PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 9, Pages 2577–2591
S 0002-9939(04)07372-1
Article electronically published on April 21, 2004

"LEBESGUE MEASURE" ON $\mathbb{R}^\infty$, II

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(Communicated by David R. Larson)

Abstract. Let $\mathbb{R}$ be the set of real numbers, and define $\mathbb{R}^\infty = \prod_{i=1}^{\infty} \mathbb{R}$. We construct a complete measure space $(\mathbb{R}^\infty, \mathcal{L}, \lambda)$ where the σ-algebra $\mathcal{L}$ contains the Borel subsets of $\mathbb{R}^\infty$, and λ is a translation-invariant measure such that for any measurable rectangle $R = \prod_{i=1}^{\infty} R_i$, if $0 \leq \prod_{i=1}^{\infty} m(R_i) < +\infty$, then $\lambda(R) = \prod_{i=1}^{\infty} m(R_i)$, where m is Lebesgue measure on $\mathbb{R}$. The measure λ is not σ-finite. We prove three Fubini theorems, namely, the Fubini theorem, the mean Fubini-Jensen theorem, and the pointwise Fubini-Jensen theorem. Finally, as an application of the measure λ, we construct, via selfadjoint operators on $L_2(\mathbb{R}^\infty, \mathcal{L}, \lambda)$, a "Schrödinger model" of the canonical commutation relations: $[P_j, P_k] = [Q_j, Q_k] = 0$, $[P_j, Q_k] = i\delta_{jk}$, $1 \leq j, k < +\infty$.

1. Introduction

Let $\mathbb{R}$ denote the set of real numbers, and define $\mathbb{R}^\infty = \prod_{i=1}^{\infty} \mathbb{R}$. In this paper we show that there exists a complete measure space $(\mathbb{R}^\infty, \mathcal{L}, \lambda)$ such that the σ-algebra $\mathcal{L}$ contains the Borel subsets of $\mathbb{R}^\infty$, and λ is a translation-invariant measure with the property that if $R = \prod_{i=1}^{\infty} R_i$ is any infinite-dimensional measurable rectangle such that the "volume", $\prod_{i=1}^{\infty} m(R_i) := \lim_{n\to\infty} \prod_{i=1}^{n} m(R_i)$, of R is a nonnegative real number, then

$$\lambda(R) = \prod_{i=1}^{\infty} m(R_i).$$

Here, m is Lebesgue measure on $\mathbb{R}$. We dub λ, *infinite-dimensional Lebesgue measure on* $\mathbb{R}^\infty$. We prove three Fubini theorems for λ, namely, the Fubini theorem, the mean Fubini-Jensen theorem, and the pointwise Fubini-Jensen theorem. Finally, as an application of the measure λ, we construct, via selfadjoint operators

Received by the editors August 16, 1994 and, in revised form, March 21, 2003.
2000 *Mathematics Subject Classification.* Primary 28A35, 28C10, 81D05.
Key words and phrases. Canonical commutation relations, Elliott-Morse measures, Fubini theorem, Fubini-Jensen theorem, infinite-dimensional Lebesgue measure, invariant measures, Schrödinger model.

on $\mathcal{H} = L_2(\mathbb{R}^\infty, \lambda)$, a "Schrödinger model" of the canonical commutation relations: $[P_j, P_k] = [Q_j, Q_k] = 0$, $[P_j, Q_j] = i\delta_{jk}$, $1 \le j, k < +\infty$. The operators P_j, Q_j are given by

$$(Q_j f)(x) = x_j f(x), \quad \mathbb{D}(Q_j) = \{\, f \in \mathcal{H} \mid x_j f \in \mathcal{H} \,\},$$
$$(P_j f)(x) = -i \frac{\partial}{\partial x_j} f(x), \quad \mathbb{D}(P_j) = \left\{ f \in \mathcal{H} \,\middle|\, \frac{\partial}{\partial x_j} f \in \mathcal{H} \right\}.$$

The measure λ is the final version of a similar measure which we introduced in the paper [B].

We will present the main properties of the measure λ as consequences of the following more general construction.

Let $(X_i, \mathcal{M}_i, \mu_i)$ be a sequence of measure spaces, and for each i let ρ_i be a metric on the set X_i. We will assume that the following conditions are satisfied.

(i) Each (X_i, ρ_i) is a locally compact metric space.
(ii) Each $\mathcal{M}_i$ contains the family, $\mathcal{B}(X_i)$, of Borel subsets of X_i and μ_i is a regular Borel measure on X_i [Rd, Def. 2.15].
(iii) For all i, and for every $\delta > 0$, there exists a sequence (A_j) of Borel subsets of X_i such that $d_i(A_j) < \delta$ and $X_i = \bigcup_{j=1}^{\infty} A_j$, where $d_i(A_j)$ is the diameter of A_j in X_i.

Define $X = \prod_{i=1}^{\infty} X_i$. Throughout the rest of this paper, we will assume that X has the product topology. We denote by $\mathcal{R}$ the family of all rectangles $R \subseteq X$ of the form

$$R = \prod_{i=1}^{\infty} R_i, \quad R_i \in \mathcal{B}(X_i), \text{ and } 0 \le \prod_{i=1}^{\infty} \mu_i(R_i) := \lim_n \prod_{i=1}^{n} \mu_i(R_i) < +\infty.$$

For $R \in \mathcal{R}$, we define $\tau(R) = \prod_{i=1}^{\infty} \mu_i(R_i)$. Let τ^* be the set function on $\mathcal{P}(X)$ defined by

$$\tau^*(E) = \inf \sum_{R_j \in \mathcal{R} \cup R_j \supseteq E} \tau(R_j), \quad E \subseteq X.$$

We use the convention that $0 \cdot +\infty = +\infty \cdot 0 = 0$, $+\infty \cdot +\infty = +\infty$, and that the infimum taken over the empty set has the value $+\infty$.

Theorem I. *The set function τ^* is an outer measure on X. Let $\mathcal{M}$ be the σ-algebra of subsets of X that are measurable with respect to τ^*, and let μ be the measure on $\mathcal{M}$ obtained by restricting τ^* to $\mathcal{M}$. Then $\mathcal{B}(X) \subseteq \mathcal{M}$, and for all $R = \prod_{i=1}^{\infty} R_i \in \mathcal{R}$, we have $\mu(R) = \prod_{i=1}^{\infty} \mu_i(R_i)$. If each space X_i contains disjoint subsets A_i, B_i such that $\mu_i(A_i) = \mu_i(B_i) = 1$, then the measure μ is not σ-finite. Finally, assume that each (X_i, ρ_i) is an $\mathcal{M}_i$-measurable group. If each μ_i is left-invariant measure on $\mathcal{M}_i$, then μ is a left-invariant measure on $\mathcal{M}$. Similarly, if each μ_i is right-invariant measure on $\mathcal{M}_i$, then μ is right-invariant measure on $\mathcal{M}$.*

Definition I. The measure λ on $\mathbb{R}^\infty$ given by Theorem I when each $X_i = \mathbb{R}$ and each $\mu_i = m$, is called *Lebesgue measure on* $\mathbb{R}^\infty$, and the σ-algebra $\mathcal{M}$ is denoted by $\mathcal{L}$. Clearly, Theorem I implies that λ is not σ-finite; moreover, λ is translation invariant. The measure λ is the final version of a similar measure introduced in [B]. We do not know whether or not the λ defined here coincides with the original version in [B].

