessary.

Suppose $w_n = f_n : \mathbb{D} \rightarrow \mathbb{D}$ is a sequence of harmonic diffeomorphisms of \mathbb{D} onto itself satisfying the conditions in Theorem 1. Let ϕ_n be the Hopf differential of f_n , i.e., $\phi_n = \rho(f_n)\partial_z f_n \partial_{\bar{z}} \overline{f_n}$. Then ϕ_n is holomorphic in \mathbb{D} . Suppose $N(r) = \sup_{n \in \mathbb{N}} \sup_{z \in \mathbb{D}_r} |\partial w_n|^2$. Then, in \mathbb{D}_r ,

$$(3.1) \quad |\phi_n(z)| = \rho(f_n)|\partial_z f_n||\partial_{\bar{z}} f_n| \leq \rho(z)|\partial w_n|^2 \leq M(r)N(r),$$

where $M(r) = 4/(1 - r^2)^2$. This implies that $\{\phi_n\}$ is normal in \mathbb{D} .

Next we show that $\{|\partial w_n|^2\}$ is normal in \mathbb{D} . It is sufficient to prove that $\{|\partial w_n|^2\}$ is equicontinuous in \mathbb{D} by virtue of the classical theory of normal families. Since $|\nabla_0 |\partial w_n|^2| = |\partial_z |\partial w_n|^2| + |\partial_{\bar{z}} |\partial w_n|^2|$, we have

$$(3.2) \quad |\nabla(\log |\partial w_n|^2)| = \frac{|\nabla |\partial w_n|^2|}{|\partial w_n|^2} = \frac{(1 - |z|^2)(|\partial_z |\partial w_n|^2| + |\partial_{\bar{z}} |\partial w_n|^2|)}{2|\partial w_n|^2}.$$

Since $|\partial w_n|^2$ is bounded from below by a positive constant σ and is locally uniformly bounded from above, we see that $|\partial_z |\partial w_n|^2|$ and $|\partial_{\bar{z}} |\partial w_n|^2|$ are uniformly bounded in \mathbb{D}_r from Lemma 2.3. This indicates that $\{|\partial w_n|^2\}$ is equicontinuous.

Note that $\mu_{f_n} = \frac{\rho^{-1}(z)\phi_n(\bar{z})}{|\partial w_n|^2}$. We know that $\{\mu_{f_n}\}$ is normal from the normality of $\{|\partial w_n|^2\}$ and $\{\phi_n\}$. So

$$(3.3) \quad \|\mu_{f_n}\|_{\mathbb{D}_r} \leq s(r) < 1, \quad K_{\mathbb{D}_r}[f_n] = \frac{1 + \|\mu_{f_n}\|_{\mathbb{D}_r}}{1 - \|\mu_{f_n}\|_{\mathbb{D}_r}} \leq \frac{1 + s(r)}{1 - s(r)},$$

where $s(r)$ is a constant only depending on r . This implies that $\{f_n\}$ is normal in \mathbb{D} and so is $\{\rho(f_n) = \frac{4}{(1 - |f_n|^2)^2}\}$ as well. Suppose f is the limit map which is not a constant. Obviously, the map f is a locally quasiconformal map with $K_{\mathbb{D}_r}[f] \leq (1 + s(r))/(1 - s(r))$.

Furthermore, we can conclude from Lemma 2.4 that (1.7) holds.

Next, we prove that f is harmonic in \mathbb{D} . For this, we only need to show that (1.8) and (1.9) hold.

First, we show that

$$\partial_z f_n \rightrightarrows \partial_z f, \text{ as } n \rightarrow \infty.$$

By computation, we have

$$(3.4) \quad 0 = \partial_{\bar{z}} \phi_n = \partial_{\bar{z}} \rho(f_n) \partial_z f_n \partial_z \bar{f}_n + \rho(f_n) (\partial_z \partial_{\bar{z}} f_n \partial_z \bar{f}_n + \partial_z f_n \partial_z \partial_{\bar{z}} \bar{f}_n).$$

It follows from (3.4) that

$$\begin{aligned} |\partial_{\bar{z}} \rho(f_n) \partial_z f_n \partial_z \bar{f}_n| &\geq \rho(f_n) |\partial_z \partial_{\bar{z}} f_n| |\partial_z f_n| (1 - \|\mu_{f_n}\|_{\mathbb{D}_r}) \\ &\geq \rho(f_n) |\partial_z \partial_{\bar{z}} f_n| |\partial_z f_n| (1 - s(r)), \quad z \in \mathbb{D}_r. \end{aligned}$$

Rewrite the above inequality in the following form:

$$(3.5) \quad |\partial_z \partial_{\bar{z}} f_n| \leq \frac{|\partial_{\bar{z}} \rho(f_n)| |\partial_z f_n|}{(1 - s(r)) \rho(f_n)}.$$

