PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 9, Pages 2647–2654
S 0002-9939(04)07420-9
Article electronically published on April 21, 2004

ON CRITERIA FOR EXTREMALITY OF TEICHMÜLLER MAPPINGS

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(Communicated by Juha M. Heinonen)

ABSTRACT. Let f be a Teichmüller self-mapping of the unit disk Δ corresponding to a holomorphic quadratic differential φ. If φ satisfies the growth condition $A(r,\varphi)=\iint_{|z|<r}|\varphi|dxdy=O((1-r)^{-s})$ (as $r\to 1$), for any given $s>0$, then f is extremal, and for any given $a\in(0,1)$, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that

$$\left\{\frac{\varphi(a^{1/2^{n_k}}z)}{\iint_\Delta|\varphi(a^{1/2^{n_k}}z)|dxdy}\right\}$$

is a Hamilton sequence. In addition, it is shown that there exists φ with bounded Bers norm such that the corresponding Teichmüller mapping is not extremal, which gives a negative answer to a conjecture by Huang in 1995.

1. INTRODUCTION

Let Δ be the unit disk $\{|z|<1\}$ in the complex plane $\mathbb{C}$. Suppose g is a quasiconformal self-mapping of Δ. We denote by $Q(g)$ the class of all quasiconformal self-mappings of Δ that agree with g on the boundary $\partial\Delta$. A quasiconformal mapping $f_0\in Q(g)$ is said to be an extremal mapping for the boundary values corresponding to $h=g|_{\partial\Delta}$ if it minimizes the maximal dilatations of $Q(g)$, i.e.,

$$K[f_0]=\inf\{K[f]:\ f\in Q(g)\},$$

where $K[f]$ is the maximal dilatation of f.

A quasiconformal mapping $f(z)$ of Δ is called a Teichmüller mapping if f has the complex dilatation of the form

$$\mu_f(z)=\frac{f_{\bar z}}{f_z}=k\frac{\overline{\varphi(z)}}{|\varphi(z)|}\quad(0<k<1), \tag{1.1}$$

where $\varphi\not\equiv 0$ is a holomorphic function in Δ and k is a constant. It is of interest to know whether f is extremal or, in particular, uniquely extremal among $Q(f)$.

Let $B(\Delta)=\{\phi:\ \text{holomorphic in }\Delta\text{ with the norm }\|\phi\|=\iint_\Delta|\phi(z)|dxdy<\infty\}$. A necessary and sufficient condition that f is extremal is that [7] there exists

Received by the editors December 3, 2002 and, in revised form, June 5, 2003.
2000 *Mathematics Subject Classification.* Primary 30C75.
Key words and phrases. Hamilton sequence, Teichmüller mapping, extremality.
This research was supported by the "973" Project Foundation of China (Grant No. TG199075105) and the Foundation for Doctoral Programme.

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a so-called Hamilton sequence, namely, a sequence $\{\phi_n \in B(\Delta):\ \|\phi_n\|=1\}$, such that

$$\lim_{n\to\infty}\iint_\Delta k\frac{\overline{\varphi(z)}}{|\varphi(z)|}\phi_n(z)dxdy=k. \tag{1.2}$$

For the convenience of subsequent discussion, we define

$$m(r,\varphi):=\frac{1}{2\pi}\int_0^{2\pi}|\varphi(re^{i\theta})|d\theta$$

for any holomorphic function φ in Δ.

In this paper, we pay more attention to the problem: when does μ_f of a Teichmüller mapping f have a Hamilton sequence such as $\{\varphi(R_nz)/\|\varphi(R_nz)\|\}$, $R_n\in(0,1)$, $\lim_{n\to\infty}R_n=1$, and hence f is extremal?

The problem has been investigated by many authors including Reich and Strebel [7], Hayman and Reich [1], Reich [6], Huang [2], Wu and Lai [8], and Yao [10].

For example, Reich and Strebel [7] proved

Theorem A. *If $\varphi(z)$ satisfies the growth condition*

$$m(r,\varphi)=O(\frac{1}{1-r}),\ r\to 1, \tag{1.3}$$

then the putative sequence $\varphi(Rz)/\|\varphi(Rz)\|$, $R\uparrow 1$ is a Hamilton sequence of μ_f and hence f is extremal. Moreover, the extremality of f is no longer implied if $O((1-r)^{-1})$ is replaced by $O((1-r)^{-s})$, for any $s>1$.