Define $\mathbb{N} = \{1, 2, \dots\}$. For any $A = \prod_{i=1}^{\infty} A_i$, $A_i \subseteq X_i$, and for $\emptyset \neq \Omega \subset \mathbb{N}$, we write $A_\Omega = \prod_{i\in\Omega} A_i$. We denote by μ_Ω the measures on X_Ω given by Theorem I, and we let $\mathcal{M}_\Omega$ denote the family of subsets of X_Ω that are μ_Ω-measurable. Finally, with $\Omega' = \mathbb{N} \setminus \Omega$, we define $\mu_\Omega \times \mu_{\Omega'}$ to be the product measure of μ_Ω and $\mu_{\Omega'}$ as constructed in [Ry, Section 12], and $\mathcal{M}_\Omega \otimes \mathcal{M}_{\Omega'}$ will stand for the family of subsets of $X_\Omega \times X_{\Omega'}$ that are $(\mu_\Omega \times \mu_{\Omega'})$–measurable.

Before we state the three Fubini theorems, the reader may want to look ahead at Lemma 4.2 to get some idea of what kind of functions are in $L_p(X, \mathcal{M}, \mu)$.

Theorem II (Fubini)**.** *Let $\emptyset \neq \Omega \subset \mathbb{N}$. Then $(X, \mathcal{M}, \mu)$ can be identified with $(X_\Omega \times X_{\Omega'}, \mathcal{M}_\Omega \otimes \mathcal{M}_{\Omega'}, \mu_\Omega \times \mu_{\Omega'})$. Moreover, let $f \in L_p(X, \mathcal{M}, \mu)$, $1 \le p < +\infty$. Then we have*

(i) *for μ_Ω-a.a. $x_\Omega \in X_\Omega$, the function f_{x_Ω} defined by $f_{x_\Omega}(x_{\Omega'}) = f(x_\Omega, x_{\Omega'})$ is a $\mu_{\Omega'}$-integrable function on $X_{\Omega'}$;*

(i′) *for $\mu_{\Omega'}$-a.a. $x_{\Omega'} \in X_{\Omega'}$, the function $f_{x_{\Omega'}}$ defined by $f_{x_{\Omega'}}(x_\Omega) = f(x_\Omega, x_{\Omega'})$ is a μ_Ω-integrable function on X_Ω.*

(ii) $\int_{X_{\Omega'}} f(x_\Omega, x_{\Omega'})\, d\mu_{\Omega'}(x_{\Omega'})$ *is a μ_Ω-integrable function on X_Ω;*

(ii′) $\int_{X_\Omega} f(x_\Omega, x_{\Omega'})\, d\mu_\Omega(x_\Omega)$ *is a $\mu_{\Omega'}$-integrable function on $X_{\Omega'}$;*

(iii) *we have*

$$\begin{aligned}\int_X f(x)\, d\mu(x) &= \int_{X_\Omega \times X_{\Omega'}} f(x_\Omega, x_{\Omega'})\, d(\mu_\Omega \times \mu_{\Omega'}) \\ &= \int_{X_\Omega} \Big[\int_{X_{\Omega'}} f(x_\Omega, x_{\Omega'})\, d\mu_{\Omega'}(x_{\Omega'}) \Big] d\mu_\Omega(x_\Omega) \\ &= \int_{X_{\Omega'}} \Big[\int_{X_\Omega} f(x_\Omega, x_{\Omega'})\, d\mu_\Omega(x_\Omega) \Big] d\mu_{\Omega'}(x_{\Omega'}).\end{aligned}$$

Theorem III (Mean Fubini-Jensen theorem)**.** *Let $\emptyset \neq \Omega \subset \mathbb{N}$ be an arbitrary finite index set. Let $R \in \mathcal{R}$, with $\mu(R) > 0$. For $1 \le p < +\infty$ and $f \in L_p(X, \mathcal{M}, \mu)$, define $f_{\Omega'}, f_\Omega$ on X by*

$$f_{\Omega'}(x_\Omega, x_{\Omega'}) = \int_{R_\Omega} f(u_\Omega, x_{\Omega'})\, d\mu_\Omega(u_\Omega),$$

$$f_\Omega(x_\Omega, x_{\Omega'}) = \int_{R_{\Omega'}} f(x_\Omega, y_{\Omega'})\, d\mu_{\Omega'}(y_{\Omega'}).$$

Then

(i) $\lim_\Omega \int_R \left| f_{\Omega'} - \int_R f\, d\mu \right|^p d\mu = 0;$

(ii) $\lim_\Omega \int_R |f_\Omega - f|^p d\mu = 0.$

Theorem IV (Pointwise Fubini-Jensen Theorem)**.** *For all* n*, let* $\Omega_n = \{1, 2, \ldots, n\}$*, and let* $R \in \mathcal{R}$*, with* $\mu(R) > 0$*. For* $f \in L_1(X, \mathcal{M}, \mu)$*, let* $f_{\Omega'_n}, f_{\Omega_n}$ *be as in Theorem* III*. Then for* μ*-almost all* $x \in R$*,*

(i) $\lim_n f_{\Omega'_n}(x) = \int_R f\, d\mu$;

(ii) $\lim_n f_{\Omega_n}(x) = f(x)$.

The proofs of Theorem III and Theorem IV which we present here are similar to the proofs of the corresponding theorems in [DS, III.11].

2. Proof of Theorem I

In this section we prove Theorem I. First, we need the fact that τ^* is an outer measure on X [Rg, Chapter 1, Theorem 4]. The first two lemmas, below, show that $\tau^*(R) = \tau(R)$ for $R \in \mathcal{R}$.

Lemma 2.1. *Let* $K_i \subseteq X_i$ *be compact,* $1 \le i < +\infty$*, and write* $K = \prod_{i=1}^{\infty} K_i$*. Suppose that* $K \in \mathcal{R}$*. Then we have* $\tau^*(K) = \tau(K)$*.*

Proof. We may assume that $\tau(K) > 0$; hence we have $\lim_n \prod_{i=n}^{\infty} \mu_i(K_i) = 1$. Let $K \subseteq \bigcup_{j=1}^{\infty} R_j$, where each R_j has the form $R_j = \prod_{i=1}^{\infty} R_{ij} \in \mathcal{R}$. Clearly, $\tau^*(K) \le \tau(K)$. Hence we want to show that

$$\tau(K) \le \sum_{j=1}^{\infty} \tau(R_j). \tag{2.1}$$

By replacing R_j by $K \cap R_j$, if necessary, we may assume that $\mu_i(R_{ij}) < +\infty$ for each i. Also, we may assume that

$$\sum_{j=1}^{\infty} \tau(R_j) < +\infty.$$

Because each μ_i is a regular Borel measure on X_i, we have, for each pair i, j,

$$\mu_i(R_{ij}) = \inf\{\, \mu_i(V) \mid R_{ij} \subseteq V,\ V \subseteq X_i \text{ open} \,\}. \tag{2.2}$$

For $j, n \ge 1$, define $D_{jn} = \prod_{i=1}^{n} R_{ij} \times \prod_{i=n+1}^{\infty} X_i$. Let $0 < \epsilon < 1$ be arbitrary, and let $\mathcal{F}$ be the family of all D_{jn} such that at least one of the following conditions holds:

(i) $\tau(R_j) = 0$, $\prod_{i=1}^{n} \mu_i(R_{ij}) < 2^{-(j+1)}\epsilon$, and $\prod_{i=n+1}^{\infty} \mu_i(K_i) < 1 + \epsilon$;

(ii) $\tau(R_j) > 0$, $\prod_{i=n+1}^{\infty} \mu_i(R_{ij}) > 1 - \epsilon$, and $\prod_{i=n+1}^{\infty} \mu_i(K_i) < 1 + \epsilon$.

