COMPUTING INFIMA ON CONVEX SETS, WITH APPLICATIONS IN HILBERT SPACES

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Abstract. Using intuitionistic logic, we prove that under certain reasonable conditions, the infimum of a real-valued convex function on a convex set exists. This result is then applied to problems of simultaneous approximation in Hilbert space $H$ and the corresponding operator space $B(H)$. This enables us to establish that a bounded, weak-operator totally bounded, convex subset of $B(H)$ is strong-operator located.

1. Introduction

In constructive mathematics—mathematics using intuitionistic, rather than classical, logic—the existence of infima of nonempty subsets of $\mathbb{R}$ that are bounded below is not guaranteed. Indeed, the constructive greatest-lower-bound principle states that a nonempty subset $S$ of $\mathbb{R}$ that is bounded below has an infimum if and only if the following condition holds:

\[ (*) \text{ for all real numbers } \alpha, \beta \text{ with } \alpha < \beta, \text{ either } s > \alpha \text{ for all } s \in S \text{ or else there exists } s \in S \text{ with } s < \beta \text{ (cf. } [2], \text{ page 37)}. \]

A simple application of this principle then shows that $S$ has an infimum if and only if for each $\varepsilon > 0$ there exists $s_0 \in S$ such that $s_0 \leq s + \varepsilon$ for all $s \in S$.

The additional requirement $(*)$ for the existence of the infimum of $S$ poses problems in many aspects of constructive analysis. For example, given an arbitrary point $x$ in a metric space $(X, \rho)$ and an arbitrary nonempty subset $S$ of $X$, we cannot hope to compute the distance

\[ \rho(x, S) = \inf \{ \rho(x, s) : s \in S \} \]

from $x$ to $S$. If we can compute that distance for all $x \in X$, then we say that the set $S$ is located.
Likewise, given a bounded linear mapping $T : X \to Y$ between normed spaces, we cannot guarantee that its norm,

$$
\|T\| = \sup \{\|Tx\| : x \in X, \|x\| \leq 1\},
$$

exists; if the norm does exist, then we say that $T$ is normable. It is worth noting—although we do not use this fact—that a nonzero bounded linear functional on a normed space is normable if and only if its kernel is located.

There are three main results in the present paper. The first, Theorem 2.2, provides conditions under which a convex function that is bounded below on a convex set in a normed space has an infimum. The next, Theorem 3.4, applies this to generalize Ishihara’s result ([8], Theorem 4) about the locatedness of convex subsets of a Hilbert space $H$. Our final theorem, Theorem 3.8, uses Theorem 2.2 to prove that a bounded, weak-operator totally bounded, convex subset of $B(H)$ is strong-operator located, a result that may prove valuable in the constructive theory of operator algebras (cf. [6, 4]).

2. Infima of convex functions

Let $X$ be a normed space, and let $X^*$ be the space of all bounded (but not necessarily normable) linear functionals on $X$. A subset $S$ of $X$ is said to be weakly totally bounded if for each normable $x^* \in X^*$ the set $\{\langle x, x^* \rangle : x \in S \}$ is totally bounded in $C$. Every ball in $X$ is weakly totally bounded. It is shown in [7] (Proposition 3) that a convex subset $C$ of $\mathbb{C}^n$ is totally bounded if and only if the supremum

$$
\sup \{\text{Re} f(x) : x \in C\}
$$

exists for each linear functional $f$ on $\mathbb{C}^n$, from which it follows that a subset $S$ of a normed space $X$ is weakly totally bounded if and only if the supremum

$$
\sup \{\text{Re} \langle x, x^* \rangle : x \in S\}
$$

exists for each normable linear functional $x^*$ on $X$.

