THE NUMBER OF HALL $\pi$-SUBGROUPS
OF A $\pi$-SEPARABLE GROUP

ALEXANDRE TURULL

(Communicated by Jonathan I. Hall)

Abstract. We observe a simple formula to compute the number $\nu_\pi(G)$ of Hall $\pi$-subgroups of a $\pi$-separable finite group $G$ in terms of only the action of a fixed Hall $\pi$-subgroup of $G$ on a set of normal $\pi'$-sections of $G$. As a consequence, we obtain that $\nu_\pi(K)$ divides $\nu_\pi(G)$ whenever $K$ is a subgroup of a finite $\pi$-separable group $G$. This generalizes a recent result of Navarro. In addition, our method gives an alternative proof of Navarro’s result.

1. Introduction

Let $G$ be a finite group, and let $\pi$ be a set of primes. We denote by $\nu_\pi(G)$ the number of Hall $\pi$-subgroups of $G$. In the case when $G$ is $\pi$-separable, we know that $\nu_\pi(G) > 0$, and $\nu_\pi(G) = [G : N_G(H)]$, where $H$ is any Hall $\pi$-subgroup of $G$. This formula involves the calculation of a normalizer, and it does not immediately allow us to compare $\nu_\pi(G)$ to $\nu_\pi(K)$ for a subgroup $K$ of $G$. We propose the following formula, which uses only the calculation of some centralizers, and provides an obvious way to relate $\nu_\pi(G)$ and $\nu_\pi(K)$. It appears not to have been explicitly observed before.

Theorem 1.1. Let $\pi$ be a set of primes, and let $G$ be a finite $\pi$-separable group. Let $H$ be a Hall $\pi$-subgroup of $G$, and let

$$1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_\ell = G$$

be a sequence of normal subgroups of $G$ such that $N_i/N_{i-1}$ is either a $\pi$-group or a $\pi'$-group for $i = 1, \ldots, \ell$. Let $\mathcal{F}$ be the set of $N_i/N_{i-1}$ that are $\pi'$-groups. Then,

$$\nu_\pi(G) = \prod_{F \in \mathcal{F}} [F : C_F(H)].$$

As a consequence, we obtain the following corollary, which is a generalization of the main result of Navarro [1]. Our proof of the corollary is based on our theorem, and provides an alternate proof of Navarro’s result.

Corollary 1.2. Let $\pi$ be a set of primes, let $G$ be a finite $\pi$-separable group, and let $K$ be a subgroup of $G$. Then $\nu_\pi(K)$ divides $\nu_\pi(G)$.
2. Proofs

Proof of Theorem 1.1 Assume the theorem is false. Among all counterexamples, choose one with \( \ell \) as small as possible. Since the result holds for \( \ell = 0 \), and it is well known if \( \ell \leq 2 \), we have \( \ell > 2 \). Set \( N = N_1 \). Let \( \mathcal{H} \) be the set of Hall \( \pi \)-subgroups of \( G \). Define an equivalence relation on \( \mathcal{H} \) by, for \( H_1, H_2 \in \mathcal{H} \), \( H_1 \sim H_2 \) if and only if \( H_1 N = H_2 N \). The equivalence classes are in one-to-one correspondence with the groups \( H_1 N / N \), that is, the Hall \( \pi \)-subgroups of \( G / N \). Hence, the number of equivalence classes in \( \mathcal{H} \) is exactly \( \nu_\pi(G / N) \). Since the elements of \( \mathcal{H} \) are all conjugate in \( G \), and \( N \) is a normal subgroup of \( G \), the groups \( H_1 N \) for \( H_1 \in \mathcal{H} \) are all isomorphic to \( H N \). In particular, \( \nu_\pi(H_1 N) = \nu_\pi(HN) \) does not depend on \( H_1 \in \mathcal{H} \). It follows that \[
\nu_\pi(H) = |\mathcal{H}| = \nu_\pi(G / N) \nu_\pi(HN).
\]

By our choice of \( \ell \), the theorem holds for both \( G / N \) and \( HN \), and this implies that the theorem holds for \( G \). This contradiction completes the proof of the theorem. \( \square \)

The following lemma appears in slightly different form in [1]. We include a proof for completeness.

Lemma 2.1. Suppose \( H \) is a finite group, acting on the finite group \( F \), and assume that \( |H| \) and \( |F| \) are relatively prime. Suppose that \( K \) is an \( H \)-invariant subgroup of \( F \). Then \( [K : C_K(H)] \) divides \( |F : C_F(H)| \).

