SUB-EXponential DECAY OF OPERATOR KERNELS FOR
FUNCTIONS OF GENERALIZED SCHröDINGER OPERATORS

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Abstract. We study the decay at large distances of operator kernels of functions of generalized Schrödinger operators. We prove sub-exponential decay for functions in Gevrey classes and exponential decay for real analytic functions.

1. Introduction and result

In a previous paper two of the authors proved that operator kernels of smooth functions of generalized Schrödinger operators decay faster than any polynomial [GK2]. The operator kernel of a bounded operator $A$ acting on $L^2(\mathbb{R}^d, dq; C^k)$ is defined as $\chi_x A \chi_y$, where

$$\chi_x(q) = \chi_0(q - x), \quad q, x \in \mathbb{R}^d,$$

and $\chi_0$ is the characteristic function of the cube centered at 0 with side 1. We use the standard notation $\langle x \rangle = (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^d$. Given a generalized Schrödinger operator $H$ (see [GK2] and Remark 1.3 below) and a $C^1$ function $f$ with sufficiently fast decay at 1, namely,

$$\int_{\mathbb{R}} |f^{(k)}(u)| \langle u \rangle^{k-1} du < \infty \quad \text{for } k = 0, 1, 2, \ldots,$$

it is shown in [GK2] Theorem 2] that for all $n = 0, 1, 2, \ldots$ one has

$$\|\chi_x f(H) \chi_y\| \leq C_{n,f} \langle x - y \rangle^{-n}, \quad x, y \in \mathbb{R}^d$$

(with a control on $C_{n,f}$ in terms of integrals as in (2)). Such kernel decay estimates are of interest from a purely mathematical point of view (see [S3, Da1, Da2]). Moreover, controls like (3) have already been quite useful in applications, e.g., [GK3, GK4, EfGr, Te].

The purpose of this paper is to improve on (3) and show that if the $C^\infty$ function $f$ belongs to a Gevrey-like class (see Definition 1.1), then

$$\|\chi_x f(H) \chi_y\| \leq Ce^{-c|x-y|^{\alpha}}, \quad x, y \in \mathbb{R}^d,$$

for some $\alpha \in (0, 1]$ depending on $f$. This extension was motivated by some applications which are discussed at the end of this section.
We recall that a function $f$ belongs to the Gevrey class of order $a \geq 1$, denoted by $G^a(\mathbb{R})$ (e.g., [10]), if $f \in C^\infty(\mathbb{R})$ and for each compact subset $K \subset \mathbb{R}$ there exists a constant $C_K$ such that

$$|f^{(k)}(u)| \leq C_K(C_K(k+1)^a)^k, \quad k = 0, 1, 2, \ldots, u \in K.$$  

Note that being Gevrey is thus a local property. The class $G^1(\mathbb{R})$ is the set of real analytic functions, and authors sometimes refer to Gevrey classes only for $a > 1$.

We emphasize that for any $a > 1$ the Gevrey class $G^a(\mathbb{R})$ contains compactly supported functions. (For example, the function $f_a(x) = \exp(-(1-x^2)^{-\frac{1}{a-1}})$ for $|x| < 1$ and $f_a(x) = 0$ for $|x| \geq 1$ belongs to $G^a(\mathbb{R})$.) In fact, most of the smooth cut-off functions that are commonly used can be chosen in $G^a(\mathbb{R})$ with $a$ arbitrarily close to 1. We will denote the class of compactly supported functions in $G^a(\mathbb{R})$ by $G^a_{comp}(\mathbb{R})$. Of course many interesting functions are non-compactly supported, especially analytic ones, and we need to consider a larger set of functions.

To allow functions supported either on $\mathbb{R}$ or on a half-line, we combine the local Gevrey estimate [16] with the condition of decay at $\infty$ given by [17], and make the following definition. (By $C_{a,b,...}$ we always denote a constant depending on $a,b,...$)

**Definition 1.1.** Let $I$ be an open interval and $a \geq 1$. A function $f \in C^\infty(\mathbb{R})$ is $L^1$-Gevrey of class $a$ on $I$ if for all $k = 0, 1, 2, \ldots$ we have

$$\left\| f^{(k)}(u)^{k-1} \right\|_{L^1(I)} = \int_I |f^{(k)}(u)|^{k-1} du \leq C_{f,1}(C_{f,1}(k+1)^a)^k.$$  

As far as the self-adjoint operator $H$ is concerned, the sole property we shall require is that a suitable Combes-Thomas estimate [CoTh] holds. We state it as a definition.