Let $C$ be a nonempty subset of $X$, and let $f$ be a mapping of $C$ into $\mathbb{R}$. We say that $f$ is uniformly differentiable on $C$ if there exists a mapping $x \mapsto x^*$, taking $C$ into the set of normable linear functionals on $X$, such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$
|f(x) - f(u) - \langle x - u, x^* \rangle| \leq \varepsilon \|x - u\|
$$

whenever $x, u \in C$ and $\|x - u\| < \delta$. The mapping $x \mapsto x^*$ of $C$ into $X^*$ is then called the derivative of $f$ (on $C$).

A mapping $f$ of a convex subset $C$ of $X$ into $\mathbb{R}$ is said to be convex if for all $x, y \in C$ and $\lambda \in [0, 1]$,

$$
f(\lambda x + (1 - \lambda y)) \leq \lambda f(x) + (1 - \lambda) f(y).
$$

**Lemma 2.1.** Let $C$ be a bounded convex subset of a normed space $X$ such that $\sup \{\langle y, x^* \rangle : y \in C\}$ exists for each $x^* \in X^*$. Let $f$ be a uniformly differentiable convex mapping of $C$ into $\mathbb{R}$. Then for each $\varepsilon > 0$ there exists $\tau > 0$ such that if $x \in C$, then

- either $f(x) \leq f(y) + \varepsilon$ for all $y \in C$,
- or there exists $z \in C$ such that $f(z) < f(x) - \tau \varepsilon$. 
Proof. Choose $M > 1$ such that $\|x - y\| \leq M$ for all $x, y \in C$. Let $\varepsilon > 0$, and choose $\delta \in (0, 2M)$ such that for each $x \in C$ there exists a normable $x^* \in X^*$ such that

$$|f(x) - f(u) - \langle x - u, x^* \rangle| \leq \frac{\varepsilon}{8M} \|x - u\|$$

whenever $u \in C$ and $\|x - u\| < \delta$. Fix $x \in C$, and construct the corresponding normable linear functional $x^*$. Either $\sup_{u \in C} \langle x - u, x^* \rangle < \varepsilon/2$ or else $\sup_{u \in C} \langle x - u, x^* \rangle > \varepsilon/4$. In the first case, given $u \in C$ and setting $\gamma = \frac{\delta}{2M}$, we have

$$(2.1) \quad \|x - y\| = \lambda \|x - u\| \leq \lambda M = \frac{\delta}{2}.$$ 

Hence

$$\frac{\varepsilon}{8} \lambda = \frac{\varepsilon}{8M} \lambda M \geq \frac{\varepsilon}{8M} \|x - y\| \geq f(x) - f(y) - \langle x - y, x^* \rangle \geq f(x) - (1 - \lambda) f(u) - \lambda \langle x - u, x^* \rangle = \lambda (f(x) - f(u) - \langle x - u, x^* \rangle).$$

Dividing throughout by $\lambda$ and rearranging, we obtain

$$f(x) \leq f(u) + \frac{\varepsilon}{8} + \langle x - u, x^* \rangle < f(u) + \frac{\varepsilon}{8} + \frac{\varepsilon}{2} < f(u) + \varepsilon.$$

In the case $\sup_{u \in C} \langle x - u, x^* \rangle > \varepsilon/4$, choose $u \in C$ with $\langle x - u, x^* \rangle > \varepsilon/4$ and define $\lambda, y$ as in (2.1). Then

$$-\frac{\varepsilon}{8} \lambda \leq f(x) - f(y) - \langle x - y, x^* \rangle.$$

So

$$f(y) \leq f(x) + \frac{\varepsilon}{8} \lambda - \lambda \langle x - u, x^* \rangle < f(x) + \lambda \left( \frac{\varepsilon}{8} - \frac{\varepsilon}{4} \right) = f(x) - \frac{\varepsilon}{8},$$

where $\tau = \delta/(16M)$. 

**Theorem 2.2.** Let $C$ be a nonempty, bounded, weakly totally bounded, convex subset of a normed space $X$, and $f$ a uniformly differentiable, convex mapping of $C$ into $\mathbb{R}$ that is bounded below. Then $\inf f$ exists.

