G-STRUCTURE ON THE COHOMOLOGY OF HOPF ALGEBRAS

MARCO A. FARINATI AND ANDREA L. SOLOTAR

Abstract. We prove that $\Ext^\bullet_k(k,k)$ is a Gerstenhaber algebra, where $A$ is a Hopf algebra. In case $A = D(H)$ is the Drinfeld double of a finite-dimensional Hopf algebra $H$, our results imply the existence of a Gerstenhaber bracket on $H^\bullet_{GS}(H,H)$. This fact was conjectured by R. Taillefer. The method consists of identifying $H^\bullet_{GS}(H,H) \cong \Ext^\bullet_k(k,k)$ as a Gerstenhaber subalgebra of $H^\bullet(A,A)$ (the Hochschild cohomology of $A$).

Introduction

The motivation of this paper is to prove that $H^\bullet_{GS}(H,H)$ has a structure of a G-algebra. The G-algebra structure is, roughly speaking, the existence of two products with compatibilities between them: one is associative graded commutative, and the other is a graded Lie bracket. We prove this result when $H$ is a finite-dimensional Hopf algebra (see Theorem 2.1 and Corollary 2.5). $H^\bullet_{GS}$ is the cohomology theory for Hopf algebras defined by Gerstenhaber and Schack in [4]. In order to obtain commutativity of the cup product we prove a general statement on Ext groups over Hopf algebras (without any finiteness assumption).

When $H$ is finite dimensional, the category of Hopf bimodules is isomorphic to a module category, over an algebra $X$ (also finite dimensional) defined by Cibils and Rosso (see [2]), and this category is also equivalent to the category of Yetter-Drinfeld modules, which is isomorphic to the category of modules over the Hopf algebra $D(H)$ (the Drinfeld double of $H$). In [11], Taillefer has defined a natural cup product in $H^\bullet_{GS}(H,H) = H^\bullet_k(H,H)$ (see [3] for the definition of $H^\bullet_k$). When $H$ is finite dimensional, she proved that $H^\bullet_k(H,H) \cong \Ext^\bullet_X(k,k)$, and using this isomorphism she showed that it is (graded) commutative. In a later work [11] she extended the result of commutativity of the cup product to arbitrary-dimensional Hopf algebras, and she conjectured the existence (and a formula) of a Gerstenhaber bracket.

Our method for giving a Gerstenhaber bracket is the following: under the equivalence of categories $X$-mod $\cong D(H)$-mod, the object $H$ corresponds to $H^{coH} = k$. So $\Ext^\bullet_X(k,k) \cong \Ext^\bullet_{D(H)}(k,k)$ (isomorphism of graded algebras); according to Štefan [8] one knows that $\Ext^\bullet_D(k,k) \cong H^\bullet(D(H),k)$. In Theorem 1.8 we prove...
that, if $A$ is an arbitrary Hopf algebra, then $H^\bullet(A, k)$ is isomorphic to a subalgebra of $H^\bullet(A, A)$—in particular, it is graded commutative—and the morphisms are defined at the complex level. In Theorem 2.1 we prove that the image of $C^\bullet(A, k)$ in $C^\bullet(A, A)$ is stable under the brace operation (if $M$ is an $A$-bimodule, $C^\bullet(A, M)$ denotes the standard Hochschild complex whose homology is $H^\bullet(A, M)$); in particular, the image of $H^\bullet(A, k)$ is closed under the Gerstenhaber bracket of $H^\bullet(A, A)$. So, the existence of the Gerstenhaber bracket on $H^\bullet_G(H, H)$ follows, at least in the finite-dimensional case, by taking $A = D(H)$. We did not know if this bracket coincides with the formula proposed in [11], but Taillefer, in a personal communication, told us that, using arguments as in [7], one can actually prove that the bracket given by us, in the finite-dimensional case, must agree with the bracket proposed by her. Nevertheless, the argument does not give a proof of existence in the infinite-dimensional case. So the problem, in that generality, remains open.