**Definition 1.2.** The self-adjoint operator $H$ on $L^2(\mathbb{R}^d, dq; \mathbb{C}^k)$ has property (CT) if there exists a constant $C$ (depending on $H$, $d$ and $k$) such that for all $z \not\in \sigma(H)$ and all $x, y \in \mathbb{R}^d$ we have

$$\|\chi_x(H-z)^{-1}\chi_y\| \leq C \eta_z \exp\left(-\frac{C\eta_z}{1+\eta_z + |z| |x-y|}\right),$$

where $\eta_z = d(z, \sigma(H))$.

**Remark 1.3.** The inequality (7) is proven in [17, Corollary 1] for generalized Schrödinger operators, a class of semibounded second-order partial differential operators of mathematical physics which includes the Schrödinger operator, magnetic Schrödinger operator, and the classical wave operators (i.e., acoustic operator, Maxwell operator, and other second-order partial differential operators associated with classical wave equations). Specific examples of operators with property (CT) are given by the Schrödinger operator

$$H = -\Delta + V^{(1)} + V^{(2)} \text{ on } L^2(\mathbb{R}^d, dx),$$

with $0 \leq V^{(1)} \in L^1_{loc}$ and $V^{(2)}$ relatively form bounded with respect to $-\Delta$ with relative bound $< 1$, the acoustic operator

$$A = -\frac{1}{\sqrt{\rho}} \nabla \cdot \frac{1}{\sqrt{\rho}} \text{ on } L^2(\mathbb{R}^d, dx),$$
where $\kappa(x)$ is the compressibility and $\varrho(x)$ is the mass density, and the Maxwell operator
\begin{equation}
M = \frac{1}{\sqrt{\mu}} \nabla \times \frac{1}{\varepsilon} \nabla \times \frac{1}{\sqrt{\mu}} \text{ on } L^2(\mathbb{R}^3, dx; \mathbb{C}^3),
\end{equation}
where $\mu(x)$ is the magnetic permeability and $\varepsilon(x)$ is the dielectric constant, with $\kappa(x)$, $\varrho(x)$, $\mu(x)$, $\varepsilon(x)$ bounded, uniformly strictly positive, measurable functions.

Our results are stated in the following theorem.

**Theorem 1.4.** Let $H$ be a self-adjoint operator on $L^2(\mathbb{R}^d, dq; \mathbb{C}^k)$ with property (CT), $I$ an open interval containing the spectrum of $H$. Then, given a real number $a \geq 1$, we have:

(i) If $a > 1$, then for each $a' > a$ there exists $C_{L^1, a,a'} > 0$, such that for all $L^1$-Gevrey functions $f$ of class $a$ on $I$,
\begin{equation}
\|\chi_{x} f(H) \chi_{y}\| \leq C_{L^1, a,a'} (eC_{f,I})^{\frac{a+a'}{a} - |x-y|} \exp \left(-\frac{a'-a}{4a'} |x-y|^{\frac{4a'}{a'}} \ln |x-y| \right)
\end{equation}
for all $x, y \in \mathbb{R}^d$. As a consequence, for each $a' > a$, there exist constants $C_{f,I,a,a'} > 0$ and $c_{a,a'} > 0$ such that
\begin{equation}
\|\chi_{x} f(H) \chi_{y}\| \leq C_{f,I,a,a'} \exp \left(-c_{a,a'} |x-y|^{\frac{4a'}{a'}} \right) \text{ for all } x, y \in \mathbb{R}^d.
\end{equation}

(ii) If $a = 1$, then there exist constants $C_{f,I} > 0$ such that for all $L^1$-Gevrey functions $f$ of class $a$ on $I$,
\begin{equation}
\|\chi_{x} f(H) \chi_{y}\| \leq C_{f,I} \frac{|x-y|}{1 + \ln C_{f,I}} \exp \left(-\frac{C_{f,I}}{1 + \ln C_{f,I}} |x-y| \right)
\end{equation}
for all $x, y \in \mathbb{R}^d$.