**Proof.** Without loss of generality, we may assume that $f(x) \geq 0$ for each $x \in C$. It is enough to find, for each $\varepsilon > 0$, a point $x \in C$ such that $f(x) \leq f(y) + \varepsilon$ for all $y \in C$. To this end, fixing $x_1 \in C$ and using Lemma 2.1, construct a sequence $(x_n)$ in $C$ and an increasing binary sequence $(\lambda_n)$ such that

$$\Rightarrow \quad \text{if } \lambda_n = 0, \text{ then } f(x_{n+1}) < f(x_n) - \tau \varepsilon;$$
if \( n = 1 \), then \( f(x_n) \leq f(y) + \varepsilon \) for all \( y \in C \), and \( x_k = x_n \) for all \( k > n \).

We may assume that \( \lambda_1 = 0 \). Choose a positive integer \( N \) such that \( f(x_1) - N\tau\varepsilon < 0 \). If \( \lambda_N = 0 \), then \( f(x_{N+1}) < f(x_1) - N\tau\varepsilon < 0 \), a contradiction. Hence \( \lambda_N = 1 \), and we are through.

3. Applications to approximation problems

In this section we show how Theorem 2.2 can be applied to solve two approximation problems. We first give a couple of elementary lemmas.

**Lemma 3.1.** Let \( a_1, \ldots, a_n \) be nonnegative numbers, and let \( \delta \) be a positive number. Then

\[
0 < \sum_{i=1}^{n} \left( a_i^2 + \delta \right)^{1/2} - \sum_{i=1}^{n} a_i \leq n\delta^{1/2}.
\]

**Proof.** For each \( i \) we have

\[
0 < \left( (a_i^2 + \delta)^{1/2} - a_i \right) \leq \frac{(a_i^2 + \delta)^{1/2} + a_i}{(a_i^2 + \delta)^{1/2} + a_i} \leq \frac{a_i^2 + \delta - a_i^2}{\delta^{1/2}} = \delta^{1/2}.
\]

Summing over \( i \), we obtain the desired inequalities.

**Lemma 3.2.** Let \( f_1, \ldots, f_n \) be mappings of a set \( S \) into \( \mathbb{R}^{0+} \) such that for each \( \delta > 0 \) the infimum

\[
\inf \left\{ \sum_{i=1}^{n} (f_i(y)^2 + \delta)^{1/2} : y \in S \right\}
\]

exists. Then \( \inf \{ \sum_{i=1}^{n} f_i(y) : y \in S \} \) exists.

**Proof.** Given \( \varepsilon > 0 \), set \( \delta = \left( \frac{\varepsilon}{2n} \right)^2 \) and choose \( y_0 \in S \) such that

\[
\sum_{i=1}^{n} (f_i(y_0)^2 + \delta)^{1/2} \leq \sum_{i=1}^{n} (f_i(y)^2 + \delta)^{1/2} + \varepsilon \quad (y \in S).
\]

For each \( y \in S \), Lemma 3.1 then shows that

\[
\sum_{i=1}^{n} f_i(y_0) < \sum_{i=1}^{n} (f_i(y_0)^2 + \delta)^{1/2}
\]

\[
\leq \sum_{i=1}^{n} (f_i(y)^2 + \delta)^{1/2} + \varepsilon \quad \leq \sum_{i=1}^{n} f_i(y) + n\delta^{1/2} + \frac{\varepsilon}{2} = \sum_{i=1}^{n} f_i(y) + \varepsilon.
\]

The desired result now follows from an observation made in the introduction.

