SKEW EXACTNESS PERTURBATION

ROBIN HARTE AND DAVID LARSON

(Communicated by Joseph A. Ball)

Abstract. We offer a perturbation theory for finite ascent and descent properties of bounded operators.

There are various degrees of “skew exactness” ([11]; [8], (10.9.0.1), (10.9.0.2)) between compatible pairs of operators, bounded and linear between normed spaces:

1. Definition. Suppose $T : X \to Y$ and $S : Y \to Z$ are bounded and linear between normed spaces. Then we may classify the pair $(S, T)$ as left skew exact if there is inclusion

$$S^{-1}(0) \cap T(X) = \{0\},$$

strongly left skew exact if there is $k > 0$ for which

$$\|T(\cdot)\| \leq k\|ST(\cdot)\|,$$

and splitting left skew exact if there is $R \in BL(Z, Y)$ for which

$$T = RST.$$

Also we may classify the pair $(S, T)$ as right skew exact if there is inclusion

$$S^{-1}(0) + T(X) = Y,$$

strongly right skew exact if there is $k > 0$ for which: for every $y \in Y$ there is $x \in X$ for which

$$Sy = STx \text{ with } \|x\| \leq k\|y\|,$$

and splitting right skew exact if there is $R \in BL(Y, X)$ for which

$$S = STR.$$

It is easy to see that

2. Theorem. In the notation of Definition 1, there exist implications

$$(1.1) \implies (1.2) \implies (1.3),$$

and

$$(1.6) \implies (1.5) \implies (1.4).$$
Proof. Most of this holds slightly more generally ([3], Theorems 10.1.2, 10.1.4), with a general operator $R' : X \to Z$ in place of the product $ST$. Note (cf. [3], (6.1)) that (1.1) holds iff
\begin{equation}
(ST)^{-1}(0) \subseteq T^{-1}(0),
\end{equation}
and that (1.4) holds iff
\begin{equation}
S(Y) \subseteq ST(X).
\end{equation}

For Hilbert spaces $X, Y, Z$ there exist ([3], Theorem 10.8.1) implications (1.2) $\implies$ (1.3) and (1.5) $\implies$ (1.6).

A slightly stronger version of condition (1.1) asks that
\begin{equation}
S^{-1}(0) \cap \overline{cl\ T(X)} = \{0\},
\end{equation}
which says that the operator $K_MJ_N$ is one-to-one, where (cf. Yang [17]; [3]) $K : Y \to Y/M$ and $J : N \to Y$ are the natural quotient and injection induced by the subspaces $M = \overline{cl\ TX}$ and $N = S^{-1}(0)$. Stronger again is the condition that there be $k > 0$ for which there is implication
\begin{equation}
y \in S^{-1}(0) \implies \|y\| \leq k \text{ dist}(y, T(X)),
\end{equation}
which says that the same operator $K_MJ_N$ is bounded below. Evidently
\begin{equation}
(1.2) \implies (2.6) \implies (2.5) \implies (1.1):
\end{equation}
if $k > 0$ satisfies (1.2) and if $Sy = 0$, then
\begin{equation}
\|y\| \leq \|y - Tx\| + \|Tx\| \leq \|y - Tx\| + k\|S(Tx - y)\| \leq (1 + k\|S\|)\|y - Tx\|. \quad \square
\end{equation}

Condition (2.6), with $k = 1$, has been noticed by Anderson [1], who describes it by calling $T(X)$ orthogonal to $S^{-1}(0)$; we will prefer to say that $S^{-1}(0)$ is orthogonal to $T(X)$. Turnsek [10] has observed that it holds for certain operators on Banach algebras:

3. Theorem. If $S \in BL(Y, Y)$, then (2.6) holds with $k = 1$ for $(S, S)$ provided
\begin{equation}
\|I - S\| \leq 1.
\end{equation}