**Remark 1.5.** The mass $\frac{a'}{a''}$ in (11) can be chosen arbitrarily close to $\frac{a'}{a''}$. This is done by modifying the proof by choosing $1 - s = \frac{1+\gamma(a'-a)}{a'}$ in (42) with $\gamma$ close to zero.

**Remark 1.6.** If $f$ has an analytic continuation to a region where one can use a suitable contour, then (13) is a rather simple consequence of the Combes-Thomas estimate and the Cauchy integral formula. In our theorem, no condition on the form of the domain of analyticity of $f$ in $\mathbb{C}$ is assumed. We nevertheless obtain exponential decay in $|x - y|$.

To discuss the range of application of Theorem 1.4 and more precisely of condition (14), we introduce the weighted Gevrey class $G^\alpha(I)$ in the following way: given $\gamma \in \mathbb{R}$, $a \geq 1$, and an open interval $I$, we say that $f \in G^\alpha(I)$ if $f \in C^\infty$ and there exists $C > 0$ such that
\begin{equation}
\sup_{u \in I} \left| f^{(k)}(u)(u)^{k+\gamma} \right| \leq C(C(k+1)^a)^k, \quad k = 0, 1, 2, \ldots
\end{equation}
The class $G^\alpha(I)$ is never trivial (i.e., reduced to 0): if $a > 1$ it contains all functions of $G^\alpha_{\text{comp}}(\mathbb{R})$ that are supported in $I$, and if $a = 1$ it can be easily seen that for $t > 0$ and $z \notin I$, the functions
\[ \exp(-tu^2), \quad \exp(-tu), \quad \exp(-iut) \exp(-u^2), \quad \frac{1}{u-z} \]
belong to $G^1_\gamma((\lambda, +\infty))$ for all $\lambda$ and all $\gamma$ for the first three, and to $G^1_1(\mathbb{R})$ for the last one.

Functions in $G^a(I)$, $\gamma > 0$, clearly satisfy (19). Moreover, if $f_1 \in G^a_{\gamma_1}(I)$ and $f_2 \in G^a_{\gamma_2}(I)$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $a_1, a_2 \geq 1$, then one can verify that

$$f_1 f_2 \in G^a(I), \quad a = \max(a_1, a_2), \quad \gamma = \gamma_1 + \gamma_2.$$  

This is of interest since in applications we often have to consider cutoffs of the resolvent (see for instance [GK3]), that is, $f(H)$ with

$$f(u) = (1 - f_1(u)) \frac{1}{u - \lambda_0}, \quad f_1 \in C^\infty_{\text{comp}}(\mathbb{R}), \quad f_1 \equiv 1 \text{ near } \lambda_0 \in \mathbb{R}.$$  

By choosing $f_1 \in C^a_{\text{comp}}(\mathbb{R})$ with $a$ arbitrarily close to 1, we can deduce from our remarks that $f \in G^a(I)$, and hence get the sub-exponential decay of $\| \chi_x f(H) \chi_y \|$ by (12).

We end this section by discussing some applications of Theorem 1.4 to the theory of Schrödinger operators:

(1) In the context of random Schrödinger operators, Theorem 1.4 can be used to extend the results of [GK3] to models where only a Wegner estimate with sub-exponential controls is available, as in the case of Bernoulli-Anderson models and polymers [CKM, KLS, DBG, DSS, JSBS]. In particular, it allows the definition of the strong insulator region in these cases, similar to what is done in [GK3] for models with a Wegner estimate with polynomial controls.

(2) Still in the context of random Schrödinger operators, Theorem 1.4, combined with [GK1, Theorem 3.8], provides a simple and quick proof of sub-exponential decay of any order of the operator kernel of the Fermi projector $P_{E_F} = \chi_{(-\infty, E_F]}(H)$, provided the Fermi energy $E_F$ lies inside the strong insulator region (equivalently, provided $E_F$ lies in the region of applicability of the multiscale analysis).