A simple computation shows

$$(3.6) \quad |\partial_{\bar{z}} \rho(f_n) \partial_z f_n| = 8 \frac{|\bar{f}_n (\partial_{\bar{z}} f_n)^2 + f_n \overline{\partial_z f_n} \partial_z f_n|}{(1 - |f_n|^2)^3} \leq 8 \frac{|f_n| |\partial_{\bar{z}} f_n|^2}{(1 - |f_n|^2)^3} + 2 \frac{|f_n| |\phi_n|}{1 - |f_n|^2}.$$

On the other hand,

$$\begin{aligned} \partial_z |\partial w_n|^2 &= \partial_z \left(\frac{\rho(f_n)}{\rho(z)} \partial_z f_n \partial_z \bar{f}_n \right) \\ &= \partial_z \left(\frac{\rho(f_n)}{\rho(z)} \right) \partial_z f_n \partial_z \bar{f}_n + \frac{\rho(f_n)}{\rho(z)} (\partial_z \bar{f}_n \partial_z \partial_z f_n + \partial_z f_n \partial_z \partial_z \bar{f}_n). \end{aligned}$$

Then

$$(3.7) \quad |\partial_z \partial_z f_n| \leq \frac{\rho(z)}{\rho(f_n) |\partial_z f_n|} [|\partial_z |\partial w_n|^2| + |\partial_z \left(\frac{\rho(f_n)}{\rho(z)} \right)| |\partial_z f_n|^2] + |\partial_z \partial_z f_n|.$$

In addition,

$$\begin{aligned} (3.8) \quad \left| \partial_z \left(\frac{\rho(f_n)}{\rho(z)} \right) \right| &= \left| \partial_z \frac{(1 - |z|^2)^2}{(1 - |f_n|^2)^2} \right| \\ &= \frac{2(1 - |z|^2)}{(1 - |f_n|^2)^3} |(1 - |f_n|^2) \bar{z} - (1 - |z|^2) (\bar{f}_n \partial_z f_n + f_n \partial_z \bar{f}_n)| \\ &\leq \frac{2(1 - |z|^2)}{(1 - |f_n|^2)^3} [(1 - |f_n|^2) + |f_n| (1 - |z|^2) (|\partial_z f_n| + |\partial_z \bar{f}_n|)]. \end{aligned}$$

Since $\{f_n\}$ is normal and $|\partial w_n|^2, |\bar{\partial} w_n|^2$ are uniformly bounded in \mathbb{D}_r , combining (3.5) (3.6) (3.7) and (3.8), one can easily verify that $|\partial_z \partial_z f_n|$ and $|\partial_z \partial_{\bar{z}} f_n|$ are uniformly bounded in \mathbb{D}_r . Therefore, $\{\partial_z f_n\}$ is equicontinuous in \mathbb{D} . Consequently, $\{\partial_z f_n\}$ is normal.

Note that $\partial_{\bar{z}} f_n = \mu_{f_n} \partial_z f_n$. We can conclude that $\{\partial_{\bar{z}} f_n\}$ is also normal in \mathbb{D} . Further application of Lemma 2.4 offers (1.8) and (1.9). In the meantime, for $|\partial w_n|^2 = \frac{\rho(f_n)}{\rho(z)} |\partial_z f_n|^2$ and $|\bar{\partial} w_n|^2 = \frac{\rho(f_n)}{\rho(z)} |\partial_{\bar{z}} f_n|^2$, (1.5) and (1.6) hold.

Finally, it follows readily that $\phi_n \rightrightarrows \phi_f$ (as $n \rightarrow \infty$) from $\phi_n = \rho(f_n) \partial_z f_n \partial_z \bar{f}_n$. Therefore, ϕ_f is holomorphic in \mathbb{D} , and hence f is a harmonic homeomorphism of \mathbb{D} . This completes the proof of Theorem 1.

Now, Corollary 1 derives from Theorem 1 and Corollary 2.1 immediately.

4. PROOF OF THEOREM 2

The following lemma is implied in Theorem 3.1 of [10]:

Lemma 4.1. *Every $\varphi \in T_0(\mathbb{D})$ has a unique quasiconformal harmonic extension H_φ to \mathbb{D} , and the Hopf differential ϕdz^2 of H_φ satisfies*

$$(4.1) \quad \begin{aligned} \phi dz^2 &\in BQD_0(\mathbb{D}) \\ &= \{ \psi(z)dz^2 \text{ is holomorphic in } \mathbb{D} : \lim_{r \rightarrow 0} \sup_{1-r < |z| < 1} \rho^{-1}(z)|\psi(z)| = 0 \}. \end{aligned}$$

Actually, to prove Theorem 2, we only need to show the following precise form.

