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GREKOS' S FUNCTION HAS A LINEAR GROWTH

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ABSTRACT. An exact additive asymptotic basis is a set of nonnegative integers such that there exists an integer h with the property that any sufficiently large integer can be written as a sum of exactly h elements of $\mathcal{A}$. The minimal such h is the exact order of $\mathcal{A}$ (denoted by $\mathrm{ord}^*(\mathcal{A})$). Given any exact additive asymptotic basis $\mathcal{A}$, we define $\mathcal{A}^*$ to be the subset of $\mathcal{A}$ composed with the elements $a \in \mathcal{A}$ such that $\mathcal{A} \setminus \{a\}$ is still an exact additive asymptotic basis. It is known that $\mathcal{A} \setminus \mathcal{A}^*$ is finite.

In this framework, a central quantity introduced by Grekos is the function $S(h)$ defined as the following maximum (taken over all bases $\mathcal{A}$ of exact order h):

$$S(h) = \max_{\mathcal{A}} \limsup_{a \in \mathcal{A}^*} \mathrm{ord}^*(\mathcal{A} \setminus \{a\}).$$

In this paper, we introduce a new and simple method for the study of this function. We obtain a new estimate from above for S which improves drastically and in any case on all previously known estimates. Our estimate, namely $S(h) \leq 2h$, cannot be too far from the truth since S verifies $S(h) \geq h+1$. However, it is certainly not always optimal since $S(2) = 3$. Our last result shows that $S(h)$ is in fact a strictly increasing sequence.

1. Introduction

Let $\mathcal{A}$ be a set of nonnegative integers. If $h \geq 1$ is an integer, we denote by $h\mathcal{A}$ the set of numbers that can be written as a sum of h (not necessarily distinct) elements of $\mathcal{A}$,

$$h\mathcal{A} = \{a_1 + \cdots + a_h \text{ such that } a_1, \ldots, a_h \in \mathcal{A}\}.$$

This paper deals with *exact additive asymptotic bases*, that is, sets of nonnegative integers $\mathcal{A}$ such that there exists some integer h satisfying $h\mathcal{A} \sim \mathbb{N}$ where the notation $\mathcal{B} \sim \mathcal{C}$ denotes the fact that the symmetric difference of $\mathcal{B}$ and $\mathcal{C}$ is finite. In other words, this corresponds to the fact that the integers, from some point on, are representable as a sum of exactly h elements of $\mathcal{A}$ for some integer h. When $\mathcal{A}$ is an exact additive asymptotic basis, the least such integer h is called the *exact order* of $\mathcal{A}$ (denoted hereafter by $\mathrm{ord}^*(\mathcal{A})$).

The general theory of additive bases is extensively presented in [9], a book to which the reader is referred. See also [5] for a huge number of questions and problems.

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Since in this paper we are uniquely interested in exact additive asymptotic bases, there will be no risk of confusion and we will simply write *bases* for them.

If $h\mathcal{A} \sim \mathbb{N}$, then we denote by $\mathcal{A}^*$ the set of elements a in $\mathcal{A}$ having the property that $\mathcal{A} \setminus \{a\}$ also is a basis (possibly with an exact order that is different). It is known [4], [7] that $\mathcal{A} \setminus \mathcal{A}^*$ is finite. Recently it has even been shown [2] that

$$\max_{h\mathcal{A}\sim\mathbb{N}} \ |\mathcal{A} \setminus \mathcal{A}^*| \asymp \sqrt{\frac{h}{\log h}}.$$

In the study of bases, a central quantity is

$$X(h) = \max_{h\mathcal{A}\sim\mathbb{N}} \ \max_{a\in\mathcal{A}^*} \ \mathrm{ord}^*(\mathcal{A} \setminus \{a\}).$$

It has been shown [4], [11] that $X(2) = 4$ and $X(3) = 7$. With the trivial $X(1) = 1$, these are the only known values of X. It is usually believed that $X(4) = 10$, but only the lower bound has been published up to now: it follows from the study of the basis

$$\mathcal{B} = \{0\} \cup \{2, 13, 24, 35, \dots\} \cup \{5, 16, 27, 38, \dots\} \tag{1.1}$$

composed of 0 and two arithmetic progressions with difference 11.

For general $h \geq 2$, only lower and upper bounds are known, namely

$$\max\left(\left[\frac{h^2+6h+1}{4}\right], \frac{h^2-3h}{3}\right) \leq X(h) \leq \frac{h^2+3h}{2}, \tag{1.2}$$

so that, though known to be quadratically increasing, even the order of magnitude of X is unknown. In (1.2), the lower bound is due to Stöhr [14] (first part) and Grekos [7] (second part of the maximum) and the upper bound to Nash [11].