We also provide a proof that the algebra $\text{Ext}_C^\bullet(k, k)$ is graded commutative when $C$ is a braided monoidal category satisfying certain exactness hypotheses (see Theorem 1.4). This gives an alternative proof of the commutativity of the cup product in the arbitrary-dimensional case by taking $C = H^\bullet H^\bullet YD$, the category of Yetter-Drinfeld modules.

In this paper $A$ will denote a Hopf algebra over a field $k$.

1. Cup products

This section has two parts. First we prove a generalization of the fact that the cup product on group cohomology $H^\bullet(G, k)$ is graded commutative. The general abstract setting is that of a braided (abelian) category with enough injectives satisfying an exactness condition (see Definition 1.2 below). The other part will concern the relation between self extensions of $k$ and Hochschild cohomology of $A$ with coefficients in $k$.

Let us recall the definition of a braided category:

**Definition 1.1.** The data $(C, \otimes, k, c)$ is called a **braided** category with unit element $k$ if

1. $C$ is an abelian category.
2. $- \otimes -$ is a bifunctor, bilinear, associative, and there are natural isomorphisms $k \otimes X \cong X \otimes k$ for all objects $X$ in $C$.
3. For all pair of objects $X$ and $Y$, $c_{X, Y} : X \otimes Y \to Y \otimes X$ is a natural isomorphism. The isomorphisms $c_{X,k} : X \otimes k \cong k \otimes X$ agree with the isomorphism of the unit axiom, and for all triples $X, Y, Z$ of objects in $C$, the Yang-Baxter equation is satisfied:

$$(\text{id}_Z \otimes c_{X,Y}) \circ (c_{X,Z} \otimes \text{id}_Y) \circ (\text{id}_X \otimes c_{Y,Z}) = (c_{Y,Z} \otimes \text{id}_X) \circ (\text{id}_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes \text{id}_Z).$$

A data $(C, \otimes, k)$ satisfying axioms 1 and 2, but not necessarily axiom 3 is called a **monoidal** category.

We will use the notion of exact functor for a monoidal structure.

**Definition 1.2.** Let $(C, \otimes, k)$ be an abelian monoidal category. We say that $\otimes$ is exact if and only if the canonical morphism

$$H_\bullet(X_*, d_X) \otimes H_\bullet(Y_*, d_Y) \to H_\bullet(X_* \otimes Y_*, d_{X \otimes Y})$$

is an isomorphism for all pairs of complexes in $C$. 

Example 1.3. Let $H$ be a Hopf algebra over a field $k$. Then $\mathcal{C} =_{H}\text{-mod}$ is a monoidal category with $\otimes = \otimes_k$, and this functor is clearly exact.

Theorem 1.4. Let $(\mathcal{C}, \otimes, k, c)$ be a braided category with enough injectives and exact tensor product. Then $\text{Ext}^\bullet_\mathcal{C}(k, k)$ is graded commutative.

Proof. We proceed as in the proof that $H^\bullet(G, k)$ is graded commutative (see for example [1], page 51, Vol. I). The proof is based on two points: first a definition of a cup product using the bifunctor $\otimes$, and second a lemma relating this construction and the Yoneda product of extensions.

Let $0 \to M \to X_p \to \ldots X_1 \to N \to 0$ and $0 \to M' \to X'_q \to \ldots X'_1 \to N' \to 0$ be two extensions in $\mathcal{C}$. Then $N_* := (0 \to M \to X_p \to \ldots X_1 \to 0)$ and $N'_* := (0 \to M' \to X'_q \to \ldots X'_1 \to 0)$ are two complexes, quasi-isomorphic to $N$ and $N'$ respectively. By the Künneth formula, $N_* \otimes N'_*$ is a complex quasi-isomorphic to $N \otimes N'$. So “completing” this complex with $N \otimes N'$ (more precisely considering the mapping cone of the chain map $N_* \otimes N'_* \to N \otimes N'$) one has an extension in $\mathcal{C}$, beginning with $M \otimes M'$ and ending with $N \otimes N'$.