Proof. Following the argument of Turnsek ([10], Theorem 1.1), write
\begin{equation}
S = I - U \text{ and } V_n = I + U + \ldots + U^n,
\end{equation}
so that
\begin{equation}
SV_n = I - U^{n+1} = V_nS.
\end{equation}
We have
\begin{equation}
Sy = 0 \implies (n + 1)y = V_ny = (I - U^{n+1})x + V_n(y - Sx),
\end{equation}
and hence
\begin{equation}
\|y\| \leq \frac{2}{n + 1}\|x\| + \|y - Sx\|;
\end{equation}
now let $n \to \infty$. \quad \square
Alternatively, Theorem 3 is an application of Sinclair’s Theorem ([15], Proposition 1; [3], Corollary 1), since here 0 is not in the interior of the “numerical range” of S. The argument of Theorem 3 suggests - wrongly - that we are using a weakened version of condition (1.3): we call the pair \((S,T)\) almost splitting left skew exact if there are \((R_n)\) in \(BL(Z,Y)\) with
\[
\|T - R_n ST\| \to 0 \text{ and } \sup_n \|R_n\| < \infty ,
\]
and almost splitting right skew exact if instead \((R_n)\) in \(BL(Y,X)\) with
\[
\|S - STR_n\| \to 0 \text{ and } \sup_n \|R_n\| < \infty .
\]
Also call \((S,T)\) almost strongly right skew exact if there is \(k > 0\) for which: for every \(y \in Y\) there is \((x_n)\) in \(X\) for which
\[
\|Sy - STR_n x_n\| \to 0 \text{ with } \sup_n \|x_n\| \le k\|y\| .
\]
Evidently (cf. [11], Theorem 10.1.2)
\[
(1.3) \implies (3.3) \implies (1.2) \quad \text{(3.6)}
\]
and
\[
(1.6) \implies (3.4) \implies (3.5) ;
\]
thus (3.3) implies (2.6). We do not, however, derive (3.3) for \((S,S)\) from condition (3.1). We also remark that, whenever the space \(Z\) is complete, there is the implication
\[
(1.4) \implies (3.5) .
\]
This ([2]; [5], Theorem 1.1; [8], Theorem 10.5.5) uses Baire’s theorem.
Under certain circumstances the “left” and “right” skew exactnesses are equivalent. We begin (cf. [4], Lemma 6.2) by extending the finite ascent/descent characterizations:

4. Theorem. Suppose, under the conditions of Definition 1, that \(W \subseteq X\) with \(T(W) \subseteq S^{-1}(0)\), and that \(V \subseteq Y\) with \(T(X) \subseteq S^{-1}(V)\). Then each of the following conditions is equivalent to (1.1):
\[
\begin{align*}
(4.1) & \quad T^\vee : X/T^{-1}(0) \to Y/S^{-1}(0) \text{ one-to-one} ; \\
(4.2) & \quad S^\wedge : T(X) \to V \text{ is one-to-one} .
\end{align*}
\]
Also each of the following conditions is equivalent to condition (1.4):
\[
\begin{align*}
(4.3) & \quad S^\wedge : T(X) \to S(Y) \text{ onto} ; \\
(4.4) & \quad T^\vee : X/W \to Y/S^{-1}(0) \text{ is onto} .
\end{align*}
\]

Proof. The equivalences (1.1) \iff (4.1) and (1.4) \iff (4.3) are clear. We claim that (1.1) is equivalent to (4.2) with \(V = Z\), and that this in turn is equivalent to (4.2) for arbitrary \(V\) for which \(T(X) \subseteq S^{-1}V\). The second equivalence is clear; for the first note that for arbitrary \(x \in X\) there is the implication
\[
Tx \in S^{-1}(0) \iff STx = 0 .
\]
We also claim that (1.4) is equivalent to (4.4) with \(W = \{0\}\), and that this in turn is equivalent to (4.4) for arbitrary \(W\) for which \(T(W) \subseteq S^{-1}(0)\). The second equivalence is clear; for the first note that for arbitrary \(y \in Y\) there is the implication
\[
y \in S^{-1}(0) + T(X) \iff Sy \in S(TX) .
\]
\(\square\)
If in particular $X = Y = Z$ and $ST = TS$, then (4.2) applies with $V = T(X)$, and (4.4) applies with $W = S^{-1}(0)$. We apply this in particular with $S = T^k$ for some $k \in \mathbb{N}$:

5. Theorem. If $X = Y = Z$ and $S = T^k : Y \to Y$, with $T$ in the “commutative closure” of the invertibles, in the sense that there are $(R_n)$ in $BL(X, X)$ with

$$(5.1) \quad R_n \in BL^{-1}(X, X), \quad R_n T = TR_n, \quad \|R_n - T\| \to 0,$$

then the following are equivalent:

$$(5.2) \quad (TS)^{-1}(0) \subseteq S^{-1}(0) \text{ and } T(X) = \text{cl } T(X),$$

$$(5.3) \quad S(Y) \subseteq ST(X) \text{ and } T(X) = \text{cl } T(X).$$

Proof. We recall ([6]; [8], Theorem 3.5.1) that for bounded linear operators $T : X \to Y$ between (possibly incomplete) normed spaces

$$(5.4) \quad T \text{ bounded below and a limit of dense range } \implies T \text{ almost open ,}$$

and hence ([6]; [8], Theorem 5.5.6) by duality

$$(5.5) \quad T \text{ almost open and a limit of bounded below } \implies T \text{ bounded below .}$$

Now if $R_n$ commutes with $T$, then it leaves both $T(X)$ and $S^{-1}(0)$ invariant, and if $R_n$ is invertible, then (cf. [8], Theorem 3.11.1) its restriction $R_n^\vee$ to $T(X)$ will be bounded below and its quotient on $Y/S^{-1}(0)$ will be onto. Thus if we assume (5.2), then by (4.1) and closed range, $T^\vee$ will be bounded below and the limit of onto $R_n^\vee$, therefore onto, giving (5.3). If instead we assume (5.3), then by (4.3) $S^\wedge$ will be onto and by closed range almost open, and the limit of bounded below $(R_n^\wedge)^\vee$, therefore bounded below, giving (5.2).

(5.2) and (5.3) are together equivalent to the condition that $T \in BL(X, X)$ is polar ([8], Definition 7.5.2) in the sense that $0 \in \mathbb{C}$ is at worst a pole of the resolvent function $(zI - T)^{-1}$. If we relax the closed range condition we can still get one of the implications, provided we further tighten the approximation by commuting invertible operators:

6. Theorem. Suppose that $S = T^k$ and that $0 \notin \text{int } \sigma(T)$. If the finite descent condition (1.4) holds, then so also does the finite ascent condition (5.2), including closed range.

Proof. This is shown on Hilbert space ([14], Lemma 2.5) by Herrero, Larson and Wogen. Alternatively, since we are assuming that $0$ is at worst on the boundary of the spectrum, then we can take the approximating invertible operators $R_n = T - \lambda_n I$ to be scalar perturbations of the operator $T$. Now if (1.4) holds, then the quotient operator $T^\vee$ on $X/S^{-1}(0)$ is (4.5) onto, and the limit of operators $(T - \lambda_n I)^\vee$, which we claim are invertible. As in Theorem 5 it is clear that the quotient $(T - \lambda_n I)^\vee$ is onto: we claim it is also one-to-one. To see this recall that the operator $T - \lambda_n I$ is one-to-one and the restriction $(T - \lambda_n I)^\wedge = (-\lambda_n I)^\wedge$ to the subspace $T^{-1}(0)$ is onto, so that ([5], Theorem 3.11.2) the induced quotient is also one-to-one. For the closed range note that $T(X)$ now has a closed complement, and appeal to the “Lemma of Neuberger” ([8], Theorem 4.8.2).
Theorem 6 does not reverse:

7. Example. If
\[ S = I - \lambda U \] or \[ S = I - \lambda V \] or \[ S = \lambda W \],
where \( |\lambda| = 1 \), \( U \) and \( V \) are the forward and backward shifts on \( \ell_2 \), and \( W \) the standard weight,
\[ (Ux)_1 = 0 \] or \[ (Ux)_{n+1} = x_n \] or \[ (Vx)_n = x_{n+1} \] or \[ (Wx)_n = (1/n)x_n \],
then \( S \) is one-to-one and not onto, therefore of finite descent and not of finite ascent, while
\[ \|I - S\| = 1 \] so that \( 0 \notin \text{int} \sigma(S) \).