**Proposition 3.3.** The norm \( ||\cdot|| \) on a Hilbert space \( H \) is uniformly differentiable on any set that is bounded away from 0. In fact, if \( 0 < \varepsilon < 1 \), \( ||x|| > r > 0 \), and \( 0 < ||h|| < r\varepsilon/2 \), then

\[
0 \leq ||x + h|| - ||x|| - \left\langle h, \frac{x}{||x||} \right\rangle \leq \varepsilon ||h||.
\]
Proof. Let $x, h$ be vectors in $H$ with $0 < \|h\| < \|x\|$. Then

$$\|x\| - \|x - h\| \leq \|x\| - \left\langle x - h, \frac{x}{\|x\|} \right\rangle$$

$$= \left\langle x - (x - h), \frac{x}{\|x\|} \right\rangle = \left\langle h, \frac{x}{\|x\|} \right\rangle$$

$$= \left\langle x + h - x, \frac{x}{\|x\|} \right\rangle$$

$$= \left\langle x + h, \frac{x}{\|x\|} \right\rangle - \|x\|$$

$$\leq \|x + h\| - \|x\|.$$  

Hence

$$(3.1) \quad 0 \leq \|x + h\| - \|x\| - \left\langle h, \frac{x}{\|x\|} \right\rangle \leq \|x + h\| - \|x - h\| - 2\|x\|.$$  

Set

$$u = \frac{x + h}{\|x + h\|}, \quad v = \frac{x - h}{\|x - h\|}.$$  

Then the right-hand side of (3.1) becomes

$$\langle x + h, u \rangle + \langle x - h, v \rangle - 2\|x\| \leq \|x\| + \langle h, u \rangle + \|x\| - \langle h, v \rangle - 2\|x\|$$

$$= \langle h, u - v \rangle.$$  

Hence

$$(3.2) \quad 0 \leq \|x + h\| - \|x\| - \left\langle h, \frac{x}{\|x\|} \right\rangle \leq \|h\| \|u - v\|.$$  

Now,

$$\frac{x + h}{\|x + h\|} - \frac{x}{\|x\|} = \left( \frac{x + h}{\|x + h\|} - \frac{x}{\|x + h\|} \right) + \left( \frac{x}{\|x + h\|} - \frac{x}{\|x\|} \right)$$

$$= \frac{h}{\|x + h\|} + \frac{\|x\| - \|x + h\|}{\|x + h\| \|x\|} x. $$

So

$$\left\| \frac{x + h}{\|x + h\|} - \frac{x}{\|x\|} \right\| \leq \frac{\|h\|}{\|x + h\|} + \frac{\|x\| - \|x + h\|}{\|x + h\|} \leq \frac{2\|h\|}{\|x + h\|}.$$  

Likewise,

$$\left\| \frac{x - h}{\|x - h\|} - \frac{x}{\|x\|} \right\| \leq \frac{2\|h\|}{\|x - h\|}.$$  

Hence

$$\|u - v\| \leq \left\| \frac{x + h}{\|x + h\|} - \frac{x}{\|x\|} \right\| + \left\| \frac{x - h}{\|x - h\|} - \frac{x}{\|x\|} \right\|$$

$$\leq 2\|h\| \left( \frac{1}{\|x + h\|} + \frac{1}{\|x - h\|} \right).$$  

If $r > 0$, $0 < \varepsilon < 1$, and $0 < \|h\| < r\varepsilon/2$, then, by the foregoing and (3.2), we have

$$0 \leq \|x + h\| - \|x\| - \left\langle h, \frac{x}{\|x\|} \right\rangle \leq \frac{4\|h\|}{r/2} \|h\| \leq 4\varepsilon \|h\|.$$  

This completes the proof. \qed
If $f, g$ are uniformly differentiable mappings of a subset $C$ of a Hilbert space $H$ into $\mathbb{R}$, with derivatives $x \rightsquigarrow x_f$ and $x \rightsquigarrow x_g$ on $C$, then $f + g$ is differentiable, and has derivative $x \rightsquigarrow x_f^* + x_g^*$ on $C$. Note that, as an example in [2] shows, it cannot be proved constructively that the sum of two normable linear functionals on a Banach space is normable; hence it also cannot be proved that the sum of two differentiable mappings in that more general context is differentiable.

We now have the first of our two approximation results.