Furthermore, Hayman and Reich [1] proved that f is also uniquely extremal if $\varphi(z)$ satisfies the growth condition (1.3).

Up to the present, the best growth condition for extremality, due to Wu and Lai [8], is as follows.

Theorem B. *Suppose $\varphi(z)$ satisfies the following growth condition:*

$$m(r,\varphi)=o(\frac{1}{(1-r)^s}),\quad r\to 1 \text{ for any given } s>1. \tag{1.4}$$

Then there exists a sequence $\{R_n\}$, $0<R_n<1$, $\lim_{n\to\infty}R_n=1$, such that $\{\frac{\varphi(R_nz)}{\|\varphi(R_nz)\|}\}$ is a Hamilton sequence, and hence f is extremal.

In [3], Lai and Wu conjectured that (1.4) is the best possible growth condition for the extremality. But our Theorem 1 below says that their conjecture is not true in the precise sense.

Theorem 1. *Set $\Delta_r=\{z\in\Delta:|z|<r<1\}$,*

$$A(r,\varphi)=\iint_{\Delta_r}|\varphi(z)|dxdy=\int_0^r tdt\int_0^{2\pi}|\varphi(te^{i\theta})|d\theta.$$

Suppose φ satisfies the growth condition:

$$A(r,\varphi)=O(\frac{1}{(1-r)^s}),\quad r\to 1 \text{ for any given } s>0. \tag{1.5}$$

Then for any given $a\in(0,1)$, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\frac{\varphi(a^{1/2^{n_k}}z)}{\|\varphi(a^{1/2^{n_k}}z)\|}\}$ is a Hamilton sequence of μ_f and hence f is extremal.

It is clear that (1.4) implies (1.5), but the converse is not true, i.e., (1.5)$\not\Rightarrow$ (1.4).

Let $BQD(\Delta)$ denote the Banach space consisting of all φ holomorphic in Δ with the Bers norm

$$\|\varphi\|_{\Delta} = \sup_{z\in\Delta} |\rho^{-1}(z)\varphi(z)| < \infty, \tag{1.6}$$

where $\rho(z)|dz|^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$ is the Poincaré metric on Δ.

In [2], Huang posed the following conjecture.

Extremal Conjecture. *If φ belongs to $BQD(\Delta)$, then every Teichmüller mapping corresponding to φ is extremal.*

As far as we know at present, any φ belonging to $BQD(\Delta)$ corresponds to an extremal Teichmüller mapping. There are even φ with unbounded Bers norms corresponding to extremal Teichmüller mappings (see an example in Section 2). It seems that the conjecture is true. However, our Theorem 2 gives a negative answer to it.

Theorem 2. *Let Γ be the covering transformation group of a hyperbolic finite type Riemann surface. Then for any φ in $BQD(\Delta,\Gamma)\backslash\{0\}$, the Teichmüller mapping corresponding to φ is not extremal, where $BQD(\Delta,\Gamma) = \{\varphi \in BQD(\Delta) : \varphi(z) = \varphi(\gamma(z))\gamma'^2(z),$ for all $\gamma \in \Gamma\}$.*

2. Proof of Theorem 1

Theorem 1 is an immediate corollary of the following theorem:

Theorem 3. *Suppose there exists some $a \in (0,1)$ such that*

$$A(\frac{1+a_n}{2},\varphi) = O((1-a_n)^{-s}), \text{ as } n\to\infty, \text{ for any given } s>0, \tag{2.1}$$

where $a_n = a^{1/2^n}$. Then there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\varphi(a_{n_k}z)/\|\varphi(a_{n_k}z)\|\}$ is a Hamilton sequence and hence f is extremal.

Theorem 3 indicates that a discrete growth condition (2.1) of φ is sufficient to induce f extremal. Meanwhile, it also makes clear that a best possible growth condition on φ for extremality can hardly be given.

The main idea of the proof of Theorem 3 comes from [8]. We need some preparation before proving it.