A simple argument shows that $\mathcal{F}$ covers K. Now let $D_{jn} \in \mathcal{F}$. If (ii) holds for the pair j, n, then we may use (2.2) to find open sets $R_{ij} \subseteq V_{ijn} \subseteq X_i$, $1 \le i < +\infty$, such that $\prod_{i=1}^{\infty} \mu_i(V_{ijn}) < \tau(R_j) + 2^{-(j+1)}\epsilon$, and $1 - \epsilon < \prod_{i=n+1}^{\infty} \mu_i(V_{ijn}) < +\infty$. On the other hand, if (i) holds, then we may use (2.2) to find open sets $R_{ij} \subseteq V_{ijn} \subseteq X_i$,

$1 \le i < +\infty$, such that $\prod_{i=1}^{n} \mu_i(V_{ijn}) < 2^{-(j+1)}\epsilon$. For $D_{jn} \in \mathcal{F}$, define $W_{jn} = \prod_{i=1}^{n} V_{ijn} \times \prod_{i=n+1}^{\infty} X_i$. Then one of the following conditions holds:

(iii) $\prod_{i=n+1}^{\infty} \mu_i(K_i) < 1+\epsilon$, $\prod_{i=1}^{\infty} \mu_i(V_{ijn}) < \tau(R_j)+\dfrac{\epsilon}{2^{j+1}}$, and $1-\epsilon < \prod_{i=n+1}^{\infty} \mu_i(V_{ijn}) < +\infty$;

(iv) $\prod_{i=n+1}^{\infty} \mu_i(K_i) < 1+\epsilon$, $\prod_{i=1}^{n} \mu_i(V_{ijn}) < \dfrac{\epsilon}{2^{j+1}}$.

Now define $\mathcal{G}$ by $\mathcal{G} = \{ W_{jn} \mid D_{jn} \in \mathcal{F} \}$. Then $\mathcal{G}$ covers K, and because K is compact, there exist $j_1, \dots, j_k$ and $n_1, \dots, n_k$ such that $K \subseteq \bigcup_{p=1}^{k} W_{j_p n_p}$. For $n > \max\{ n_1, \dots, n_k \}$ and $1 \le p \le k$, set $K^{(n)} = \prod_{i=1}^{n} K_i$, $S_{np} = \prod_{i=1}^{n_p} V_{ij_p n_p} \times \prod_{i=n_p+1}^{n} K_i$. Then $K^{(n)} \subseteq \bigcup_{p=1}^{k} S_{np}$. Hence, with $\mu^{(n)} = \mu_1 \times \cdots \times \mu_n$, we have

(2.3)
$$\prod_{i=1}^{n} \mu_i(K_i) = \mu^{(n)}(K^{(n)}) \le \sum_{p=1}^{k} \mu^{(n)}(S_{np}) \le \sum_{p=1}^{k} \left\{ \prod_{i=1}^{n_p} \mu_i(V_{ij_p n_p}) \cdot \prod_{i=n_p+1}^{n} \mu_i(K_i) \right\}.$$

Now let $\sum'$ be the sum over those $1 \le p \le k$ for which (iii) holds for j_p and n_p. Let $\sum''$ be the sum over those $1 \le p \le k$ for which (iv) holds for j_p and n_p. Then, taking the limit $n \to +\infty$ in (2.3), we have

$$\begin{aligned}
\tau(K) = \prod_{i=1}^{\infty} \mu_i(K_i) &\le \sum_{p=1}^{k} \left\{ \prod_{i=1}^{n_p} \mu_i(V_{ij_p n_p}) \cdot \prod_{i=n_p+1}^{\infty} \mu_i(K_i) \right\} \\
&\le (1+\epsilon) \sum{}' \left\{ \prod_{i=1}^{\infty} \mu_i(V_{ij_p n_p}) \Big/ \prod_{i=n_p+1}^{\infty} \mu_i(V_{ij_p n_p}) \right\} + \epsilon(1+\epsilon) \sum{}'' 2^{-(j_p+1)} \\
&\le \left(\frac{1+\epsilon}{1-\epsilon} \right) \sum{}' \prod_{i=1}^{\infty} \mu_i(V_{ij_p n_p}) + \epsilon(1+\epsilon) \\
&\le \left(\frac{1+\epsilon}{1-\epsilon} \right) \sum{}' [\tau(R_{j_p}) + \epsilon 2^{-(j_p+1)}] + \epsilon(1+\epsilon) \\
&\le \left(\frac{1+\epsilon}{1-\epsilon} \right) \sum_{j=1}^{\infty} \tau(R_j) + \left(\frac{1+\epsilon}{1-\epsilon} \right) + \epsilon(1+\epsilon).
\end{aligned}$$

Because $0 < \epsilon < 1$ is arbitrary, we see that (2.1) holds. □

Lemma 2.2. *For $R = \prod_{i=1}^{\infty} R_i \in \mathcal{R}$, we have $\tau^*(R) = \tau(R)$.*

Proof. Because $\tau^*(R) \le \tau(R)$, we may assume that $\tau(R) > 0$. Then we have $0 < \prod_{i=1}^{\infty} \mu_i(R_i) < +\infty$. Therefore, for each i, $\mu_i(R_i) < +\infty$. Because each μ_i is a regular Borel measure on X_i, we have

(2.4) $$\mu_i(R_i) = \sup\{ \mu_i(K) \mid K \subseteq R_i,\ K\text{compact} \}.$$

Let $\epsilon > 0$ be arbitrary. Using (2.4), we can find compact sets $K_i \subseteq X_i$ such that $\prod_{i=1}^{\infty} \mu_i(R_i) < \prod_{i=1}^{\infty} \mu_i(K_i) + \epsilon < +\infty$. Define $K = \prod_{i=1}^{\infty} K_i$. Then $K \subseteq R$, and hence by Lemma 2.1, we have $\tau(K) = \tau^*(K) \leq \tau^*(R) \leq \tau(R) < \tau(K) + \epsilon$. Thus, $0 \leq \tau(R) - \tau^*(R) < \epsilon$, and because $\epsilon > 0$ is arbitrary, we see that $\tau(R) = \tau^*(R)$. □

Define a set function ν^* on X by

$$\nu^*(E) = \sup_{\delta>0} \tau^*_\delta(E), \quad \tau^*_\delta(E) = \inf \sum_{\substack{R_j \in \mathcal{R}, d(R_j) \leq \delta \\ \cup R_j \supseteq E}} \tau(R_j), \quad E \subseteq X,$$

where $d(R_j)$ is the diameter of R_j with respect to the metric on X given by $\rho(x, y) = \sum_{i=1}^{\infty} 2^{-i} \rho_i(x_i, y_i)/[1+\rho_i(x_i, y_i)]$. Then by [Rg, Chapter 1, Theorem 15], ν^* is an outer measure on X, and by [Rg, Chapter 1, Theorem 19], all Borel subsets of X are ν^*-measurable. Lemmas 2.3 and 2.4, below, show that $\tau^*(E) = \nu(E)$ for all $E \subseteq X$, and hence the σ-algebra $\mathcal{M}$ contains $\mathcal{B}(X)$. □

Lemma 2.3. *Let $n \geq 2$, and define $X^{(n)} = \prod_{i=1}^{n} X_i$. Let $\rho^{(n)}$ be the metric on $X^{(n)}$ defined by $\rho^{(n)}(x, y) = \sum_{i=1}^{n} 2^{-i} \rho_i(x_i, y_i)/[1 + \rho_i(x_i, y_i)]$. For $A \subseteq X^{(n)}$, define $d^{(n)}(A)$ to be the diameter of A with respect to the metric $\rho^{(n)}$. Let $R = \prod_{i=1}^{n} R_i$, where $R_i \in \mathcal{B}(X_i)$, $\prod_{i=1}^{n} \mu_i(R_i) < +\infty$, are arbitrary. Then for every $\delta > 0$, there exists a disjoint sequence (D_j) of subsets $D_j \subseteq X^{(n)}$ such that the following conditions hold:*

- (i) $R = \bigcup_{j=1}^{\infty} D_j$, *and for each* j, $D_j = \prod_{i=1}^{n} D_{ij}$, *where* $D_{ij} \in \mathcal{B}(X_i)$ *for* $i = 1, \ldots, n$;
- (ii) $d^{(n)}(D_j) < \delta$ *for* $i = 1, \ldots, n$;
- (iii) $\prod_{i=1}^{n} \mu_i(R_i) = \sum_{j=1}^{\infty} \prod_{i=1}^{n} \mu_i(D_{ij}) = \sum_{j=1}^{\infty} \mu^{(n)}(D_j)$.