A remarkable feature in Grekos' method for proving his lower bound in the previously stated result is that bases (and elements to be removed) used in his proof are of a very special kind. Indeed, generalizing the construction of $\mathcal{B}$ in (1.1), Grekos considers sets $\mathcal{A}$ that are the union of some arithmetic progressions with the same difference together with zero; when one removes zero from $\mathcal{A}$, the exact order increases drastically from h to at least $(h^2 - 3h)/3$. In view of this construction, Grekos was led to formulate the idea that, for any basis $\mathcal{A}$ of exact order h, very few elements a of $\mathcal{A}^*$ are such that, when removed, $\mathrm{ord}^*(\mathcal{A} \setminus \{a\}) = X(h)$. Therefore, Grekos introduced the following related quantity:

$$S(h) = \max_{h\mathcal{A}\sim\mathbb{N}} \ \limsup_{a\in\mathcal{A}^*} \ \mathrm{ord}^*(\mathcal{A} \setminus \{a\})$$

and formulated the conjecture that, except for $h = 1$ where trivially $S(1) = X(1) = 1$, $S(h)$ is much smaller than $X(h)$. More modestly, he asked for a proof of

$$S(h) < X(h), \tag{1.3}$$

for any $h \geq 2$, which is tantamount to asking for a proof that only finitely many $a \in \mathcal{A}^*$ have $\mathrm{ord}^*(\mathcal{A} \setminus \{a\}) = X(h)$.

Motivated by this conjecture, Grekos himself proved that $S(3) \leq 6 < X(3)$ (see [8]) and announced, as a consequence of [6], that $S(2) = 3$ (actually, in [6], this result is proved under an additional minimality condition). But Grekos' proof was never published. That is why, as a first result, we present our own proof of $S(2) = 3$ (which is based on the results of [6]).

Theorem 1.

$$S(2) = 3.$$

In the general case $h \geq 2$, until quite recently the best that was known on S was

$$h + 1 \leq S(h) \leq X(h). \tag{1.4}$$

The upper bound is immediate, and this was Grekos' conjecture that it was improvable. Concerning the lower bound, it was first proved by Härtter [10], in a non-constructive manner. It also follows from the following fact: for any integer $h \geq 2$, the set $\mathcal{A}_h$ defined as

$$\Big\{ \sum_{f \in \mathcal{F}} 2^f, \mathcal{F} \text{ being a finite nonempty set of distinct integers congruent mod } h \Big\}$$

is a basis with $\mathrm{ord}^*(\mathcal{A}_h) = h$ such that $\mathrm{ord}^*(\mathcal{A}_h \setminus \{a\}) > h$ for infinitely many $a \in \mathcal{A}_h$. Actually much more is true: $\mathcal{A}_h$ is a minimal basis in the sense that $\mathrm{ord}^*(\mathcal{A}_h \setminus \{a\}) > h$ for any $a \in \mathcal{A}_h$ (as proved in [12]). The subject of minimal bases was initiated by Erdős [3] and Härtter [10].

The first improvement on (1.4), which as a by-product settles Grekos' conjecture for large h (namely for $h \geq 61$ in view of the lower bound (1.2) for $X(h)$), was obtained in [13] where it is shown that for any positive integer h, one has

$$S(h) \leq \frac{h^2}{4} + 4h + 2.$$

In an unpublished paper, de la Bretèche [1] refines the linear term in the above estimate (he obtains $5h/4 + 1$ instead of $4h + 2$) and shows the validity of (1.3) for the cases $4 \leq h \leq 60$, with the effect of proving completely Grekos' conjecture. However, de la Bretèche's estimate has a main term $\sim h^2/4$, still far from the lower bound $h + 1$.

Consequently, the main remaining question was to determine whether the behaviour of S was closer to that of $h+1$ or to that of $h^2/4$. In this paper, we answer this question by proving that the growth order of S is *linear*. More precisely, our new estimate is the following.

Theorem 2. *For any positive integer h,*

$$S(h) \leq 2h.$$

This upper bound overcomes all previous results in this direction. Moreover, its proof adopts a viewpoint drastically different from that used in all the preceding papers on the subject. In addition to the result in itself, the main advantage of the present approach is its simplicity, since any technicality (a usual feature in the subject) is avoided.

To summarize the situation for $h \geq 2$, we now know that

$$h + 1 \leq S(h) \leq 2h, \tag{1.5}$$

and this is the best that is known in any case except for $h = 2$, where Theorem 1 proves that the lower bound is in fact an equality. This seems far from being sufficient to conjecture that $S(h) = h + 1$ (a formula false for $h = 1$) rather than $S(h) = 2h - 1$ (for instance). Also, we may notice that the upper estimate in (1.5) gives a new proof of $S(3) \leq 6$, an upper bound first proved in [8]. It is still undecided whether $S(3)$ is 4, 5 or 6. Maybe, the question of determining $S(3)$ is as difficult as the general problem of computing $S(h)$ for any h.

Finally, to throw some additional light on the behaviour of S, a natural question to investigate is: Is the sequence $S(h)$ strictly increasing? Our final result answers this question positively.