**Theorem 3.4.** Let $C$ be a bounded, weakly totally bounded, convex subset of a Hilbert space $H$, and let $x_1, \ldots, x_n$ be points of $H$. Then

$$\inf \left\{ \sum_{i=1}^{n} \|x_i - y\| : y \in C \right\}$$

exists.

**Proof.** Assume, to begin with, that $\rho(x_i, C) > 0$ for each $i$. Then Proposition 3.3 shows that each of the convex functions $y \mapsto \|x_i - y\|$ is uniformly differentiable on $C$, as therefore is the convex function $y \mapsto \sum_{i=1}^{n} \|x_i - y\|$ . Applying Theorem 2.2 to the latter function, we obtain the desired infimum.

It remains to remove the condition that $\rho(x_i, C) > 0$ for each $i$. We accomplish this as follows. Let $H'$ be the Hilbert space $H \oplus H$. Picking a unit vector $e$ in $H$, and given any $\delta > 0$, define $C' = C \oplus \{ \delta^{1/2}e \}$ and $x'_i = (x_i, 0)$ ($1 \leq i \leq n$). Then $C'$ is a bounded, weakly totally bounded, convex subset of $H'$. Also, for each $i$ and each $y' = (y, \delta^{1/2}e)$ in $C'$,

$$\|x'_i - y'\| = \sqrt{\|x_i - y\|^2 + \delta} \geq \delta^{1/2};$$

so $\rho(x'_i, C') > 0$. By the first part of the proof, the quantity

$$m_\delta = \inf \left\{ \sum_{i=1}^{n} \|x'_i - y'\| : y' \in C' \right\}$$

$$= \inf \left\{ \sum_{i=1}^{n} \left( \|x_i - y\|^2 + \delta \right)^{1/2} : y \in C \right\}$$

exists. The desired conclusion now follows from Lemma 3.2. \qed

In the special case $n = 1$, we obtain the following result from [8].

**Corollary 3.5.** Let $C$ be a bounded, convex subset of a Hilbert space $H$ such that

$$\sup \{ \langle x, x' \rangle : x \in C \}$$

exists for each $x' \in X$. Then $C$ is located in $H$.

Our second application of Theorem 2.2 deals with linear subsets of $B(H)$, where $H$ is a Hilbert space. It requires one more lemma and some constructions.

**Lemma 3.6.** Let $H$ be a direct sum $\bigoplus_{i=1}^{n} H_i$ of finitely many Hilbert spaces, and denote by $pr_i$ the mapping of $H$ onto $H_i$ that takes $x = (x_1, \ldots, x_n)$ to $x_i$. Then $\|pr_i(.)\|$ is uniformly differentiable on each subset $S$ of $H$ for which $pr_i(S)$ is bounded away from $0$.

\footnote{The inequality $\rho(x_i, C) > 0$ is not predicated on the existence of the distance from $x_i$ to $C$. Rather, it is a shorthand for the statement that $x_i$ is bounded away from $C$.}
Proof. Let $S$ be such a subset of $H$, and let $\varepsilon > 0$. We see from Proposition 3.3 that there exists $\delta > 0$ such that if $x_i, u_i \in H_i$ and $\|x_i - u_i\| < \delta$, then
\[
\left\| x_i - u_i - \left( x_i - u_i, \frac{x_i}{\|x_i\|} \right) \right\| \leq \varepsilon \|x_i - u_i\|.
\]
Let $x, u \in H$ and $\|x - u\| < \delta$. Then $\|x_i - u_i\| < \delta$, and so
\[
\left\| x_i - u_i - \left( x - u, \left( 0, \ldots, \frac{x_i}{\|x_i\|}, 0, \ldots, 0 \right) \right) \right\| \leq \varepsilon \|x_i - u_i\| \leq \varepsilon \|x - u\|.
\]
This is what we want. \qed

Recall that
- the strong-operator topology on $B(H)$ is the locally convex one induced by the seminorms $\|\cdot\|_F$, where $F = \{x_1, \ldots, x_n\}$ is a finitely enumerable subset of $H$ and
  \[
  \|T\|_F = \left( \sum_{i=1}^{n} \|Tx_i\|^2 \right)^{1/2};
  \]
- the weak-operator topology on $B(H)$ is the locally convex one induced by the seminorms
  \[
  T \mapsto \sum_{i=1}^{n} |\langle Tx_i, y_i \rangle|,
  \]
  where $x_1, \ldots, x_n, y_1, \ldots, y_n$ are elements of $H$.