So, we have defined a cup product:

$$\text{Ext}^p_\mathcal{C}(N, M) \times \text{Ext}^q_\mathcal{C}(N', M') \to \text{Ext}^{p+q}_\mathcal{C}(N \otimes N', M \otimes M').$$

We will denote this product by $\otimes$, and the Yoneda product by $\sim$. The lemma relating this product and the Yoneda one is the following:

Lemma 1.5. If $\eta \in \text{Ext}^p_\mathcal{C}(M, N)$ and $\xi \in \text{Ext}^q_\mathcal{C}(M', N')$, then

$$\eta \otimes \xi = (\eta \otimes \text{id}_{N'}) \sim (\text{id}_M \otimes \xi).$$

Proof of the Lemma. Interpreting the elements $\eta$ and $\xi$ as extensions, it is clear how to define a morphism of complexes $(\eta \otimes \text{id}_{N'}) \sim (\text{id}_M \otimes \xi)$, and by the Künneth formula, it is a quasi-isomorphism.

In the particular case that $M = M' = N = N' = k$, the lemma implies that $\eta \otimes \xi = \eta \sim \xi$ for all $\eta$ and $\xi$ in $\text{Ext}^p_\mathcal{C}(k, k)$. Now the theorem is a consequence of the isomorphism $(X_* \otimes X_*, d_{X \otimes Y}) \cong (Y_* \otimes X_*, d_{Y \otimes X})$, valid for every pair of complexes in $\mathcal{C}$, defined by

$$(-1)^{pq}c_{X_*, Y_*} : X_p \otimes Y_q \to Y_q \otimes X_p.$$

Note that the differentials are morphisms in the category $\mathcal{C}$. So the map defined above commutes with the differentials because of the bifunctoriality of the braiding.

Example 1.6. Let $H$ be a cocommutative Hopf algebra. Then $\_H$-mod is braided with $c$ the usual flip. When $H = k[G]$ we recover that $H^\bullet(G, k)$ is graded commutative. The other typical example is $H = U(\mathfrak{g})$, the enveloping algebra of a Lie algebra $\mathfrak{g}$. It is known that $\text{Ext}^\bullet_{U(\mathfrak{g})}(k, k) = \Lambda^\bullet(\mathfrak{g})$, is graded commutative.

Example 1.7. Let $H$ be an arbitrary Hopf algebra with bijective antipode and $\mathcal{C} = \text{mod}_H \text{YD}$ the category of Yetter-Drinfeld modules over $H$. It is well known (see [6], p. 214) that the map $M \otimes N \to N \otimes M$ defined by $m \otimes n = m_{-1} n \otimes m_0$ is a braiding on $\text{mod}_H \text{YD}$. So $\text{Ext}^\bullet_{\text{mod}_H \text{YD}}(k, k)$ is graded commutative.

Theorem 1.8. If $A$ is a Hopf algebra, then $\text{Ext}^\bullet_A(k, k) \cong H^\bullet(A, k)$. Moreover, $H^\bullet(A, k)$ is isomorphic to a subalgebra of $H^\bullet(A, A)$. 

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Proof. After Štefan [8], since \( A \) is an \( A \)-Hopf Galois extension of \( k \), \( H^\bullet(A, M) \cong \text{Ext}^\bullet_A(k, M^{ad}) \) for all \( A \)-bimodules \( M \).

Here, \( M^{ad} \) denotes the left \( H \)-module with underlying vector space \( M \), but with structure \( h_{\text{ad}} m := h_1 m S(h_2) \). The notation \( (S \text{ for the antipode, and the Sweedler-type summation}) \text{ is the standard one.} \)

In particular, \( H^\bullet(A, k) = \text{Ext}^\bullet_A(k, k) \). But one can give, for this particular case, an explicit morphism at the complex level. In order to do this, we will choose a specific resolution of \( k \) as a left \( A \)-module. Notice that, in particular, our argument will give an alternative proof of Stefan’s result for this case.