For $\frac{1}{2} < t^2 < t < 1$, we write

$$\frac{\iint_\Delta (\overline{\varphi(z)}/|\varphi(z)|)\varphi(tz)dxdy}{\iint_\Delta |\varphi(tz)|} = t^2 + t^2\frac{\beta(t)+\gamma(t)}{\alpha(t)}, \tag{2.2}$$

where

$$\alpha(t) = A(t,\varphi),\ \beta(t) = \iint_{t<|z|<1} \frac{\overline{\varphi(z)}}{|\varphi(z)|}\varphi(tz)dxdy,$$

$$\gamma(t) = \iint_{\Delta_t} \frac{\overline{\varphi(z)}}{|\varphi(z)|}[\varphi(tz)-\varphi(z)]dxdy.$$

Claim 1. $m(\varrho,\varphi')=\frac{1}{2\pi}\int_0^{2\pi}|\varphi'(\varrho e^{i\theta})|d\theta\le\frac{R}{R^2-\varrho^2}m(\varphi,R)$, where $R=\frac{1+\varrho}{2}$.

Actually, by the Cauchy formula

$$\varphi'(\varrho e^{i\theta})=\frac{1}{2\pi i}\int_{|z|=R}\frac{\varphi(z)}{(z-\varrho e^{i\theta})^2}dz=\frac{R}{2\pi}\int_0^{2\pi}\frac{\varphi(Re^{i(t+\theta)})e^{i(t-\theta)}}{(Re^{it}-\varrho)^2}dt,$$

we obtain

$$m(\varrho,\varphi')\le\frac{1}{2\pi}\int_0^{2\pi}\frac{R\,m(R,\varphi)}{R^2-2R\varrho\cos t+\varrho^2}dt=\frac{R\,m(R,\varphi)}{R^2-\varrho^2}. \tag{2.3}$$

Claim 2. Set $\zeta(t)=\int_{\frac12}^t m(r,\varphi)dr$, $\eta(t)=\int_{\frac12}^{\frac{1+t}{2}}\frac{1+t-2R}{1-R}m(R,\varphi)dR$, $\varepsilon(t)=\eta(t)-\eta(t^2)$. Then we have

$$\alpha(t)\ge\pi\zeta(t), \tag{2.4}$$

$$|\beta(t)|\le 4\pi[\zeta(t)-\zeta(t^2)], \tag{2.5}$$

$$|\gamma(t)|\le 4\pi\varepsilon(t). \tag{2.6}$$

Inequality (2.4) is obvious. Since $|\beta(t)|\le\frac{2\pi}{t^2}\int_{t^2}^t rm(r,\varphi)dr$, we get (2.5). Using Claim 1 and changing the order of integration, we find

(2.7)

$$\begin{aligned}
|\gamma(t)|&\le\iint_{\Delta_t}|\varphi(tz)-\varphi(z)|dxdy\\
&\le\int_0^t rdr\int_0^{2\pi}d\theta\int_{tr}^r|\varphi'(\varrho e^{i\theta})|d\varrho\le 2\pi t\int_0^t dr\int_{tr}^r\frac{R}{R^2-\varrho^2}m(R,\varphi)d\varrho\\
&\le 4\pi\int_0^t dr\int_{\frac{1+tr}{2}}^{\frac{1+r}{2}}\frac{R}{(3R-1)(1-R)}m(R,\varphi)dR\le 4\pi\int_0^t dr\int_{\frac{1+tr}{2}}^{\frac{1+r}{2}}\frac{m(R,\varphi)}{1-R}dR\\
&=4\pi\int_0^t dr\int_{\frac12}^{\frac{1+r}{2}}\frac{m(R,\varphi)}{1-R}dR-4\pi\int_0^t dr\int_{\frac12}^{\frac{1+tr}{2}}\frac{m(R,\varphi)}{1-R}dR\\
&=4\pi\int_{\frac12}^{\frac{1+t}{2}}\frac{1+t-2R}{1-R}m(R,\varphi)dR-4\pi\int_{\frac12}^{\frac{1+t^2}{2}}\frac{1+t^2-2R}{t(1-R)}m(R,\varphi)dR\\
&\le 4\pi[\eta(t)-\eta(t^2)]=4\pi\varepsilon(t).
\end{aligned}$$

The second equality in (2.7) comes from changing the order of the first two integrals.