Lemma 4.2. *For any given $\varphi \in T_0(\mathbb{D})$, let $w = H_\varphi$ denote the quasiconformal harmonic extension of φ to \mathbb{D} with respect to the Poincaré metric. Then for any $\varepsilon > 0$ there is a compact set $E \subset \mathbb{D}$ such that*

$$(4.2) \quad (1 + \varepsilon)^{-1}d_{\mathbb{D}}(x, y) \leq d_{\mathbb{D}}(H_\varphi(x), H_\varphi(y)) \leq (1 + \varepsilon)d_{\mathbb{D}}(x, y)$$

for all $(x, y) \in (\mathbb{D} \times \mathbb{D}) \setminus (E \times E)$.

Before the proof of Lemma 4.2, We apply Corollary 1 to prove an infinitesimal version of Lemma 4.2:

Lemma 4.3. *Let $\varphi \in T_0(\mathbb{D})$, and let $(H_\varphi)_*$ be the map of tangent bundles induced by H_φ . Let $\{z_n\}$ be a sequence of points in \mathbb{D} such that $z_n \rightarrow \infty$, and let v_n be a tangent vector of hyperbolic length one at z_n for each n . Then*

$$(4.3) \quad \lim_{n \rightarrow \infty} \|(H_\varphi)_*v_n\|_{\mathbb{D}} = 1,$$

where $\|\cdot\|_{\mathbb{D}}$ denotes the hyperbolic length of some tangent vector to \mathbb{D} .

Proof. Let μ be the complex dilatation of H_φ . For each n choose an automorphism g_n of \mathbb{D} such that g_n maps 0 to z_n , and set $\mu_n = g_n^*(\mu) = (\mu \circ g_n)\overline{g_n'}/g_n'$. Since $\mu(z) = \frac{\rho^{-1}(z)\overline{\phi(z)}}{|\partial w|^2}$, Lemma 4.1 implies that μ vanishes at infinity on \mathbb{D} . We conclude from Lemma 2.5 and the definition of μ_n that

$$(4.4) \quad \lim_{n \rightarrow \infty} \|\mu_n\|_{\overline{B}(0,r)} = \lim_{n \rightarrow \infty} \|\mu\|_{\overline{B}(z_n,r)} = 0, \quad \forall r > 0.$$

For each n there is an automorphism γ_n of \mathbb{D} such that

$$(4.5) \quad \varphi_n = \gamma_n \circ H_\varphi \circ g_n|_{\partial\mathbb{D}} \in T_0(\mathbb{D}).$$

Since the harmonicity is conformally invariant, we have $H_{\varphi_n} = \gamma_n \circ H_\varphi \circ g_n$ and hence H_{φ_n} are all K -quasiconformal harmonic, with $K = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$ independent of n , and μ_n is just the complex dilatation of H_{φ_n} . It follows readily from Corollary 1 and (4.4), by passing to a subsequence if necessary, that H_{φ_n} converges uniformly on compact sets in \mathbb{D} to the identity map as $n \rightarrow \infty$. Moreover, $\partial_z H_{\varphi_n} \rightarrow 1$ and $\partial_{\bar{z}} H_{\varphi_n} \rightarrow 0$ as $n \rightarrow \infty$, uniformly on compact sets in \mathbb{D} . Thus we have proved that for any $\delta > 0$ there is an integer N such that

$$(1 + \delta)^{-1} \leq \|(H_{\varphi_n})_*(v)\|_{\mathbb{D}} \leq 1 + \delta$$

whenever $n > N$ and v is a unit tangent vector to \mathbb{D} at the point 0.

Because all automorphisms of \mathbb{D} are hyperbolic isometries, we have

$$(4.6) \quad (1 + \delta)^{-1} \leq \|(H_\varphi)_*(v_n)\|_{\mathbb{D}} = \|(H_{\varphi_n})_*v\|_{\mathbb{D}} \leq 1 + \delta$$

whenever $n > N$ and v_n is a unit tangent vector to \mathbb{D} at the point z_n . □

Remark 1. The above proof imitates that of Lemma 13.2 of [4].

The proof of Lemma 4.2 almost takes word by word from that of Lemma 13.1 in [4]. For completeness and for convenience of the reader, we reproduce the proof as follows.

We denote the hyperbolic length of a smooth curve $c(t)$ in \mathbb{D} by $L(c)$.