(3) In [GKT], an approximation lemma shows that the weight given to balls of radius $\varepsilon$ by spectral measures of Hamiltonians that coincide on a region of size $O(\varepsilon^{-1})$ are close up to a polynomial term in $\varepsilon$. Combining the methods of the present paper with the ones of [GKT] improves the latter by showing that the error term is actually sub-exponentially small. It moreover allows one to treat more general potentials, since they could be assumed to be sub-exponentially bounded rather than polynomially.

(4) We note that if $f$ is a real analytic function, then it follows from (13) that the moments $\| \langle X \rangle^{p/2} e^{-iHt} f(H) \chi_0 \|^2$ cannot grow faster than $(t(\ln t)^3)^p$. (This can be seen by cutting $\mathbb{R}^d$ into two regions: $F(|x|) \leq t \ln t$ and $F(|x|) \geq t \ln t$, where $F(u) = u(\ln u)^{-2}$.)

2. Proof

In order to use the Combes-Thomas estimate (20), we need to express $f(H)$ in terms of the resolvent $R(z) = (H - z)^{-1}$. This is done by the Helffer-Sjöstrand formula [HeSj, DiSj]

$$f(H) = \frac{1}{\pi} \int \int \tilde{\partial} \tilde{f}(u + iv) R(u + iv) dudv, \quad \tilde{\partial} = \frac{1}{2} (\partial_u + i\partial_v),$$  

(16)
where \( \tilde{f} \) is a quasi-analytic extension of \( f \), i.e., \( \tilde{\partial}\tilde{f} = O(v) \) near \( v = 0 \), with conditions ensuring the convergence of the integral. Here we use an extension \( \tilde{f} \) of \( f \) similar to the one in [GK2], with one difference: we add a smooth cutoff \( \varphi(u) \) around the spectrum of \( H \) (we can thus allow for a larger class of functions \( f \), such as \( e^{-x} \) if the operator \( H \) is bounded from below):

\[
\tilde{f}_n(u + iv) = \varphi(u)\tau(v/\langle u \rangle)S_n\tilde{f}(u + iv),
\]

where \( \tau \) is smooth such that \( \tau(t) = 1 \) for \( |t| \leq 1/2 \) and \( \tau(t) = 0 \) for \( |t| \geq 1 \); the function \( \varphi \) is smooth as well and satisfies \( \varphi(u) = 1 \) near \( \sigma(H) \), and \( \text{supp } \varphi \subset I \).

In all that follows, we set \( \rho(u, v) = \varphi(u)\tau(v/\langle u \rangle) \). Then we have

\[
\tilde{\partial}\tilde{f}_n(u + iv) = \tilde{\partial}\rho(u, v)\sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(u)(iv)^k + \frac{\rho(u, v)}{2n!}f^{(n+1)}(u)(iv)^n.
\]

Note that we have

\[
\tilde{\partial}\tau(v/\langle u \rangle) = \frac{1}{2\langle u \rangle}\left(-\frac{u}{\langle u \rangle}v\tau(v/\langle u \rangle) + iv\tau'(v/\langle u \rangle)\right).
\]

The main idea of the proof is to write

\[
f(H) = \frac{1}{\pi} \int_{|v| < \epsilon(u)} \tilde{\partial}\tilde{f}_n(u + iv)R(u + iv)dudv + \frac{1}{\pi} \int_{|v| > \epsilon(u)} \tilde{\partial}\tilde{f}_n(u + iv)R(u + iv)dudv
\]

(22)

\[
= I + II,
\]

with appropriate choices of \( \epsilon \) and \( n \).

In all that follows, the constants \( C, c \) may depend on \( I \) but never on \( \epsilon, n, f \) or \( x, y \).

**Lemma 2.1.** There exist \( C, c > 0 \) such that

\[
\left\| \chi_x \mathbb{1} \chi_y \right\| \leq Ce^{-c|x-y|} \sum_{k=0}^{n} \frac{1}{k!} \left\| f^{(k)}(u)^{k-1} \right\|_{L^1(I)} + \frac{C\epsilon^n}{n!} \left\| f^{(n+1)}(u)^n \right\|_{L^1(I)}
\]

for all \( n \geq 1 \), \( 0 < \epsilon \leq 1/2 \) and \( x, y \in \mathbb{R}^d \). The constant \( c \) in the exponent depends only on the choice of \( \varphi \) (through its support and the distance of its support to the open interval \( I \)).