Proof. This is easily checked: note that, extended to all sequences, there is equivalence, for arbitrary \( x \in X_N \),
\[ x \in (I - \lambda U)^{-1} \iff x \in (I - \lambda V)^{-1} \iff x = x_1(1, \lambda, \lambda^2, \ldots) \]. □

We need some auxiliary subspaces:

8. Definition. If \( T \in BL(X, X) \) write
\[ T^{-\infty}(0) = \bigcup_{n=1}^{\infty} T^{-n}(0) \quad \text{and} \quad T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X) \]
for the hyperkernel and the hyperrange of \( T \), and
\[ E_X(T) = \sum_{\lambda \in \mathbb{C}} (T - \lambda I)^{-\infty}(0) \quad \text{and} \quad F_X(T) = \bigcap_{\lambda \in \mathbb{C}} (T - \lambda I)^{\infty}(X) \].

Each of the subspaces in Definition 8 is linear, not necessarily closed, and hyperinvariant under \( T \). We recall that \( T \in BL(X, X) \) is called algebraic if there is a nontrivial polynomial \( 0 \neq p \in \text{Poly} \) for which
\[ p(T) = 0 \]
mORE generally \( T \) is said to be locally algebraic if
\[ X = \bigcup \{ p(T)^{-1}(0) : 0 \neq p \in \text{Poly} \} \].
For the record

9. Theorem. If \( T \in BL(X, X) \) for a Banach space \( X \), then
\[ T \text{ locally algebraic} \implies T \text{ algebraic} \].

Necessary and sufficient for \( T \) to have finite descent is that
\[ E_X(T) + T(X) = X \].

Proof. The first part of this is known as Kaplansky’s Lemma; the proof is a combination of Baire’s theorem and the Euclidean algorithm for polynomials. The Euclidean algorithm also gives equality
\[ E_X(T) = \bigcup \{ p(T)^{-1}(0) : 0 \neq p \in \text{Poly} \} = \{ x \in X : \dim \text{Poly}(T)x < \infty \} \],
and dually
\[ F_X(T) = \bigcap \{ p(T)(X) : 0 \neq p \in \text{Poly} \} \].
Then again with a combination of Baire’s theorem and the Euclidean algorithm, if $T \in BL(X, X)$ there is ([13], Lemma 2.4) $k \in \mathbb{N}$ for which

\begin{equation}
E_X(T) + T(X) = T^{-\infty}(0) + T(X) = T^{-k}(0) + T(X). \quad \Box
\end{equation}

Dually, using the Euclidean algorithm, we get half way:

\begin{equation}
F_X(T) \cap T^{-1}(0) = T^{\infty}(X) \cap T^{-1}(0).
\end{equation}

For the essence of a possible spectral mapping theorem (cf. [11]), we have

**10. Theorem.** If $S, T \in BL(X, X)$ satisfy $ST = TS$ and either

\begin{equation}
S \in \{T^k : k \in \mathbb{N}\}
\end{equation}

or

\begin{equation}
VS - TU = I \quad \text{with } \{U, V\} \subseteq \text{comm}(S, T),
\end{equation}

then there is the equivalence

\begin{equation}
ST \text{ of finite ascent } \iff S, T \text{ of finite ascent},
\end{equation}

and the equivalence

\begin{equation}
ST \text{ of finite descent } \iff S, T \text{ of finite descent}.
\end{equation}