Let $\varepsilon > 0$ be given. We shall derive the desired inequality (4.2) from Lemma 4.3. By that lemma we can choose a compact set $E_0 \subset \mathbb{D}$ so that

$$(4.7) \quad (1 + \frac{\varepsilon}{2})^{-1} \|v\|_{\mathbb{D}} \leq \|w_*(v)\|_{\mathbb{D}} \leq (1 + \frac{\varepsilon}{2}) \|v\|_{\mathbb{D}}$$

whenever v is a tangent vector to \mathbb{D} at a point of $\mathbb{D} \setminus E_0$.

Let R be the hyperbolic diameter of $w(E_0)$. We claim that if $x \in \mathbb{D}$ satisfies the condition $d_{\mathbb{D}}(x, E_0) > 2R/\varepsilon$, then

$$(4.8) \quad d_{\mathbb{D}}(w(x), w(y)) \leq (1 + \varepsilon)d_{\mathbb{D}}(x, y)$$

for all $y \in \mathbb{D}$.

To prove this claim, we consider a distance minimizing geodesic $c : [0, 1] \rightarrow \mathbb{D}$ that joins $x = c(0)$ to $y = c(1)$.

If $c(t) \in \mathbb{D} \setminus E_0$ for all $t \in [0, 1]$, then inequality (4.7) gives

$$L(w \circ c) \leq (1 + \frac{\varepsilon}{2})L(c) = (1 + \frac{\varepsilon}{2})d_{\mathbb{D}}(x, y).$$

Since the curve $w \circ c$ joins $w(x)$ to $w(y)$, we obtain

$$(4.9) \quad d_{\mathbb{D}}(w(x), w(y)) \leq (1 + \frac{\varepsilon}{2})d_{\mathbb{D}}(x, y).$$

If $c(t) \in E_0$ for some $t \in [0, 1]$, set

$$a = \min\{t \in [0, 1] : c(t) \in E_0\} \quad \text{and} \quad b = \max\{t \in [0, 1] : c(t) \in E_0\}.$$

The curves $c : [0, a] \rightarrow \mathbb{D}$ and $c : [b, 1] \rightarrow \mathbb{D}$ are distance minimizing geodesics, and the sum of their lengths is at most $d_{\mathbb{D}}(x, y)$. Since (apart from their endpoints) they avoid E_0 , the reasoning that gave (4.9) yields

$$d_{\mathbb{D}}(w(x), w(c(a))) + d_{\mathbb{D}}(w(c(b)), w(y)) \leq (1 + \frac{\varepsilon}{2})d_{\mathbb{D}}(x, y).$$

Since both $c(a)$ and $c(b)$ belong to E_0 , the definition of R and our condition on x give

$$d_{\mathbb{D}}(w(c(a)), w(c(b))) \leq R < \frac{\varepsilon}{2}d_{\mathbb{D}}(x, E_0) \leq \frac{\varepsilon}{2}d_{\mathbb{D}}(x, c(a)) \leq \frac{\varepsilon}{2}d_{\mathbb{D}}(x, y).$$

Adding these inequalities and applying the triangle inequality, we obtain (4.8), proving our claim.

By the same reasoning, we have

$$d_{\mathbb{D}}(x, y) \leq (1 + \varepsilon)d_{\mathbb{D}}(w(x), w(y)), \quad \forall y \in \mathbb{D},$$

if $x \in \mathbb{D}$ satisfies the condition $d_{\mathbb{D}}(w(x), w(E_0)) > 2D/\varepsilon$, where D is the hyperbolic diameter of E_0 . We will not go into detail.

We conclude that (4.2) holds unless both x and y belong to the compact set

$$E = \{x \in \mathbb{D} : d_{\mathbb{D}}(x, E_0) \leq 2R/\varepsilon \text{ or } d_{\mathbb{D}}(w(x), w(E_0)) \leq 2D/\varepsilon\}.$$

If $\varphi \in T(\mathbb{D})$ admits a quasiconformal harmonic extension H_{φ} to \mathbb{D} , then H_{φ} is a hyperbolic quasi-isometry on \mathbb{D} [16]. Our Lemma 4.2 gives a somewhat more subtle form for $\varphi \in T_0(\mathbb{D})$.

Remark 2. Another further application of our Corollary 1 shows [18] that the harmonic embedding map from the Bers-norm bounded Hopf differentials space to the Teichmüller space $T(X)$ induces an embedding from an appropriate quotient Banach space into the asymptotic Teichmüller space $AT(X)$, where X is a hyperbolic Riemann surface. In addition, if the Schoen conjecture is true, i.e., every $\varphi \in T(\mathbb{D})$ has a quasiconformal harmonic extension to \mathbb{D} , then our Theorem 2 can be generalized to any hyperbolic Riemann surface.

Remark 3. If the quasiconformal map f is an asymptotic hyperbolic isometry on \mathbb{D} , then $f|_{\partial\mathbb{D}}$ is a symmetric map of the unit circle.

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