**Proof.** Assuming that \( \epsilon \leq 1/2 \) implies that the term involving \( \tau' \) in (19) does not contribute. Actually we only have to consider \( \varphi'(u)\tau(v/\langle u \rangle)S_n\tilde{f} \) and the last term of (19). We first estimate the contribution of the last term by using the basic
estimate $\|R(u + iv)\|^2 \leq |v|^{-1}$, and we get

$$
\int \int_{|v| < \epsilon(u)} \frac{\rho(u,v)}{n!} f^{(n+1)}(u)(iv)^n R(u + iv) dudv \leq C \frac{C}{n!} \int_{\epsilon(u)}^{\epsilon(\epsilon(u))} f^{(n+1)}(u) \|v\|^{n-1} dudv 
$$

$$
\leq \frac{2C}{n!} \int_{\epsilon(u)}^{\epsilon(\epsilon(u))} f^{(n+1)}(u) \|v\|^n du 
$$

Note that we have not used the Combes-Thomas estimate for this part. Next we consider the term involving $\varphi'(u)$. This term is compactly supported outside $\sigma(H)$ with respect to $z = u + iv$, so that $\eta_z = \text{dist}(z, \sigma(H))$ is uniformly bounded away from 0 and $\|u\|$ is bounded. The Combes-Thomas estimate (7) implies that

$$
\int \int_{|v| < \epsilon(u)} \varphi'(u) \tau(v/\langle u \rangle) S_n f(u + iv) \chi_x R(u + iv) \chi_y dudv \leq Ce^{-c|x-y|} \int \int_{\text{supp } \varphi'} (0 \leq \epsilon(u) \leq \langle u \rangle) \sum_{k=0}^{n-1} \frac{1}{k!} \left\| f^{(k)}(u) \right\| |v|^k dudv 
$$

$$
\leq Ce^{-c|x-y|} \left( \sum_{k=0}^{n-1} \frac{1}{k!} \left\| f^{(k)}(u) \right\|^{k} \right) \left\| \langle u \rangle \right\| L^1(I), 
$$

where we used the fact that $\varphi'$ is compactly supported and $\epsilon \leq 1/2$. \qed

We consider now the estimates for $\Pi$.

**Lemma 2.2.** There exist $C, c > 0$ such that

$$
\|\chi_x \Pi \chi_y\| \leq Ce^{-c|x-y|} \sum_{k=0}^{n} \frac{1}{k!} \left\| f^{(k)}(u) \right\|^{k-1} \left\| \langle u \rangle \right\| L^1(I) + \frac{C}{n!} e^{-c|x-y|} \left\| f^{(n+1)}(u) \right\| L^1(I), 
$$

for all $\epsilon > 0$, $n \geq 1$, $x, y \in \mathbb{R}^d$.

**Proof.** On the domain of integration that we are considering, one has $|v| \leq \eta = \text{dist}(u + iv, \sigma(H))$ and $\epsilon(u) \leq |v| \leq \langle u \rangle$. Thus the Combes-Thomas estimate (7) reduces to

$$
\|\chi_x R(u + iv) \chi_y\| \leq \frac{C}{|v|} e^{-c|x-y|} \leq \frac{C}{|v|} e^{-c|x-y|}, 
$$

for all $x, y \in \mathbb{R}^d$, with $C, c > 0$ constants independent of $x, y, u, v$. To evaluate $\|\chi_x \Pi \chi_y\|$ we use (19) and treat separately the sum and the last term. We use (29) only for the latter, getting

$$
\|\chi_x \Pi \chi_y\| \leq \frac{1}{\pi} \int \int |\tilde{\varphi}(u,v)| \left\{ \sum_{k=0}^{n} \frac{1}{k!} \left\| f^{(k)}(u) \right\| |v|^k \right\} \|\chi_x R(u + iv) \chi_y\| dudv 
$$

$$
+ Ce^{-c|x-y|} \int \int \left\{ \frac{1}{n!} \left\| f^{(n+1)}(u) \right\| |v|^{n-1} \right\} dudv. 
$$
The last term clearly yields the second term in (28).