**Proof.** The backward implications are easy ([8], Theorem 7.9.2): if $S$ and $T$ commute and satisfy $S^{-k}(0) = S^{-k-1}(0)$ and $T^{-k}(0) = T^{-k-1}(0)$, then

\begin{equation}
(ST)^{-k}(0) = S^{-k}T^{-k}(0) = S^{-k}T^{-k-1}(0) = T^{-k-1}S^{-k}(0) = T^{-k-1}S^{-k-1}(0) = (ST)^{-k-1}(0).
\end{equation}

If instead $ST = TS$ with $S^kX = S^{k+1}X$ and $T^kX = T^{k+1}X$, then

\begin{equation}
(ST)^kX = S^kT^k(X) = S^kT^{k+1}X = T^{k+1}S^kX = T^{k+1}S^{k+1}X = (ST)^{k+1}X.
\end{equation}

Also the forward implications are clear when (10.1) $S = T^k$ is a power of $T$. If instead we assume (10.2), then we argue

\begin{equation}
(ST)^{-1}(0) \subseteq T^{-1}(0) + T(X) \quad \text{and } (ST)X \supseteq T^{-1}(0) \cap T(X),
\end{equation}

while if $(U, V)$ satisfies (10.2), then for arbitrary $k \in \mathbb{N}$

\begin{equation}
V_kS^k - T^kU_k = I \quad \text{with } \{U_k, V_k\} \subseteq \text{comm}(S^k, T^k).
\end{equation}

To verify (10.5) argue

\begin{equation}
STx = 0 \Rightarrow x + TUx = VSx \in T^{-1}(0);
T(Tx) = 0 \Rightarrow Tx = TVSx - TUTx = (ST)(Vx).
\end{equation}

For (10.6) note that for arbitrary $k \in \mathbb{N}$

\begin{equation}
VS - TU = I \Rightarrow V^{k+1}S^{k+1} - TU(I + VS + \ldots + V^kS^k) = I. \quad \Box
\end{equation}

For an induced “spectrum” to be a closed set we have

**11. Theorem.** $T \in BL(X, X)$ is of finite descent; then so is $T - \lambda I$ for sufficiently small $\lambda \in \mathbb{C}$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof. This has been shown on Hilbert space by Han, Larson, and Pan ([13], Lemma 2.2, Theorem 2.4). It is clear from the open mapping theorem (applied to condition (4.4) with \( W = \{0\} \)) that if condition (1.4) holds, then also
\[ S^{-1}(0) + (T - U)(X) = Y \]
whenever \( T - U \in BL(X, Y) \) is sufficiently close to \( T \in BL(X, Y) \). The problem is that we must also perturb \( S \). However if \( S = T^k \) and \( U = I \), so that \( E_X(T - U) = E_X(T) \), then we can argue
\[ E_X(T - U) + (T - U)(X) = E_X(T) + (T - U)(X) \supseteq S^{-1}(0) + (T - U)(X) = X. \]

The subspaces of Definition 8 lead to certain special kinds of operators:

12. Definition. We shall call \( T \in BL(X, X) \) triangular if the subspace \( E_X(T) \) is dense:
\[ (12.1) \quad \text{cl} \ E_X(T) = X. \]
Dually \( T \in BL(X, X) \) is co-triangular if the subspace \( F_X(T) \) is trivial:
\[ (12.2) \quad F_X(T) = \{0\}. \]

The shifts of Example 7 are either triangular or co-triangular:

13. Example. On each of the spaces \( c_0 \) and \( \ell_p \) (\( 1 \leq p < \infty \)), the forward shift \( U \) is triangular, the backward shift \( V \) is co-triangular and the standard weight \( W \) is both triangular and co-triangular.