Claim 3. Suppose (2.1) holds. Then there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that

$$\lim_{k\to\infty}\frac{\eta(a_{n_k})}{\eta(a_{n_k}^2)}=1. \tag{2.8}$$

In fact, it is sufficient to show that $\underline{\lim}_{n\to\infty}\frac{\eta(a_n)}{\eta(a_n^2)}=1$ holds. Otherwise, there exists some constant $c>1$ such that $\eta(a_n)>c\eta(a_n^2)$, for all $n\in\mathbb{N}$. Then we have $\eta(a_n)>c^n\eta(a)$, $n\in\mathbb{N}$. By virtue of $\lim_{n\to\infty}a_n=1$, it is obvious that

$$\eta(a_n)>c^n\eta(a)=\eta(a)\Big(\frac{\log a}{\log a_n}\Big)^{\log_2 c}\sim\widetilde{c}\Big(\frac{1}{1-a_n}\Big)^{\log_2 c},\ n\to\infty, \tag{2.9}$$

where $\widetilde{c} = \eta(a)(\log \frac{1}{a})^{\log_2 c}$. However, (2.1) gives

$$\begin{aligned} \eta(a_n) &\leq 2a_n \int_{\frac{1}{2}}^{\frac{1+a_n}{2}} m(r,\varphi)dr \leq 4 \int_{\frac{1}{2}}^{\frac{1+a_n}{2}} rm(r,\varphi)dr \\ &= \frac{2}{\pi}[A(\frac{1+a_n}{2},\varphi) - A(\frac{1}{2},\varphi)] \leq \frac{2}{\pi}A(\frac{1+a_n}{2},\varphi) \\ &= O(\frac{1}{(1-a_n)^s}),\ n\to\infty,\ \text{for any given } s>0. \end{aligned}$$

This contradicts (2.9), proving our claim.

Claim 4. Suppose $\{a_{n_k}\}$ is obtained from Claim 3. Let $b_k = a_{n_k}$. Then

$$\lim_{k\to\infty} \frac{\varepsilon(b_k)}{\zeta(\frac{1+b_k^2}{2})} = 0 \tag{2.10}$$

and

$$\lim_{k\to\infty} \frac{\zeta(b_k^2)}{\zeta(\frac{1+b_k^2}{2})} = 1. \tag{2.11}$$

Since $\eta(b_k^2) \leq 2b_k^2 \int_{\frac{1}{2}}^{\frac{1+b_k^2}{2}} m(r,\varphi)dr = 2b_k^2\zeta(\frac{1+b_k^2}{2})$, we get

$$\frac{\varepsilon(b_k)}{\zeta(\frac{1+b_k^2}{2})} = \frac{\eta(b_k^2)}{\zeta(1+b_k^2/2)}[\frac{\eta(b_k)}{\eta(b_k^2)} - 1] \leq 2b_k^2[\frac{\eta(b_k)}{\eta(b_k^2)} - 1].$$

By (2.8), (2.10) is obtained. Notice that

$$\begin{aligned} \frac{t}{1+t}[\zeta(\frac{1+t^2}{2}) - \zeta(t^2)] &= \frac{t}{1+t}\int_{t^2}^{\frac{1+t^2}{2}} m(r,\varphi)dr \\ &\leq \int_{t^2}^{\frac{1+t^2}{2}} \frac{t(1-t)}{1-r}m(r,\varphi)dr \leq \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{t(1-t)}{1-r}m(r,\varphi)dr \\ &= \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{1+t-2r}{1-r}m(r,\varphi)dr - \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{1+t^2-2r}{1-r}m(r,\varphi)dr \\ &\leq \int_{\frac{1}{2}}^{\frac{1+t}{2}} \frac{1+t-2r}{1-r}m(r,\varphi)dr - \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{1+t^2-2r}{1-r}m(r,\varphi)dr \\ &= \varepsilon(t). \end{aligned}$$

Set $t = b_k$. By virtue of (2.10), we get (2.11).