Now the term $|\partial\rho| = |\varphi' + \varphi \partial \tau|$ will be yielding a factor $1/|\langle u \rangle|$, coming either from (20) or from the fact that $\varphi'$ is compactly supported. In other terms, we will use the bound

$$|\partial \rho(u, v)| \leq \frac{C}{|\langle u \rangle|}. \quad (31)$$

Moreover, on the support of $\partial \rho$ the distance to the spectrum of $H$ is uniformly bounded away from zero. Indeed, on the support of $\varphi'$, a compact set disjoint from the spectrum of $H$, we have $\eta = \text{dist}(u + iv, \sigma(H)) \geq C|\langle u \rangle|$ for some constant $C > 0$. On the support of $\partial \tau$ we have $\eta \geq |v| \geq \langle u \rangle/2$. As a consequence, it follows by (7) that on the support of $\partial \rho$ we have

$$\|\chi_x R(u + iv) \chi_y\| \leq \frac{C}{|\langle u \rangle|} e^{-|x - y|}. \quad (32)$$

Plugging (31) and (32) into the first integral on the right-hand side of (30) leads to

$$\int \int |\partial \rho(u, v)| \left\{ \sum_{k=0}^n \frac{1}{k!} |f^{(k)}(u)| |v|^k \right\} \|\chi_x R(u + iv) \chi_y\|dvdu$$

$$\leq \ C e^{-c|x-y|} \int I \int_{-I} \frac{1}{|\langle u \rangle|^2} \left\{ \sum_{k=0}^n \frac{1}{k!} |f^{(k)}(u)| |v|^k \right\} dvdu$$

$$\leq \ C e^{-c|x-y|} \sum_{k=0}^n \frac{1}{k!} \left\|f^{(k)}(u) - 1\right\|_{L^1(I)}^{k-1} \quad (33)$$

$$\leq \ C e^{-c|x-y|} \sum_{k=0}^n \frac{1}{k!} \left\|f^{(k)}(u) - 1\right\|_{L^1(I)}^{k-1}, \quad (34)$$

giving the first term in (28).

The next lemma gives an estimate on the following quantity, which appears in the last two lemmas, after controlling the norms by (19):

$$A_{n, a} = \sum_{k=0}^n \frac{1}{k!} C_{f, I} (C_{f, I}(k + 1)a)^k + \frac{1}{n!} C_{f, I}(C_{f, I}(n + 2)a)^{n+1}. \quad (35)$$

It will motivate the choice of $\epsilon$ and $n$ in the proof of Theorem 1.4.

**Lemma 2.3.** For all $a \geq 1$, one has

$$A_{n, a} \leq 3 e^{na} (eC_{f, I})^{n+1} a^{n(a-1)}, \quad \forall \ n \geq 1.$$  

**Proof.** First note that $k! \geq (k/\epsilon)^k$ for $k \geq 1$, and estimate $e^k C_{f, I}^k$ by $(eC_{f, I})^n$ for $k \leq n$ (recall that $C_{f, I} \geq 1$). Moreover, we have $(k + 1)a^k/k \leq e^a k^{a-1}k$ and, recalling that $a \geq 1$, we can therefore estimate each term of the sum by the one for $k = n$. In a similar way, $(n + 2)a^{n+1}/n^a \leq e^{na} a^{n(a-1) - n}$. The result now follows easily.

**Proof of Theorem 1.4.** The last three lemmas show clearly that we have to consider, with $K_{f, I} = eC_{f, I}$,

$$n^a(K_{f, I})^{n+2} \exp((a - 1)n \ln n - c|x - y|), \quad (36)$$

$$n^a(K_{f, I})^{n+2} \exp((a - 1)n \ln n + n \ln \epsilon), \quad (37)$$

$$n^a(K_{f, I})^{n+2} \exp((a - 1)n \ln n - c|\epsilon - y|). \quad (38)$$
To simplify, and with no loss of generality, we may assume that the constants $c$ appearing in (36) and (38) are the same. In both cases $a = 1$ and $a > 1$, we will choose $n = \mathcal{O}(|x-y|^a)$; thus, if we can show that the exponential terms listed above have the decay announced in the theorem, we shall get the result.