Proof. The hyperkernel of the backward shift is dense, since it includes all the “terminating” sequences:
\[ (13.1) \quad V^{-\infty}(0) \supseteq c_0. \]
Thus
\[ (13.2) \quad E(V) \supseteq V^{-\infty}(0) \text{ is dense} \]
and also
\[ (13.3) \quad F(V) = \bigcap_{|\lambda|=1} (V - \lambda I)^{-\infty}(X) \supseteq \sum_{|\lambda|<1} (V - \lambda I)^{-\infty}(0) \supseteq V^{-\infty}(0) \text{ is dense}. \]
Since \( U - \lambda I \) is one-to-one for every \( \lambda \in \mathbb{C} \) we have
\[ (13.4) \quad E(U) = \{0\} \text{ is trivial} \]
and also
\[ (13.5) \quad F(U) \subseteq U^{\infty}(X) = \{0\} \text{ is trivial}. \]
Finally we notice that the weight \( W \) commutes with the projection \( UV \), and more generally
\[ (13.6) \quad WU^nV^n = U^nV^nW \quad (n \in \mathbb{N}) ; \]
also for each \( n \in \mathbb{N} \)
\[ (13.7) \quad \left( \frac{1}{n} I - W \right)^{-1}(0) = U^{n-1}(I - UV)V^{n-1}(X), \]
\[ \left( \frac{1}{n} I - W \right)(X) = (U^{n-1}(I - UV)V^{n-1})^{-1}(0), \]
so that \( E(W) \) is dense and \( F(W) \) is trivial. \( \square \)
Triangularity and Fredholmness co-operate to generate finite ascent or descent:

14. Theorem. If $T \in BL(X,X)$, then

(14.1) $T$ upper semi-Fredholm and co-triangular $\implies$ $T$ of finite ascent and

(14.2) $T$ lower semi-Fredholm and triangular $\implies$ $T$ of finite descent.

Proof. If $T \in BL(X,X)$ is upper semi-Fredholm, then the finite ascent condition can be written in the form

(14.3) $F_X(T) \cap T^{-1}(0) = \{0\}$.

Indeed since ([8], Theorem 6.12.2) each power $T^m$ is also upper semi-Fredholm, then $T^{-m}(0)$ is finite dimensional for each $m \in \mathbb{N}$ and $T^{-m}(X)$ is closed; thus if for each $m \in \mathbb{N}$ we have $T^m(X) \cap T^{-1}(0) \neq \{0\}$, then there is $(x_m)$ in $X$ for which

$$\|T^m(x_m)\| = 1$$

and $T^{m+1}x_m = 0$.

By local compactness there is a subsequence

$$(y_m) = T^{\phi(m)}(x_{\phi(m)}) \to y_\infty \in T^\infty(X),$$

using the closedness of all the ranges, so that $\|y_\infty\| = 1$ and $y_\infty \in F_X(T) \cap T^{-1}(0)$.

This proves (14.1); towards (14.2) we claim that for subspaces $Y, Z \subseteq X$ (14.4) $Y$ closed of finite codimension and $Z$ dense $\implies$ $Y + Z = X$,

because if $\dim(X/Y) = n$ find successively $e_1, e_2, \ldots, e_n$ with

$$e_{j+1} \in Z \setminus (Y + Ce_1 + Ce_2 + \ldots + Ce_j).$$

Applying this with $Y = T(X)$ and $Z = E_X(T)$ gives (14.2).

It is clear that in (14.1) we can replace the “co-triangular” condition (12.2) by the weaker condition (14.3); dually in (14.2) we can replace the triangular condition (12.1) by the weaker condition

(14.5) $\text{cl } E_X(T) + T(X) = X$.

We cannot however remove the semi-Fredholm conditions: for example if $T = U \otimes V$ is the tensor product of the shifts, then (14.3) holds but we never get $T^k(X) \cap T^{-1}(0) = \{0\}$. For operators which are both upper semi-Fredholm and of finite ascent, or lower semi-Fredholm of finite descent (“semi Browder” in the sense of [8], Definition 7.9.1) the conditions of Theorem 10 can be replaced by simple commutivity ([8], Theorem 7.9.2).

References


School of Mathematics, Trinity College, Dublin 2, Ireland

E-mail address: rharte@maths.tcd.ie

Department of Mathematics, Texas A & M University, College Station, Texas 77843-3368

E-mail address: larson@math.tamu.edu