Let \( C_n(A, b') \) be the standard resolution of \( A \) as an \( A \)-bimodule, namely \( C_n(A, b') = A \otimes A^{\otimes n} \otimes A \) and \( b'(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1} (a_i \in A) \). This resolution splits on the right. So \( (C_n(A) \otimes A, b' \otimes \text{id}_k) \) is a resolution of \( A \otimes_A k = k \) as a left \( A \)-module. Using this resolution, \( \text{Ext}_A^\bullet(k, k) \) is the cohomology of the complex \( (\text{Hom}_A(C_n(A) \otimes A, k), (b' \otimes \text{id}_k)^*) \cong (\text{Hom}(A^{\otimes n}, k), \partial) \). Under this isomorphism, the differential \( \partial \) is given by

\[
(\partial f)(a_1 \otimes \ldots \otimes a_n) = \epsilon(a_1) f(a_2 \otimes \ldots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) + (-1)^n f(a_1 \otimes \ldots \otimes a_{n-1}) \epsilon(a_n),
\]

which is precisely the formula of the differential of the standard Hochschild complex computing \( H^\bullet(A, k) \).

One can easily check that the cup product on \( \text{Ext}_A^\bullet(k, k) \) which, by Lemma [13], equals the Yoneda product, corresponds to the cup product on \( H^\bullet(A, k) \). So this isomorphism is an algebra isomorphism.

Now we will give two multiplicative maps \( H^\bullet(A, k) \rightarrow H^\bullet(A, A) \) and \( H^\bullet(A, A) \rightarrow H^\bullet(A, k) \). Consider the counit \( \epsilon : A \rightarrow k \). It is an algebra map, and so the induced map \( \epsilon_* : H^\bullet(A, A) \rightarrow H^\bullet(A, k) \) is multiplicative. We will define a multiplicative section of this map.

Let \( f : A^{\otimes p} \rightarrow k \) be a Hochschild cocycle, and define \( \widehat{f} : A^{\otimes p} \rightarrow A \) by the formula

\[
\widehat{f}(a^1 \otimes \ldots \otimes a^p) := a_1^1 \ldots a_1^p \cdot f(a_2^1 \otimes \ldots \otimes a_2^p)
\]

where we have used the Sweedler-type notation with summation symbol omitted: \( a_1^1 \otimes a_1^2 = \Delta(a^1), \) for \( a^1 \in A \).

Let us check that \( \widehat{f} \) is a Hochschild cocycle with values in \( A \),

\[
\partial(\widehat{f})(a^0 \otimes \ldots \otimes a^p) = a^0 \widehat{f}(a^1 \otimes \ldots \otimes a^p) + \sum_{i=0}^{p-1} (-1)^{i+1} \widehat{f}(a^0 \otimes \ldots \otimes a^i a^i+1 \otimes \ldots \otimes a^p) + (-1)^{p+1} \widehat{f}(a^0 \otimes \ldots \otimes a^{p-1}) a^p
\]

\[
= a^0 \cdot a_1^1 \ldots a_1^p \cdot f(a^2_1 \otimes \ldots \otimes a^p_2) + (-1)^{p+1} a^0_1 \ldots a_1^{p-1} \cdot f(a^0_2 \otimes \ldots \otimes a_2^{p-1}) a^p + \sum_{i=0}^{p-1} (-1)^{i+1} a^0_1 \ldots a_1^i a^i+1 \ldots a^p_1 \cdot f(a^0_2 \otimes \ldots \otimes a_2^i a_2^{i+1} \otimes \ldots \otimes a_2^p).
\]
Using that \( f \) is a Hochschild cocycle with values in \( k \), we know that
\[
0 = \epsilon(a^0)f(a^1 \otimes \ldots \otimes a^p) + \sum_{i=0}^{p-1} (-1)^{i+1} f(a^0 \otimes \ldots \otimes a^i.a^{i+1} \otimes \ldots \otimes a^p)
+ (-1)^{p+1} f(a^0 \otimes \ldots \otimes a^{p-1}) \epsilon(a^p).
\]