Proof. For each $1 \leq i \leq n$, property (iii) of the introduction implies that there exists a sequence (A_{ij}) of Borel subsets of X_i such that $X_i = \bigcup_{j=1}^{\infty} A_{ij}$, $A_{ij} \in \mathcal{B}(X_i)$, and $d_i(A_{ij}) < \delta/n$. For $1 \leq i, j < +\infty$, set $B_{ij} = A'_{i1} \cap \cdots \cap A'_{ij-1} \cap A_{ij}$, $D_{ij} = B_{ij} \cap R_i$. Then for each $1 \leq i \leq n$, $1 \leq j < +\infty$, we have $d_i(D_{ij}) < \delta/n$, $R_i = \bigcup_{j=1}^{\infty} D_{ij}$, and each D_{ij} is a Borel subset of X_i. Now, for $1 \leq j_1, \ldots, j_n < +\infty$, define $D_{j_1,\ldots,j_n} = D_{1j_1} \times \cdots \times D_{nj_n}$. Then the $D_{j_1,\ldots,j_n}$ are disjoint, and $d^{(n)}(D_{j_1,\ldots,j_n}) < \delta$. Moreover, we have

$$\prod_{i=1}^{n} \mu_i(R_i) = \mu^{(n)}(R) = \sum_{j_1=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} \mu^{(n)}(D_{j_1,\ldots,j_n}) = \sum_{j_1=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} \prod_{i=1}^{n} \mu_i(D_{ij_i}).$$

This proves the lemma. □

Lemma 2.4. *For each $R \in \mathcal{R}$, we have $\tau^*(R) = \nu^*(R)$.*

Proof. Write $R = \prod_{i=1}^{\infty} R_i$. By Lemma 2.2, $\tau^*(R) = \tau(R)$. It is clear that $\tau^*(R) \leq \nu^*(R)$. Hence it will suffice to prove that $\nu^*(R) \leq \tau(R)$. To this end, let $\delta, \epsilon > 0$ be arbitrary. Select n so large that $\sum_{i=n+1}^{\infty} 2^{-i} < \delta/2$. By Lemma 2.3, we can write $R^{(n)} = \prod_{i=1}^{n} R_i = \bigcup_{j=1}^{\infty} D_j$, where each D_j has the form $D_j = \prod_{i=1}^{n} D_{ij}$, $D_{ij} \in \mathcal{B}(X_i)$, with $d^{(n)}(D_j) < \delta/2$ and $\mu^{(n)}(R^{(n)}) = \sum_{j=1}^{\infty} \mu^{(n)}(D_j)$. Define $B_j = D_j \times \prod_{i=n+1}^{\infty} R_i$, $1 \leq j < +\infty$. Then $R = \bigcup_{j=1}^{\infty} B_j$, and for all j, $d(B_j) < \delta$. We now consider the following two cases.

Case I. In this case, there exists an i such that $\mu_i(R_i) = 0$, and therefore $\tau(R) = \lim_p \prod_{i=1}^{p} \mu_i(R_i) = 0$. We may assume that n is so large that $\mu^{(n)}(R^{(n)}) = 0$. Then for each j, $B_j \in \mathcal{R}$, with $\tau(B_j) = 0$. By definition, $\tau_\delta^*(R) \leq \sum_{j=1}^{\infty} \tau(B_j) = 0 = \tau(R)$. Hence, because $\delta > 0$ is arbitrary, we have $\nu^*(R) \leq \tau(R)$.

Case II. In this case, $\mu_i(R_i) > 0$ for all i. Therefore, $\prod_{i=n+1}^{\infty} \mu_i(R_i) < +\infty$. Hence, $\tau_\delta^*(R) \leq \sum_{j=1}^{\infty} \tau(B_j) = \tau(R)$. We conclude that $\nu^*(R) \leq \tau(R)$. This completes the proof of the lemma. □

Proof of Theorem I. Lemmas 2.2 and 2.4 prove the first part of Theorem I. The invariance statements in Theorem I easily follow from the definition of μ. Finally, the non-σ-finiteness statement in Theorem I is a consequence of [Rg, Chapter 2, Theorem 58]. □

3. Proof of Theorem II

Lemma 3.1. *Let $\emptyset \neq \Omega \subset \mathbb{N}$, and let $\mu_\Omega, \mu_{\Omega'}$ be as in Theorem* II. *Define the mapping $\varphi : X \to X_\Omega \times X_{\Omega'}$ by $x \mapsto (x_\Omega, x_{\Omega'})$. Then φ maps X one-to-one onto $X_\Omega \times X_{\Omega'}$, such that $\varphi(\mathcal{M}) = \mathcal{M}_\Omega \otimes \mathcal{M}_{\Omega'}$. Moreover, for all $E \in \mathcal{M}$, we have $\mu(E) = \mu_\Omega \times \mu_{\Omega'}(\varphi(E))$. Consequently,*

$$\int_X f(x)\,d\mu(x) = \int_{X_\Omega \times X_{\Omega'}} f(x_\Omega, x_{\Omega'})\,d(\mu_\Omega \times \mu_{\Omega'})(x_\Omega, x_{\Omega'}),$$

for all μ-integrable functions f.

Proof. If σ^* is the set function on $X_\Omega \times X_{\Omega'}$ given by $\sigma^*(E) = \inf \sum_{j=1}^{\infty} \sigma(A_j \times B_j)$, the inf being taken over all sequences $(A_j \times B_j)$ of measurable rectangles in $X_\Omega \times X_{\Omega'}$, and $\sigma(A_j \times B_j) = \mu_\Omega(A_j) \cdot \mu_{\Omega'}(B_j)$, then σ^* is an outer measure such that $\mathcal{M}_\Omega \otimes \mathcal{M}_{\Omega'}$ is the family of σ^*-measurable sets, and $\mu_\Omega \times \mu_{\Omega'}$ is σ^* restricted to this family [Ry, Section 12.4]. The function φ is clearly bijective. We first show that for $E \subseteq X$, $\tau^*(E) = \sigma^*(\varphi(E))$. To this end, let $E \subseteq \bigcup_{j=1}^{\infty} R_j$, where $R_j \in \mathcal{R}$ and

$\sum_{j=1}^{\infty} \tau(R_j) \leq \tau^*(E)+\epsilon$, $\epsilon > 0$ arbitrary. For each j, write $R_j = \prod_{i=1}^{\infty} R_{ij}$. Then for all j, we have $\varphi(R_j) = R_\Omega \times R_{\Omega'}$. Consequently, $\sigma(\varphi(R_j)) = \mu_\Omega(R_\Omega)\cdot\mu_{\Omega'}(R_{\Omega'})$. Now, let $1 \leq j < +\infty$. If $\mu_i(R_{ij}) = 0$ for some i, then either $\mu_\Omega(R_\Omega) = 0$ or $\mu_{\Omega'}(R_{\Omega'}) = 0$, and consequently, $\sigma(\varphi(R_j)) = 0 = \tau(R_j)$. On the other hand, if j is such that $\mu_i(R_{ij}) > 0$ for all i, then $\sigma(\varphi(R_j)) = \prod_{i\in\Omega} \mu_i(R_{ij})\cdot \prod_{i\in\Omega'} \mu_i(R_{ij}) = \tau(R_j)$. Therefore, $\sigma^*(\varphi(E)) \leq \sum_{j=1}^{\infty} \sigma(\varphi(R_j)) \leq \tau^*(E)+\epsilon$. We conclude that $\sigma^*(\varphi(E)) \leq \tau^*(E)$.