Let $H_n$ be the direct sum $\bigoplus_{i=1}^{n} H$ of $n$ copies of $H$. To each $T \in B(H)$ there corresponds an element $\overline{T}$ of $B(H_n)$ defined by
\[
\overline{T}x = (Tx_1, \ldots, Tx_n)
\]
for each $x = (x_1, \ldots, x_n) \in H_n$. The mapping $T \mapsto \overline{T}$ is uniformly continuous on $B(H)$ relative to the weak-operator topologies on $B(H)$ and $B(H_n)$: this readily follows from the identity
\[
\langle \overline{T}x, y \rangle = \sum_{i=1}^{n} \langle Tx_i, y_i \rangle.
\]

For each subset $A$ of $B(H)$, define
\[
\overline{A} = \{ \overline{T} : T \in A \}.
\]
Note that if $A$ has any of the properties “bounded”, “convex”, or “weak-operator totally bounded”, then $\overline{A}$ has the same property. For example, if $A$ is weak-operator totally bounded, then the uniform continuity of the mapping $T \mapsto \overline{T}$ relative to the weak-operator topologies ensures that $\overline{A}$ is weak-operator totally bounded (\[8\], Proposition 2.1.7). Incidentally, the unit ball $B_1(H)$ of $B(H)$ is weak-operator totally bounded [8].

Similarly, we define $H_{\infty}$ to be the direct sum $\bigoplus_{i=1}^{\infty} H$ of a sequence of copies of $H$, and we define $\overline{T}$ (for each $T \in B(H)$) and $\overline{A}$ in the obvious way. In this case
also, the properties “bounded”, “convex”, and “weak-operator totally bounded” transfer from $\mathcal{A}$ to $\bar{\mathcal{A}}$.

**Lemma 3.7.** Let $H$ be a Hilbert space, and let $\mathcal{A}$ be a nonempty subset of $\mathcal{B}(H)$ that is (uniformly) bounded, weak-operator totally bounded, and convex. Let $x_1, \ldots, x_n$ be elements of $H$, and let $S$ be an element of $\mathcal{B}(H)$. Then the infimum

$$\inf \left\{ \sum_{i=1}^{n} \| Sx_i - Tx_i \| : T \in \mathcal{A} \right\}$$

exists.

**Proof.** To begin with, suppose that $\rho(Sx_i, Ax_i) > 0$ for each $i$. (Note that $Ax_i$ is located, by Corollary 3.5.) Using the foregoing construction of $H_n$, $\bar{T}$, and $\bar{\mathcal{A}}$, we see that $\bar{\mathcal{A}}$ is bounded, convex, and weak-operator totally bounded. Let $x = (x_1, \ldots, x_n) \in H_n$, and let

$$C = \{ ((S - T)x_1, \ldots, (S - T)x_n) : T \in \mathcal{A} \} = \left\{ \left( \tilde{S} - \tilde{T} \right) x : T \in \mathcal{A} \right\}.$$

Then $C$ is a bounded, weakly totally bounded, convex subset of $H_n$. Define $f : C \to \mathbb{R}^+$ by

$$f \left( \left( \tilde{S} - \tilde{T} \right) x \right) = \sum_{i=1}^{n} \| Sx_i - Tx_i \|.$$

Then $f$ is convex and, in view of Lemma 3.6 and the extra supposition at the start of this proof, uniformly differentiable on $C$. So, by Theorem 2.2, $\inf_{\xi \in C} f(\xi)$ exists. But this infimum is precisely the quantity that we want.