So, the summation term in \( \partial(\hat{f}) \) can be replaced using the equality
\[
\sum_{i=0}^{p-1} (-1)^{i+1} a^0 \ldots a^i a^{i+1} \ldots a^p f(a^0 \otimes \ldots \otimes a^p)
= -a^0 \ldots a^p \left( \epsilon(a^0)f(a^1 \otimes \ldots \otimes a^p) + (-1)^{p+1} f(a^0 \otimes \ldots \otimes a^{p-1})\epsilon(a^p) \right)
= -(a^0 a^1 \ldots a^p f(a^1 \otimes \ldots \otimes a^p) + (-1)^{p+1} a^0 \ldots a^{p-1}.a^p f(a^0 \otimes \ldots \otimes a^{p-1})
\]

and this finishes the computation of \( \partial(\hat{f}) \).

Clearly \( \epsilon \hat{f} = f \); so \( \epsilon_* \) is a split epimorphism. To check that \( f \mapsto \hat{f} \) is multiplicative is straightforward:

Let \( g : A^{\otimes q} \to k \) be a cocycle and \( \hat{g} : A^{\otimes q} \to A \) the cocycle with values in \( A \) corresponding to \( g \). We can check the following:

\[
\hat{f} \sim g(a^1 \otimes \ldots \otimes a^{p+q}) = a_1^1 \ldots a_{p+q}^1 \cdot (f \sim g)(a_2^1 \otimes \ldots \otimes a_2^{p+q})
= a_1^1 \ldots a_{p+q}^1 . f(a_2^1 \otimes \ldots \otimes a_2^{p+q}) g(a_3^{p+1} \otimes \ldots \otimes a_3^{p+q})
= (\hat{f} \sim \hat{g})(a^1 \otimes \ldots \otimes a^{p+q}).
\]

\( \square \)

2. Brace operations

In this section we prove our main theorem, stating that the map \( H^\bullet(A,k) \to H^\bullet(A,A) \) is “compatible” with the brace operations, and as a consequence with the Gerstenhaber bracket. Note that the map \( H^\bullet(A,k) \to H^\bullet(A,A) \) is defined at the standard complex level. Let us define \( C^p(A,M) := \text{Hom}(A^{\otimes p}, M) \).

**Theorem 2.1.** The image of the map \( C^\bullet(A,k) \to C^\bullet(A,A) \) is stable under the brace operation. Moreover, if \( \hat{f} \) and \( \hat{g} \) are the images in \( C^\bullet(A,A) \) of two elements \( f \) and \( g \) belonging to \( C^\bullet(A,k) \), then \( \hat{f} \circ_1 \hat{g} = \hat{f} \circ_1 \hat{g} \).

**Proof.** Let us recall the definition of the brace operations (see (3)). If \( F : A^{\otimes p} \to M \) and \( G : A^{\otimes q} \to A \) and \( 1 \leq i \leq p \), then \( F \circ_i G : A^{\otimes p+q-1} \to M \) is defined by
\[
(F \circ_i G)(a^1 \otimes \ldots \otimes a^i \otimes b^1 \otimes \ldots \otimes b^q \otimes a^{i+1} \otimes \ldots \otimes a^p)
= F(a^1 \otimes \ldots \otimes a^i \otimes G(b^1 \otimes \ldots \otimes b^q) \otimes a^{i+1} \otimes \ldots \otimes a^p).
\]

Assume now that \( f : A^{\otimes p} \to k \), \( g : A^{\otimes q} \to k \) and \( F = \hat{f} \) and \( G = \hat{g} \), namely
\[
F(a^1 \otimes \ldots \otimes a^p) = a_1^1 \ldots a_p^1 . f(a_2^1 \otimes \ldots \otimes a_2^p)
\]
and similarly for $G$ and $g$. Then (denoting $(a \otimes b)$ by $(a, b)$),