To show that $\tau^*(E) \leq \sigma^*(\varphi(E))$, we may assume that $\sigma^*(\varphi(E)) < +\infty$. Let $\varphi(E) \subseteq \bigcup_{j=1}^{\infty} A_j \times B_j$, where $A_j \in \mathcal{M}_\Omega$ and $B_j \in \mathcal{M}_{\Omega'}$, with $\sum_{j=1}^{\infty} \sigma(A_j\times B_j) \leq \sigma^*(\varphi(E)) + \epsilon/2$, $\epsilon > 0$ arbitrary. For each j, we can find $A_{jk} \in \mathcal{R}_\Omega, B_{jk} \in \mathcal{R}_{\Omega'}$, $1 \leq k < +\infty$, such that $A_j \subseteq \bigcup_{k=1}^{\infty} A_{jk}$, $B_j \subseteq \bigcup_{k=1}^{\infty} B_{jk}$, and $\sum_{k,p=1}^{\infty} \tau_\Omega(A_{jk})\cdot \tau_{\Omega'}(B_{jp}) \leq \mu_\Omega(A_j)\cdot\mu_{\Omega'}(B_j)+\epsilon/2^{j+1}$ (the notation "R_Ω", "$R_{\Omega'}$", "τ_Ω", and "$\tau_{\Omega'}$", is self-explanatory). Now, for each triple j,k,p, we have $\tau(\varphi^{-1}(A_{jk}\times B_{jp})) = \tau_\Omega(A_{jk})\cdot\tau_{\Omega'}(B_{jp})$. Moreover, $E \subseteq \bigcup_{j,k,p=1}^{\infty} \varphi^{-1}(A_{jk}\times B_{jp})$. Therefore, we have

$$\tau^*(E) \leq \sum_{j,k,p=1}^{\infty} \tau(\varphi^{-1}(A_{jk}\times B_{jp})) \leq \sum_{j=1}^{\infty} \left[\mu_\Omega(A_j)\cdot\mu_{\Omega'}(B_j)+\epsilon 2^{-(1+j)}\right] \leq \sigma^*(\varphi(E))+\epsilon.$$

Thus, $\tau^*(E) \leq \sigma^*(\varphi(E))$, and we conclude that $\tau^*(E) = \sigma^*(\varphi(E))$. The equality $\tau^*(E) = \sigma^*(\varphi(E))$, $E \subseteq X$, easily implies the rest of the lemma. □

Proof of Theorem II. To prove Theorem II, let $\emptyset \neq \Omega \subset \mathbb{N}$. Then by Lemma 3.1, we may identify $(X, \mathcal{M}, \mu)$ with $(X_\Omega \times X_{\Omega'}, \mathcal{M}_\Omega \otimes \mathcal{M}_{\Omega'}, \mu_\Omega \times \mu_{\Omega'})$—throughout the remainder of this paper, we will make this identification. Then Theorem II follows from [Ry, 12.19]. □

Lemma 3.2. *Let* $R = \prod_{i=1}^{\infty} R_i \in \mathcal{R}$, *with* $\mu(R) > 0$. *Let* (f_i) *be a sequence of functions such that for each* i, $f_i \in L_1(X_i, \mathcal{M}_i, \mu_i)$, *and* $0 \leq f_i \leq \chi_{R_i}$. *Then* $f = \prod_{i=1}^{\infty} f_i \in L_1(X, \mathcal{M}, \mu)$, *with* $\int_R f\,d\mu = \prod_{i=1}^{\infty} \int_{R_i} f_i\,d\mu_i$.

Proof. By Theorem II and dominated convergence, we have

$$\lim_n \prod_{i=1}^{n} \int_{R_i} f_i\,d\mu_i = \lim_n \left[\int_R \prod_{i=1}^{n} f_i\,d\mu \Big/ \prod_{i=n+1}^{\infty} \mu_i(R_i)\right] = \int_R f\,d\mu.$$

□

4. Proof of Theorem III

Lemma 4.1 (Tonelli)**.** *Let* $\emptyset \neq \Omega \subset \mathbb{N}$. *Define* φ *as in Lemma* 3.1. *Let* $A \in \mathcal{M}_\Omega$ *be* σ*-finite with respect to* μ_Ω, *and let* $B \in \mathcal{M}_{\Omega'}$ *be* σ*-finite with respect to* $\mu_{\Omega'}$.

Let f be a nonnegative μ-measurable function on X. Then we have

(i) for μ_Ω-a.a. $x_\Omega \in X_\Omega$, the function f_{x_Ω} defined by $f_{x_\Omega}(x_{\Omega'}) = f(x_\Omega, x_{\Omega'})$ is a $\mu_{\Omega'}$-measurable function on $X_{\Omega'}$;

(i′) for $\mu_{\Omega'}$-a.a. $x_{\Omega'} \in X_{\Omega'}$, the function $f_{x_{\Omega'}}$ defined by $f_{x_{\Omega'}}(x_\Omega) = f(x_\Omega, x_{\Omega'})$ is a μ_Ω-measurable function on X_Ω;

(ii) $\int_A f(x_\Omega, x_{\Omega'})\, d\mu_{\Omega'}(x_{\Omega'})$ is a μ_Ω-measurable function on X_Ω;

(ii′) $\int_B f(x_\Omega, x_{\Omega'})\, d\mu_\Omega(x_\Omega)$ is a $\mu_{\Omega'}$-measurable function on $X_{\Omega'}$;

(iii) we have

$$\begin{aligned}\int_{\varphi^{-1}(A\times B)} f(x)\, d\mu(x) &= \int_{A\times B} f(x_\Omega, x_{\Omega'})\, d(\mu_\Omega \times \mu_{\Omega'}) \\ &= \int_A \left[\int_B f(x_\Omega, x_{\Omega'})\, d\mu_{\Omega'}(x_{\Omega'})\right] d\mu_\Omega(x_\Omega) \\ &= \int_B \left[\int_A f(x_\Omega, x_{\Omega'})\, d\mu_\Omega(x_\Omega)\right] d\mu_{\Omega'}(x_{\Omega'}).\end{aligned}$$

Proof. Conditions (i), (i′), (ii), and (ii′) are proved in the same way that the corresponding conditions are proved in [Ry, 12.20].Thus, we need only prove condition (iii). To this end, write $A = \bigcup_{j=1}^{\infty} A_j$, $B = \bigcup_{j=1}^{\infty} B_j$, where the A_j are disjoint and the B_j are disjoint, with $\mu_\Omega(A_j), \mu_{\Omega'}(B_j) < +\infty$ for all j. Then $A \times B = \bigcup_{j=1}^{\infty} A_j \times B_j$. Consequently, we may assume that $\mu_\Omega(A), \mu_{\Omega'}(B) < +\infty$. For each n, define $f_n(x) = f(x)$, if $f(x) \le n$, otherwise, $f_n(x) = 0$. By B. Levi's theorem and Theorem II,

$$\begin{aligned}\int_{\varphi^{-1}(A\times B)} f(x)\, d\mu(x) &= \lim_n \int_{A\times B} f_n(x_\Omega, x_{\Omega'})\, d(\mu_\Omega \times \mu_{\Omega'}) \\ &= \int_{A\times B} f(x_\Omega, x_{\Omega'})\, d(\mu_\Omega \times \mu_{\Omega'}) \\ &= \lim_n \int_A \left[\int_B f_n(x_\Omega, x_{\Omega'})\, d\mu_{\Omega'}(x_{\Omega'}\right] d\mu_\Omega(x_\Omega) \\ &= \int_A \left[\int_B f_(x_\Omega, x_{\Omega'})\, d\mu_{\Omega'}(x_{\Omega'}\right] d\mu_\Omega(x_\Omega).\end{aligned}$$