Now, we complete the proof of Theorem 3. By Claim 2, it suffices to show that

$$\lim_{k\to\infty} \frac{\zeta(b_k) - \zeta(b_k^2)}{\zeta(b_k)} = 0 \tag{2.12}$$

and

$$\lim_{k\to\infty} \frac{\varepsilon(b_k)}{\zeta(b_k)} = 0. \tag{2.13}$$

In view of Claim 4, they can be deduced from

$$\frac{\zeta(b_k)-\zeta(b_k^2)}{\zeta(b_k)} \le \frac{\zeta(\frac{1+b_k^2}{2})-\zeta(b_k^2)}{\zeta(\frac{1+b_k^2}{2})} = 1-\frac{\zeta(b_k^2)}{\zeta(\frac{1+b_k^2}{2})}$$

and

$$\frac{\varepsilon(b_k)}{\zeta(b_k)} \le \frac{\varepsilon(b_k)}{\zeta(\frac{1+b_k^2}{2})}\cdot\frac{\zeta(\frac{1+b_k^2}{2})}{\zeta(b_k^2)}.$$

Thus, Theorem 3 follows.

In particular, if $A(r,\varphi)=O(\log^q\frac{1}{1-r})$ as $r\to 1$ for some $q>0$, then $\varphi(z)$ is associated with extremal Teichmüller mappings.

Example. Let $\varphi(z)=\frac{\log^q(1-z)}{(1-z)^2}$, $q>0$. The function φ corresponds to extremal Teichmüller mappings since

$$\begin{aligned} A(r,\varphi) &= \int_0^r tdt\int_0^{2\pi}|\varphi(te^{i\theta})|d\theta \\ &= \int_0^r tdt\int_0^{2\pi}\frac{|\log\frac{1}{1-te^{i\theta}}|^q}{|1-te^{i\theta}|^2}d\theta \le 2\pi\log^{q+1}\frac{1}{1-r} \\ &= o(\frac{1}{(1-r)^s}),\ r\to 1,\ \text{for any given } s>0. \end{aligned}$$

Here, we have chosen a suitable univalent branch for φ in Δ. Obviously, the Bers norm $\|\varphi\|_\Delta$ of φ is infinite, i.e., $\varphi\notin BQD(\Delta)$.

Remark. Note that $\varepsilon(t)=\eta(t)-\eta(t^2)\le\eta(t)-\eta(t^p)$ for $p\ge 2$. The same reasoning allows us to take $a_n=a^{1/p^n}$ in Theorem 3. So, the Hamilton sequence in Theorem 1 can be replaced by $\{\varphi(a^{1/p^{n_k}}z)/\|\varphi(a^{1/p^{n_k}}z)\|\}$ $(p\ge 2)$.

Finally, we end this section with the following problem.[1]

Problem. Let φ be holomorphic in Δ. If φ corresponds to an extremal Teichmüller mapping f, can we say that μ_f has a Hamilton sequence such as $\{\varphi(t_nz)/\|\varphi(t_nz)\|:\lim_{n\to\infty}t_n=1,\ t_n\in(0,1)\}$?

3. Proof of Theorem 2

A Riemann surface M is said to be of finite analytic type (g,n) if and only if M is obtained from a closed Riemann surface of finite genus g by deleting n points, $n\in\mathbb{N}$. A surface of finite analytic type is hyperbolic if and only if the inequality

$$3g-3+n>0$$

holds.

First, we state a result by Mcmullen in [5]:

Theorem C. *Let $f:\ X\to X'$ be a Teichmüller mapping between Riemann surfaces of hyperbolic finite type. Then the mapping $\widetilde{f}:\ \Delta\to\Delta$ obtained by lifting f to the universal covers of X and X' is not extremal among quasiconformal mappings with the same boundary values (unless f is conformal).*

[1] *Added in proof.* The problem has a negative answer for which the counterexample will be given in [11].

Let Γ be the covering transformation group of a hyperbolic Riemann surface $X = \Delta/\Gamma$ of finite type (g, n). Suppose $\widetilde{f}: \Delta \to \Delta$ is a Teichmüller mapping with $\mu_{\widetilde{f}} = k\frac{\overline{\varphi}}{|\varphi|}$, where $\varphi \in BQD(\Delta, \Gamma)\backslash\{0\}$. Then $\widetilde{f}$ induces a new covering transformation group $\Gamma' = \widetilde{f} \circ \Gamma \circ \widetilde{f}^{-1}$ which produces a new hyperbolic finite-type Riemann surface $X' = \Delta/\Gamma'$. Therefore, $\widetilde{f}$ can be projected to a Teichmüller mapping $f: \Delta/\Gamma \to \Delta/\Gamma'$.