To remove our extra supposition, let $H' = H \oplus H$. Given $\delta > 0$, define $\mathcal{A}' = \mathcal{A} \oplus \{ \delta^{1/2}I \}$, $S' = (S, 0)$, and $x'_i = (x_i, e)$ $(1 \leq i \leq n)$. For each $T \in \mathcal{A}$ we have

$$\| S'x'_i - \left( T, \delta^{1/2}I \right) x'_i \| = \left( \| Sx_i - Tx_i \|^2 + \delta \right)^{1/2} \geq \delta^{1/2};$$

so $\rho(S'x'_i, Ax'_i) > 0$. It follows from the first part of the proof that the quantity

$$m_\delta = \inf \left\{ \sum_{i=1}^{n} \| S'x'_i - \left( T, \delta^{1/2}I \right) x'_i \| : T \in \mathcal{A} \right\}$$

exists. The desired conclusion now follows from Lemma 3.2. \[\square\]

Let $X$ be a vector space over $\mathbb{C}$, and $(\| \cdot \|_i)_{i \in I}$ a family of seminorms defining a locally convex structure on $X$. We say that a subset $S$ of $X$ is **located** if for each finitely enumerable subset $F$ of $I$ and for each $x \in X$ the infimum

$$\inf \left\{ \sum_{i \in F} \| x - s \|_i : s \in S \right\}$$

exists. In other words, $S$ is located in $X$ if and only if, for each finitely enumerable subset $F$ of $I$, it is located with respect to the pseudometric derived from the seminorm $\sum_{i \in F} \| \cdot \|_i$. 
Theorem 3.8. Let $H$ be a Hilbert space, and let $A$ be a nonempty subset of $B(H)$ that is (uniformly) bounded, weak-operator totally bounded, and convex. Then $A$ is strong-operator located in $B(H)$.

Proof. Let $H_{\infty} = \bigoplus_{m=1}^{\infty} H$ be the Hilbert space direct sum of a sequence of copies of $H$, and denote the inner product and norm on $H_{\infty}$ by $\langle \cdot, \cdot \rangle_{\infty}$ and $\|\cdot\|_{\infty}$ respectively. For $1 \leq k \leq m$, let

$$F_k = \{x_1^k, \ldots, x_n^k\}$$

be a finitely enumerable subset of $H$, and let

$$\xi_k = (x_1^k, \ldots, x_n^k, 0, 0, 0, \ldots) \in H_{\infty}.$$ 

For each $S \in B(H)$ define $\tilde{S} \in B(H_{\infty})$, and also define $\tilde{A}$, as indicated immediately before the statement of Lemma 3.7. Then $\tilde{A}$ is (uniformly) bounded, weak-operator totally bounded, and convex in $B(H_{\infty})$. Consider any $S \in B(H)$. Applying Lemma 3.7 in $H_{\infty}$, we see that

$$\inf \left\{ \sum_{k=1}^{n} \| \tilde{S}\xi_k - \tilde{T}\xi_k \| : T \in \tilde{A} \right\}$$

exists. But this number equals

$$\inf_{T \in A} \left( \sum_{k=1}^{n} \left( \sum_{i=1}^{n_k} \| Sx_i^k - Tx_i^k \|^2 \right)^{1/2} \right) = \inf_{T \in A} \sum_{k=1}^{n} \| S - T \|_{F_k}.$$ 

This is what we want. \qed

In the classical theory of algebras of operators on a Hilbert space $H$, any subset of $B_1(H)$ is weak-operator totally bounded. Since this is not the case constructively, it seems sensible in the constructive theory of operators to require of a subspace, or subalgebra, $A$ of $B(H)$ that its unit ball $A_1$ be weak-operator totally bounded. In that case, although in the infinite-dimensional case it is false that $A_1$ is strong-operator compact, Theorem 3.8 shows that $A_1$ has the potentially useful (if classically vacuous) property of strong-operator locatedness.

References


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