Proof. Assume the lemma is false, and choose a counterexample with \( |F| \) as small as possible, and among all such, with \( |F : K| \) as small as possible. Since \( |F| \) and \( |H| \) are relatively prime, for each prime \( p \), the \( H \)-invariant Sylow \( p \)-subgroups of \( F \) (respectively of \( K \)) are the maximal \( H \)-invariant \( p \)-subgroups of \( F \) (respectively of \( K \)), and they are conjugate by elements of \( C_F(H) \) (respectively by elements of \( C_K(H) \)). Hence, for each prime \( p \), we may take some \( H \)-invariant Sylow \( p \)-subgroup \( P \) of \( F \) that contains an \( H \)-invariant Sylow \( p \)-subgroup of \( K \). This implies that \( P \) also contains a Sylow \( p \)-subgroup of \( C_F(H) \), and a Sylow \( p \)-subgroup of \( C_K(H) \). Hence, \( P, C_P(P) \), \( P \cap K \) and \( K \cap C_P(H) \) are respectively Sylow \( p \)-subgroups of \( F, C_F(H) \), \( K \) and \( C_K(H) \). It follows that the \( p \)-part of \( [K : C_K(H)] \) is \( [P \cap K : P \cap C_K(H)] \), and the \( p \)-part of \( [F : C_F(H)] \) is \( [P : P \cap C_F(H)] \). Hence, the minimality of our counterexample implies that \( F \) is a \( p \)-group for some prime \( p \). The minimality of \( [F : K] \) in our counterexample implies that \( K \) is maximal among the \( H \)-invariant subgroups of \( F \). Let \( \Phi \) be the Frattini subgroup of \( F \). Since \( K \) is a proper subgroup, \( K \Phi \) is a proper \( H \)-invariant subgroup of \( F \), and it follows that \( \Phi \subseteq K \). Hence, \( K \) is a normal subgroup of \( F \). It follows that \( H \) acts on the group \( F / K \) and that \( C_F(H) / C_K(H) \) is isomorphic to the centralizer of the action of \( H \) on \( F / K \). This implies that \( [C_F(H) : C_K(H)] \) divides \( |F : K| \). Hence the conclusion of the lemma holds. This final contradiction completes the proof of the lemma. \( \square \)

Proof of Corollary 1.2 Let \( H \) be a Hall \( \pi \)-subgroup of \( G \) that contains a Hall \( \pi \)-subgroup of \( K \). Hence, \( K \cap H \) is a Hall \( \pi \)-subgroup of \( K \). Let \[
1 = N_0 \lhd N_1 \lhd \cdots \lhd N_\ell = G
\]
be a sequence of normal subgroups of \( G \) such that \( N_i / N_{i-1} \) is either a \( \pi \)-group or a \( \pi' \)-group for \( i = 1, \ldots, \ell \). Let \( \mathcal{F} \) be the set of \( N_i / N_{i-1} \) that are \( \pi' \)-groups. Then,
by Theorem 1.1, we have
\[(2.1) \quad \nu_\pi(G) = \prod_{F \in \mathcal{F}} [F : C_F(H)].\]

Furthermore, \(\nu_\pi(K)\) can be computed in a similar way by considering the action of \(H \cap K\) on the appropriate quotients of the sequence of normal subgroups of \(K\),
\[1 = N_0 \cap K \triangleleft N_1 \cap K \triangleleft \cdots \triangleleft N_t \cap K = K.\]

Notice that \((N_i \cap K)/(N_{i-1} \cap K)\) is \((H \cap K)\)-isomorphic to \((N_i \cap K)/N_{i-1}/N_{i-1}\), an \((H \cap K)\)-invariant subgroup of \(N_i/N_{i-1}\). For each \(F \in \mathcal{F}\), we set \(K(F)\) to be \((N_i \cap K)/N_{i-1}/N_{i-1}\). Then we have
\[\nu_\pi(K) = \prod_{F \in \mathcal{F}} [K(F) : C_{K(F)}(H \cap K)].\]

It then follows from Lemma 2.1 that \(\nu_\pi(K)\) divides
\[\prod_{F \in \mathcal{F}} [F : C_F(H \cap K)].\]

Since this clearly divides our expression (2.1) for \(\nu_\pi(G)\) above, the corollary follows. \(\square\)

REFERENCES


Department of Mathematics, University of Florida, Gainesville, Florida 32611-8105
E-mail address: turull@math.ufl.edu

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use