Proof in the case $a = 1$. We choose $n \sim \delta|x-y|$ with $\delta > 0$ small enough and a fixed $\epsilon > 0$ small enough as well. Then (37) gives the following first condition:

$$\epsilon K_{f,t} < 1.$$  
(39)

Now, by choosing $n$ such that $\delta|x-y| - 1 \leq n + 2 < \delta|x-y|$, (38) yields the second condition:

$$\delta \ln K_{f,t} - c \epsilon < 0.$$  
(40)

Similarly, we obtain the third condition from (36):

$$\delta \ln K_{f,t} - c < 0.$$  
(41)

The existence of $\epsilon, \delta > 0$ such that these three conditions are satisfied is clear since we may first choose $\epsilon < 1$ satisfying (39) and then $\delta$ small enough satisfying (40) as well. We may choose $\epsilon$ such that $\epsilon K_{f,t} = e^{-1}$ and $\delta = (c\epsilon)/(2 \ln K_{f,t})$. Since $K_{f,t} \geq e$, each term (36), (37), (38) is bounded by the bound of (57), namely $nK_{f,t}^2 \exp(-n)$. The estimate (13) follows.  

Proof in the case $a > 1$. In this part $n$ and $\epsilon$ will both depend on $|x-y|$. Pick $a' > a$ as in the theorem. Precisely, we write $\nu = a' - a > 0$, and set

$$\epsilon = |x-y|^{-s}, \quad |x-y|^\delta \leq n < |x-y|^{\delta + 1},$$  
(42)

with

$$\delta = \frac{1}{a}, \quad 1 - s = \frac{1 + \frac{1}{2} \nu}{a + \nu}.$$  

Note that $|x-y| \sim n^{1/\delta}$ and $\epsilon \leq n^{-s/\delta}$; moreover, since we consider large $|x-y|$, we may assume that $|x-y| \geq (n/2)^{1/\delta}$. We now have, for $n$ large enough (depending only on $s, \delta, a$),

$$n^a (K_{f,t})^{n+2} \exp \left( -\frac{c}{2} n^{\frac{\delta}{2}} + (a - 1) n \ln n \right)$$  
(43)

$$\leq (K_{f,t})^{n+2} \exp \left( -\frac{c}{4} n^{\frac{\delta}{2}} \right),$$  
(44)

$$n^a (K_{f,t})^{n+2} \exp \left( -\frac{s}{\delta} n^{\frac{\delta}{2}} - (a - 1) n \ln n \right)$$  
(45)

$$\leq (K_{f,t})^{n+2} \exp \left( -\frac{1}{2} \frac{s}{\delta} n^{\frac{\delta}{2}} - (a - 1) n \ln n \right),$$  
(46)

$$n^a (K_{f,t})^{n+2} \exp \left( -\frac{c}{2} n^{\frac{\delta}{2}} + (a - 1) n \ln n \right)$$  
(47)

$$\leq (K_{f,t})^{n+2} \exp \left( -\frac{c}{4} n^{\frac{\delta}{2}} \right),$$  
(48)

provided

$$\frac{1 - s}{\delta} > 1 \quad \text{and} \quad \frac{s}{\delta} - a + 1 > 0.$$
But (42) ensures the latter, since
\[
\frac{1 - s}{\delta} = 1 + \nu \quad \text{and} \quad \frac{s}{\delta} - a + 1 = \frac{1}{2}\nu.
\]
Thus the estimate of the theorem depends only on the slowest term (46), and we obtain
\[
\| \chi x f(H) \chi y \| \leq C_{a,a'} (K_{f,1})^{n+2} \exp \left( \frac{a' - a}{4} n \ln n \right).
\]
We get the result since \( n \sim |x - y|^s = |x - y|^\frac{1}{\nu} \) and \( K_{f,1} = e^{C_{f,1}} \).

This finishes the proof of Theorem 1.4.

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References


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