$$(F \circ_i G)(a^1, \ldots, a^i, b^1, \ldots, b^q, a^{i+1}, \ldots, a^p)$$

$$= F(a^1, \ldots, a^i, G(b^1, \ldots, b^q), a^{i+1}, \ldots, a^p)$$

$$= F(a^1, \ldots, a^i, b_1^{1} \ldots b_1^{q} g(b_2^{1}, \ldots, b_2^{q}), a^{i+1}, \ldots, a^p)$$

$$= a_1^1 \ldots a_i^1 b_1^1 \ldots b_1^q d_1^{i+1} \ldots a_1^p f(a_2^1, \ldots, a_2^1, b_2^1 \ldots b_2^q g(b_3^1, \ldots, b_3^q), a_2^{i+1}, \ldots, a_2^p)$$

$$= \tilde{F} \circ_i G(a^1, \ldots, a^i, b_1^1 \ldots b_1^q, a^{i+1}, \ldots, a^p).$$

Recall that the brace operations define a “composition” operation $F \circ G = \sum_{i=1}^p (-1)^{q(i-1)} F \circ_i G$, where $F \in \mathcal{C}^p(A, A)$ and $G \in \mathcal{C}^q(A, A)$. The Gerstenhaber bracket is defined as the commutator of this composition. So we have the desired corollary:

**Corollary 2.2.** If $A$ is a Hopf algebra, then $H^\bullet(A, k)$ is a Gerstenhaber subalgebra of $H^\bullet(A, A)$.

**Example 2.3.** Let $A$ be a Hopf algebra. Then $\text{Ext}_A^1(k, k) \cong \text{Der}(A, k) = \text{Prim}(A^*)$, where $\text{Prim}(A^*) = \{x \in A^* \text{ such that } m^*(x) = x \otimes 1 + 1 \otimes x\}$. It is easy to check that the Lie bracket given in the above theorem coincides with the commutator of the convolution product, viewing $\text{Der}(A, k)$ as a subset of $A^*$.

**Example 2.4.** Let $G$ be a connected affine algebraic group and $g := \text{Ker}(e)/\text{Ker}(e)^2$ its tangent Lie algebra. One has that $HH^\bullet(\mathcal{O}(G), \mathcal{O}(G)) = \Lambda_{\mathcal{O}(G)}^* \text{Der}(\mathcal{O}(G)) \cong \mathcal{O}(G) \otimes \Lambda^* g$, where the Gerstenhaber structure here is the Schouten-Nijenhuis bracket. Also $\text{Ext}_{\mathcal{O}(G)}^\bullet(k, k) = \Lambda^* g$, and it is generated (as an algebra) in degree one. So the bracket is determined by its values on $\text{Ext}_{\mathcal{O}(G)}^1(k, k) = g$, which is the bracket of $g$ as a Lie algebra. This $G$-algebra structure is also well known.

Consider $H$ a finite-dimensional Hopf algebra and $X = X(H)$ the algebra defined by Cibils and Rosso (see [2]). We can prove, at least in the finite-dimensional case, the conjecture of [11] that $H_{GS}(H, H)$ is a Gerstenhaber algebra:

**Corollary 2.5.** Let $H$ be a finite-dimensional Hopf algebra. Then $H_{GS}^\bullet(H, H)$ is a Gerstenhaber algebra.

**Proof.** The isomorphism $H_{GS}^\bullet(H, H) \cong \text{Ext}_{X}^\bullet(H, H)$ was proved in [10].

Let $A$ denote $D(H)$, the Drinfeld double of $H$. One knows that $A$-mod via $M \mapsto M^{\text{co}H}$. Then $\text{Ext}_{X}^\bullet(H, H) \cong \text{Ext}_{A}^\bullet(H^{\text{co}H}, H^{\text{co}H}) = \text{Ext}_{A}^\bullet(k, k)$, and this a Gerstenhaber subalgebra of $H^\bullet(A, A)$.

**References**


Departamento de Matemática Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria Pab I. 1428, Buenos Aires, Argentina

E-mail address: mfarinat@dm.uba.ar

Departamento de Matemática Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria Pab I. 1428, Buenos Aires, Argentina

E-mail address: asolotar@dm.uba.ar