The rest of the lemma is proved in a similar manner. □

Lemma 4.2. *The family of functions of the form $\chi_R, R \in \mathcal{R}$, has dense linear span in $L_p(X, \mathcal{M}, \mu)$, $1 \le p < +\infty$.*

Proof. Because functions of the form χ_A, $A \in \mathcal{M}$, $\mu(A) < +\infty$, have dense linear span in $L_p(X, \mathcal{M}, \mu)$, it suffices to show that any function of this form can be approximated arbitrarily close by functions of the form χ_R, $R \in \mathcal{R}$. To this end, let $\epsilon > 0$ be arbitrary, and select $R_j \in \mathcal{R}$ such that $A \subseteq B := \bigcup_{j=1}^{\infty} R_j$ and $\sum_{j=1}^{\infty} \tau(R_j) < \mu(A) + [\epsilon/2]^p$. Then we have $\|\chi_B - \chi_A\|_p = [\mu(B - A)]^{\frac{1}{p}} < \epsilon/2$.

For $1 \leq j < +\infty$, define $C_j = R_1' \cap \cdots \cap R_{j-1}' \cap R_j$. Then the C_j are disjoint, with $B = \bigcup_{j=1}^{\infty} C_j$. Now, for each j, $C_j = (1 - \chi_{R_1}) \cdots (1 - \chi_{R_{j-1}})\chi_{R_j}$. Therefore, χ_{C_j} is a linear combination of functions of the form $\chi_{R_{j_1}} \cdots \chi_{R_{j_k}} = \chi_{R_{j_1} \cap \cdots \cap R_{j_k}}$, where $j_1, \ldots, j_k$ are positive integers. But each set of the form $R_{j_1} \cap \cdots \cap R_{j_k}$ is in $\mathcal{R}$. Hence each function χ_{C_j} is a linear combination of functions of the form χ_R, $R \in \mathcal{R}$.

Select n so large that $\int_X \sum_{j=n+1}^{\infty} \chi_{C_j}\, d\mu = \mu\left(\bigcup_{j=1}^{\infty} C_j\right) < [\epsilon/2]^p$. Then we have $\| \sum_{j=1}^{n} \chi_{C_j} - \chi_A \|_p < \epsilon$. Therefore, with $f = \sum_{j=1}^{n} \chi_{C_j}$, we see that $f \in \left\langle \chi_R \mid R \in \mathcal{R} \right\rangle$ and $\|f - \chi_A\|_p < \epsilon$. $\square$

Lemma 4.3. *Let $\emptyset \neq \Omega \subset \mathbb{N}$, and let $1 \leq p < +\infty$. Let $R \in \mathcal{R}$, with $\mu(R) > 0$. Suppose that $f \in L_p(X, \mathcal{M}, \mu)$. For $(x_\Omega, x_{\Omega'}) \in X_\Omega \times X_{\Omega'}$, define*

$$f_\Omega(x_\Omega, x_{\Omega'}) = \int_{R_{\Omega'}} f(x_\Omega, y_{\Omega'})\, d\mu_{\Omega'}(y_{\Omega'}), \quad f_{\Omega'}(x_\Omega, x_{\Omega'}) = \int_{R_\Omega} f(u_\Omega, x_{\Omega'})\, d\mu_\Omega(u_\Omega).$$

Then we have

$$\int_{R_\Omega \times R_{\Omega'}} |f_\Omega|^p\, d(\mu_\Omega \times \mu_{\Omega'}) \leq \left[\mu_\Omega(R_\Omega)\right]^2 \cdot \|f\|_p^p,$$
$$\int_{R_\Omega \times R_{\Omega'}} |f_{\Omega'}|^p\, d(\mu_\Omega \times \mu_{\Omega'}) \leq \left[\mu_{\Omega'}(R_{\Omega'})\right]^2 \cdot \|f\|_p^p.$$

Proof. Using Hölder's inequality, we see that $\int_R |f|\, d\mu < +\infty$. Thus, by Theorem II, the function

$$x_\Omega \mapsto \int_{R_{\Omega'}} \chi_{R_\Omega}(x_\Omega) f(x_\Omega, y_{\Omega'})\, d\mu_{\Omega'}(y_{\Omega'}) = \chi_{R_\Omega}(x_\Omega) f_\Omega(x_\Omega, y_{\Omega'})$$

is in $L_p(X_\Omega, \mathcal{M}_\Omega, \mu_\Omega)$. In particular, this function is $\mathcal{M}_\Omega$-measurable, and because the function $(x_\Omega, x_{\Omega'}) \mapsto \chi_{R_\Omega}(x_\Omega) f_\Omega(x_\Omega, x_{\Omega'})$ does not depend on $x_{\Omega'}$, this latter function is clearly $\mathcal{M}$-measurable. Therefore, by Theorem II and Tonelli's theorem (Lemma 4.1), we have

$$\begin{aligned}
\int_{R_\Omega \times R_{\Omega'}} |f_\Omega|^p\, d(\mu_\Omega \times \mu_{\Omega'}) &= \int_{R_{\Omega'}} \left[\int_{R_\Omega} \left| \int_{R_{\Omega'}} f(x_\Omega, z_{\Omega'})\, d\mu_{\Omega'} \right|^p d\mu_\Omega \right] d\mu_{\Omega'} \\
&\leq \mu_{\Omega'}(R_{\Omega'}) \cdot \int_{R_{\Omega'}} \left[\int_{R_\Omega} \int_{R_{\Omega'}} |f(x_\Omega, z_{\Omega'})|^p\, d\mu_{\Omega'} d\mu_\Omega \right] d\mu_{\Omega'} \\
&= \mu_{\Omega'}(R_{\Omega'}) \cdot \int_{R_{\Omega'}} \left[\int_{R_\Omega \times R_{\Omega'}} |f(x_\Omega, z_{\Omega'})|^p\, d(\mu_\Omega \times \mu_{\Omega'}) \right] d\mu_{\Omega'} \\
&\leq \left[\mu_{\Omega'}(R_{\Omega'})\right]^2 \cdot \|f\|_p^p.
\end{aligned}$$

The rest of the lemma is proved in a similar manner. $\square$

Proof of Theorem III. To prove Theorem III, let $R \in \mathcal{R}$, with $\mu(R) > 0$, $R = \prod_{i=1}^{\infty} R_i$. We first prove the theorem for functions of the form $f = \chi_S$, $S \in \mathcal{R}$. To this end, let $S = \prod_{i=1}^{\infty} S_i \in \mathcal{R}$. Define $f = \chi_S$.