Recall that the Bers space $BQD(X)$ is the space of all holomorphic quadratic differentials $\varphi(z)dz^2$ on X that are bounded in the following sense:

$$\|\varphi\|_X = \sup_{p\in X} |\varphi(p)|\sigma^{-1}(p) < \infty,$$

where $\sigma(p)$ denotes the Poincaré metric density on X. It is well known that $BQD(X)$ is canonically identified with $BQD(\Delta, \Gamma)$. From the Riemann-Roch Theorem it readily follows that $BQD(\Delta, \Gamma)$ is a complex Banach space of $3g - 3 + n$ dimensions. The results on harmonic maps ([4], [9]) also show that there is a proper homeomorphism of $BQD(X)$ onto the Teichmüller space $T(X)$ of X. In the sense of not distinguishing $BQD(X)$ from $BQD(\Delta, \Gamma)$, we have $\|\varphi\|_\Delta = \|\varphi\|_X$.

Now, we can conclude that $\widetilde{f}$ is not extremal from Theorem C. This completes the proof of Theorem 2.

Combining Theorems 1 and 2, it is not difficult to see that $\varphi \in BQD(\Delta, \Gamma)$ has the property:

Corollary. *Let Γ be the covering transformation group of a hyperbolic finite-type Riemann surface. Suppose $\varphi(z)$ is in $BQD(\Delta, \Gamma)\backslash\{0\}$. Then*

$$\overline{\lim_{r\to 1}} \frac{A(r, \varphi)}{\log^s(1/(1-r))} = \infty, \quad \textit{for any given } s > 0. \tag{3.1}$$

On the other hand, it is evident that $A(r, \varphi) = O(\frac{1}{1-r})$ (as $r \to 1$) for all φ in $BQD(\Delta)$.

Acknowledgment

The author would like to thank the referee for valuable comments and suggestions.

References

[1] W. K. Hayman and E. Reich, *On the Teichmüller mappings of the disk*, Complex Variables Theory Appl. **1** (1982), 1–12. MR **84e:**30031

[2] X. Z. Huang, *On the extremality for Teichmüller mappings*, J. Math. Kyoto Univ. **35** (1995), 115–132. MR **97a:**30029

[3] W. C. Lai and Z. M. Wu, *On extremality and unique extremality of Teichmüller mappings*, Chinese Ann. Math. (Ser. B) **16** (1995), 399–406. MR **97b:**30027

[4] V. Marković, *Harmonic diffeomorphism of noncompact surfaces and Teichmüller spaces*, J. London Math. Soc. **65** (2002), 103-114. MR **2002k:**32015

[5] C. Mcmullen, *Amenability, Poincaré series and quasiconformal maps*, Invent. Math. **97** (1989), 95-127. MR **90e:**30048

[6] E. Reich, *Construction of Hamilton sequences for certain Teichmüller mappings*, Proc. Amer. Math. Soc. **103** (1988), 789–796. MR **89m:**30045

[7] E. Reich and K. Strebel, *Extremal quasiconformal mappings with given boundary values*, Contributions to Analysis, A Collection of Papers Dedicated to Lipman Bers, Academic Press, New York, 1974, pp. 375-392. MR **50:**13511

[8] Z. M. Wu and W. C. Lai, *On criterion of the extremality and construction of Hamilton sequences for a class of Teichmüller mappings*, Chinese Ann. Math. (Ser. B) **21** (2000), 339–342. MR **2002c:**30031

[9] M. Wolf, *The Teichmüller theory of harmonic maps*, J. Differential Geom. **29** (1989), 449-479. MR **90h:**58023

[10] G. W. Yao, *Hamilton sequences and extremality for certain Teichmüller mappings*, Ann. Acad. Sci. Fenn. Math. **29** (2004), 185–194.

[11] G. W. Yao, *A remark on Hamilton sequences of extremal Teichmüller mappings*, to appear.

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