Case I. In this case, $\prod_{i=1}^{\infty} \mu_i(R_i \cap S_i) = 0$. Then $\int_R f\,d\mu = 0$ and $\lim_\Omega f_{\Omega'}(x_\Omega, x_{\Omega'}) = 0$ for all $(x_\Omega, x_{\Omega'}) \in X_\Omega \times X_{\Omega'}$. Therefore, $\lim_\Omega \int_R \left|f_{\Omega'} - \int_R f\,d\mu\right|^p = 0$. We have

$$\int_R |f_\Omega - f|^p\,d\,u = \int_R \chi_{S_\Omega}(x_\Omega)|\mu_{\Omega'}(R_{\Omega'}) \cap S_{\Omega'} - \chi_{S_{\Omega'}}(x_{\Omega'})|^p\,d\mu$$
$$\leq [\mu_{\Omega'}(R_{\Omega'}) + 1]^p \cdot \mu_{\Omega'}(R_{\Omega'}) \cdot \prod_{i\in\Omega} \mu_i(R_i \cap S_i).$$

Thus, $\lim_\Omega \int_R |f_\Omega - f|^p\,d\mu = 0$.

Case II. In the case $\prod_{i=1}^{\infty} \mu_i(R_i \cap S_i) > 0$, note that the limit $\lim_{i=1}^{n} \mu_i(R_i \cap S_i)$ exists. Therefore, $\lim_\Omega \mu_{\Omega'}(R_{\Omega'} \cap S_{\Omega'}) = 1$. Then, by Theorem II, we have

$$\int_R \left|f_{\Omega'} - \int_R f\,d\mu\right|^p d\mu \leq [\mu_\Omega(R_\Omega \cap S_\Omega)]^p \cdot \Bigg\{[\mu_\Omega(R_\Omega)]^{\frac{1}{p}}$$
$$\cdot \left[\mu_{\Omega'}(R_{\Omega'}) - \mu_{\Omega'}(R_{\Omega'} \cap S_{\Omega'})\right]^{\frac{1}{p}} + \left[\int_R |1 - \mu_{\Omega'}(R_{\Omega'} \cap S_{\Omega'})|^p\,d\mu\right]^{\frac{1}{p}}\Bigg\}^p.$$

Therefore, $\lim_\Omega \int_R \left|f_\Omega - \int_R f\right|^p d\mu = 0$. A similar argument shows that $\lim_\Omega \int_R |f_\Omega - f|^p\,d\mu = 0$. Cases I and II show that Theorem III holds for functions of the form $f = \chi_S, S \in \mathcal{R}$.

To prove Theorem III for all $f \in L_p(X, \mathcal{M}, \mu)$, let $\mathcal{F}$ be the family of all $f \in L_p(X, \mathcal{M}, \mu)$ for which conditions (i) and (ii) hold. Then $\mathcal{F}$ is a linear space such that $\chi_S \in \mathcal{F}$ for all $S \in \mathcal{R}$. Hence by Lemma 4.2, $\mathcal{F}$ is dense in $L_p(X, \mathcal{M}, \mu)$. Now let $f \in L_p(X, \mathcal{M}, \mu)$, and let $\epsilon > 0$ be arbitrary. Then select $g \in \mathcal{F}$ such that $\|f - g\|_p < \epsilon/\{2\{[\mu(R)]^{\frac{2}{p}}\}$. For all functions h such that $\chi_R h \in L_p(X, \mathcal{M}, \mu)$, define $\|h\|_{p,R} = \left[\int_R |h|^p\,d\mu\right]^{\frac{1}{p}}$. Using Hölder's inequality and Lemma 4.3, we have

$$\left\|f_{\Omega'} - \int_R f\,d\mu\right\|_{p,R} \leq \{[\mu(R)]^{\frac{2}{p}} + [\mu_\Omega(R_\Omega)]^{\frac{2}{p}}\} \cdot \|f - g\|_p + \left\|g_{\Omega'} - \int_R g\,d\mu\right\|_{p,R}.$$

Therefore, $\lim_\Omega \left\|f_{\Omega'} - \int_R f\,d\mu\right\|_{p,R} \leq \epsilon$, and because $\epsilon > 0$ is arbitrary, we see that condition (i) of Theorem III holds for f. Condition (ii) of Theorem III for f is proved in a similar manner. □

5. Proof of Theorem IV

Lemma 5.1. *Let $1 \le p < +\infty$, let $R \in \mathcal{R}$, with $\mu(R) > 0$. For $f \in L_p(X, \mathcal{M}, \mu)$, let $f_\Omega, f_{\Omega'}$ be defined as in Lemma 4.3. Then the family of all those $f \in L_p(X, \mathcal{M}, \mu)$ for which the following conditions hold is dense in $L_p(X, \mathcal{M}, \mu)$:*

$$(5.1)\quad \lim_{m,n} |f_{\Omega_n}(x) - f_{\Omega_m}(x)| = 0, \quad x \in X; \quad \lim_{m,n} |f_{\Omega'_n}(x) - f_{\Omega'_m}(x)| = 0, \quad x \in X.$$

Proof. By Lemma 4.2, it suffices to prove (5.1) for functions of the form χ_S, $S \in \mathcal{R}$. To this end, let $f = \chi_S$, $S \in \mathcal{R}$. We will prove (5.1) for f_{Ω_n}; the proof for $f_{\Omega'_n}$ is similar. For all n and all $x \in X$, we have $f_{\Omega_n}(x) = \chi_{S_{\Omega_n}}(x_{\Omega_n})\mu_{\Omega'_n}(R_{\Omega'_n} \cap S_{\Omega'_n})$. Fix $x \in X$. If $x \in S$, then for all n, $f_{\Omega_n}(x) = \mu_{\Omega'_n}(R_{\Omega'_n})$, and hence (5.1) holds. If there exists $x \notin S$, then for sufficiently large n, we have $f_{\Omega_n}(x) = 0$, and consequently, (5.1) still holds. □

Theorem 5.2. *Let $R \in \mathcal{R}$, with $\mu(R) > 0$. For $f \in L_1(X, \mathcal{M}, \mu)$ and $x \in X$, let $g_n(x)$ be either*

$$\left| \int_{R_{\Omega'_n}} f(x_{\Omega_n}, y_{\Omega'_n})\, d\mu_{\Omega'_n}(y_{\Omega'_n}) \right| \textit{ or } \left| \int_{R_{\Omega_n}} f(u_{\Omega_n}, x_{\Omega'_n})\, d\mu_{\Omega_n}(u_{\Omega_n}) \right|.$$

For all $\delta > 0$, let $A_\delta = \{\, x \in R \mid \sup_n g_n(x) > \delta \,\}$. Then we have $\delta/M\mu(A_\delta) \le \int_{A_\delta} |f(x)|\, d\mu(x)$, where $M = \max_n\{\, \mu_{\Omega_n}(R_{\Omega_n}), \mu_{\Omega'_n}(R_{\Omega'_n}) \,\}$.

Proof. Without loss of generality, we may assume that $f \ge 0$. Define

$$B_j = \{\, x \mid \sup_{1 \le n \le j} g_n(x) > \delta, 1 \le n \le j \,\}, \quad C_j = B'_1 \cap \cdots \cap B'_{j-1} \cap B_j.$$

Then the C_j are disjoint and $A_\delta = \bigcup_{j=1}^{\infty} C_j$. Assume that g_n is the first of the above functions. Then for each j, the function χ_{C_j} is independent of $x_{\Omega'_j}$. Consequently, we have, by Fubini's theorem,

$$\begin{aligned}\int_{C_j} f(x)\, d\mu(x) &= \int_{R_{\Omega_j}} \chi_{C_j}(x_{\Omega_j}, x_{\Omega'_j}) g_j(x_{\Omega_j}, x_{\Omega'_j})\, d\mu_{\Omega_j}(x_{\Omega_j}) \\ &= \int_{R_{\Omega'_j}} g_j(x_{\Omega_j}, x_{\Omega'_j})\, d\mu_{\Omega'_j}(x_{\Omega_j}) \Big/ \mu_{\Omega'_j}(R_{\Omega'_j}) \ge \frac{\delta\mu(C_j)}{M}.\end{aligned}$$

It follows that $\int_{A_\delta} f(x)\, d\mu(x) \ge \delta\mu(A_\delta)/M$. The same type of argument applies to the other choice of g_n. □

Proof of Theorem IV. Let $\delta > 0, \epsilon > 0$ be arbitrary. Then by Lemma 5.1, there exists $g \in L_1(X, \mathcal{M}, \mu)$ such that $\|f - g\|_1 < \epsilon$ and (5.1) holds for g. Define $r(x) = \lim_{q \to +\infty} \sup_{m,n > q} |f_{\Omega_n}(x) - f_{\Omega_m}(x)|$. Then we have $|f_{\Omega_n}(x) - f_{\Omega_m}(x)| \le$

$|g_{\Omega_n}(x) - g_{\Omega_m}(x)| + 2\sup_k |h_{\Omega_k}(x)|$, where $h = f - g$. Therefore, by (5.1), $r(x) \leq 2\sup_k |h_{\Omega_k}(x)|$. With $A_\delta = \{x \in R \mid \sup_n |h_{\Omega_n}(x)| > \delta\}$, it follows from Lemma 5.2 that

$$\mu(\{\, x \in R \mid r(x) > \delta \,\}) \leq \mu(A_\delta) \leq \left(\frac{M}{\delta}\right) \int_{A_\delta} |h| \, d\mu \leq \frac{M\epsilon}{\delta}.$$

Thus, $r(x) = 0$ μ-a.e. on R, which implies that $\lim_n f_{\Omega_n}(x)$ exists μ-a.e. on R. Then Theorem III and Fatou's lemma imply that $\int_R |\lim_n f_{\Omega_n} - f| \, d\mu = \lim_n \int_R |f_{\Omega_n} - f| \, d\mu = 0$, and consequently $f(x) = \lim_n f_{\Omega_n}(x)$ μ-a.e. on R. The remainder of Theorem IV is proved in a similar manner. □

As an application of Theorem IV, let $R \in \mathcal{R}$, with $\mu(R) > 0$. Let (f_n) be a sequence of functions in $L_1(X, \mathcal{M}, \mu)$ such that $f(x) = \prod_{i=1}^{n} f_i(x_i) = \lim_n \prod_{i=1}^{\infty} f_i(x_i)$ exists and is finite and nonzero for μ-a.a. $x \in X$. Suppose, moreover, that $f \in L_1(X, \mathcal{M}, \mu)$. Then Theorem IV easily implies that $\int_R f \, d\mu = \prod_{i=1}^{\infty} \int_{R_i} f_i \, d\mu_i$.

6. The canonical commutation relations

Lemma 6.1. *Let $1 \leq p < +\infty$, and let $\mathcal{C}_0^\infty(\mathbb{R})$ be the space of infinitely differentiable functions on $\mathbb{R}$ of compact support. Then the family of functions of the form $f = \prod_{i=1}^{\infty} f_i$, $f_i \in \mathcal{C}_0^\infty(\mathbb{R})$, has dense linear span in $L_p(\mathbb{R}^\infty, \mathcal{L}, \lambda)$.*

Proof. By Lemma 4.2, it suffices to prove that every function of the form $f = \chi_R$, $R \in \mathcal{R}$, can be approximated arbitrarily close by a function $g = \prod_{i=1}^{\infty} g_i$, $g_i \in \mathcal{C}_0^\infty(\mathbb{R})$ in $L_p(\mathbb{R}^\infty, \mathcal{L}, \lambda)$. But this proposition follows easily from Lemma 3.2 and the fact that for any open $V \subseteq \mathbb{R}$, if V has finite Lebesgue measure, then χ_V can be approximated arbitrarily close by functions $g \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $0 \leq g \leq \chi_V$. □

Theorem 6.2. *Let $\mathcal{H}$ be the Hilbert space $L_2(\mathbb{R}^\infty, \mathcal{L}, \lambda)$. For each j, define unbounded operators P_j, Q_j on $\mathcal{H}$ by*

$$(Q_j f)(x) = x_j f(x), \quad \mathbb{D}(Q_j) = \{\, f \in \mathcal{H} \mid x_j f \in \mathcal{H} \,\},$$
$$(P_j f)(x) = -i\frac{\partial}{\partial x_j} f(x), \quad \mathbb{D}(P_j) = \left\{\, f \in \mathcal{H} \,\middle|\, \frac{\partial}{\partial x_j} f \in \mathcal{H} \,\right\}.$$

Then P_j and Q_j are densely defined selfajoint operators on $\mathcal{H}$ that satisfy the **canonical commutation relations***: $[Q_j, Q_k] = [P_j, P_k] = 0$, $[Q_j, P_k] = i\delta_{jk}$, $1 \leq j, k < +\infty$.*

Proof. The fact that P_j and Q_j are densely defined is easily deducible from Theorem II, Lemma 3.2, and Lemma 6.1. It is clear that Q_j is a symmetric operator. By imitating a standard argument [P, III.4.5], we see that the Cayley transform $(Q_j - i)(Q_j + i)^{-1}$ is unitary. Therefore Q_j is selfadjoint.

For $f \in L_1(\mathbb{R}, dx)$, let $\hat{f}$ be the Fourier transform of f. Let $\mathcal{G}$ be the family of functions $f = \prod_{k=1}^{\infty} f_k$, $f_k \in \mathcal{C}_0^\infty(\mathbb{R})$, given in Lemma 6.1, and define $\mathcal{F}_j f$ by

$(\mathcal{F}_j f)(p) = \hat{f}_j(p_j) \prod_{k \neq j}^{\infty} f_k(p_k)$, $p \in \mathbb{R}^\infty$. Then, using Theorem II, Lemma 3.2, and Lemma 6.1, it is not hard to see that $\mathcal{F}_j$ extends to a unitary operator $U_j : \mathcal{H} \to \mathcal{H}$ such that for $f \in \mathcal{G}$, $P_j f = U_j^{-1} Q_j U_j f$, and hence, by standard results from operator theory [P, III.4.5], P_j is a densely defined selfadjoint operator on $\mathcal{H}$. □

Postscript

It has been brought to the author's attention that measures related to the construction in Theorem I have been studied since the 1940s. For example, Oxtoby [O] presents a theory of translation-invariant Borel measures on Polish spaces. The measures most closely related to the construction in Theorem I are presented in Elliott and Morse [EM], and Ritter and Hewitt [RH]. The measures constructed in [EM] are called *Elliott-Morse measures*, and theorems similar to Theorem I and Theorem II are known for Elliott-Morse measures that are based on a countable family of measures. Theorem III and Theorem IV apparently have no counterparts in the literature. The Elliott-Morse measures that are based on a countable family of measures are constructed with the use of the concept of *plus-product*. The plus-product of a countable family of nonnegative extended real numbers is defined as follows:

$$\prod_{k \in I}{}^{+} a_k = \Bigg(\prod_{\substack{k \in I \\ a_k \leq 1}} a_k \Bigg) \Bigg(\prod_{\substack{k \in I \\ a_k > 1}} a_k \Bigg).$$

On the other hand, the "infinite product" used in the present paper is defined as the following limit, provided that the limit exists as an extended real number:

$$\prod_{k=1}^{\infty} a_k = \lim_{n \to \infty} \prod_{k=1}^{n} a_k.$$

Note that this product may diverge to 0. The measure constructed in Theorem I may have the pathology of there being too many sets of infinite measure. Elliott-Morse measures may suffer from the same pathology, but the use of the plus-product produces less of this pathology [RH]. The following interesting passage appears in [RH]: This paper [EM] was written in the explicit but terse and uncompromising style, more easily accessible by a computer than by a human mind, that has become known as "Morse code." Few people have apparently read it. This is a pity, as the paper contains a wealth of information.

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