Theorem 3. *For any positive integer h,*

$$S(h+1) \geq S(h) + 1.$$

2. Proof of Theorem 1

Let us begin with a general remark. Let $\mathcal{A}$ be a set of integers. For any integer x, $k(\mathcal{A}+x) = k\mathcal{A} + kx$. Thus a set is a basis if and only if any of its translates is a basis. Also for any $a \in \mathcal{A}$,

$$k\big((\mathcal{A}+x) \setminus \{a+x\}\big) = k(\mathcal{A} \setminus \{a\}) + kx.$$

Therefore, replacing $\mathcal{A}$ with $\mathcal{A} - \min(\mathcal{A})$ does not change anything on the quality of being a basis, on the exact order of $\mathcal{A}$ and on

$$\limsup_{a \in \mathcal{A}^*} \ \mathrm{ord}^*(\mathcal{A} \setminus \{a\}).$$

As a consequence, without loss of generality, we may assume that $\mathcal{A}$ contains 0.

Our proof is by contradiction. Suppose the result of Theorem 1 is false. Since $S(2) \geq 3$ by (1.4), we then have $S(2) > 3$. We can thus consider $\mathcal{A}$, a basis with exact order

$$\mathrm{ord}^*(\mathcal{A}) = 2$$

such that

$$\mathcal{B} = \{b \in \mathcal{A} \text{ such that } b > 0 \text{ and } \mathrm{ord}^*(\mathcal{A} \setminus \{b\}) \geq 4\}$$

is infinite.

For any $b \in \mathcal{B}$, define

$$\mathcal{C}_b = \mathbb{N} \setminus (3(\mathcal{A} \setminus \{b\})).$$

By definition of $\mathcal{B}$, the set $\mathcal{C}_b$ is infinite.

Lemma 1. *Let b be any fixed element in $\mathcal{B}$. For any sufficiently large $c \in \mathcal{C}_b$, the integer $c - b$ belongs to $\mathcal{A} \setminus \{b\}$.*

Proof. Let $b \in \mathcal{B}$. Consider a c in $\mathcal{C}_b$ large enough. Since $\mathcal{A}$ is a basis of exact order 2, c belongs to $2\mathcal{A}$ and can thus be written as $a + a'$ with a and a' in $\mathcal{A}$. If a and a' are both different from b, then $c \in 2(\mathcal{A} \setminus \{b\}) \subseteq 3(\mathcal{A} \setminus \{b\})$, a contradiction. Therefore, assuming $c > 2b$, this shows that c can be written in the form $a + b$ with $a \in \mathcal{A} \setminus \{b\}$. This proves the lemma. □

Lemma 2. *Let $b \in \mathcal{B}$, $a \in \mathcal{A}$ such that $a \neq b$. If $c \in \mathcal{C}_b$ is large enough, then*
(i) $c - a \notin 2(\mathcal{A} \setminus \{b\})$,
(ii) $c - a - b \in \mathcal{A} \setminus \{b\}$.

Proof. If $c - a \in 2(\mathcal{A} \setminus \{b\})$, then

$$c \in a + 2(\mathcal{A} \setminus \{b\}) \subseteq 3(\mathcal{A} \setminus \{b\}),$$

a contradiction since $c \in \mathcal{C}_b$. This proves (i).

If c is large enough, $c - a$ belongs to $2\mathcal{A}$. If $c > a + 2b$, then in view of (i), the only possibility is that $c - a \in b + (\mathcal{A} \setminus \{b\})$. This proves (ii). □

The two preceding lemmas are essentially contained in Grekos' paper [6]. Let us now deduce our key lemma.

Lemma 3. *Let $b \in \mathcal{B}$, $a \in \mathcal{A}$ such that $a \neq b$. Then*

$$a + b \notin 2(\mathcal{A} \setminus \{b\}).$$

Proof. Assume $a + b \in 2(\mathcal{A} \setminus \{b\})$. Choose a sufficiently large c in $\mathcal{C}_b$. Since $c = (c - a - b) + (a + b)$, Lemma 2 (ii) yields $c \in 3(\mathcal{A} \setminus \{b\})$, in contradiction again with the definition of $\mathcal{C}_b$. □

We now proceed to the completion of the proof of Theorem 1.

Fix b and b' in $\mathcal{B}$, with $b > b'$. Let $c \in \mathcal{C}_b$ be large enough (in particular, we suppose $c > 2b' + b$). By Lemma 2 (ii),

$$d = c - b - b' \in \mathcal{A} \setminus \{b\}. \tag{2.1}$$

Observe that $d > b'$; therefore, $d \in \mathcal{A} \setminus \{b, b'\}$.

Applying Lemma 1 gives

$$e = c - b \in \mathcal{A} \setminus \{b\}. \tag{2.2}$$

Since $c > b + b'$, $e \neq b'$. Therefore, by (2.1) and (2.2), we get

$$d + b' = e = e + 0 \in 2(\mathcal{A} \setminus \{b'\}),$$

in contradiction with Lemma 3.

3. Proof of Theorem 2

For the sake of clarity, our proof will again be decomposed in a few lemmas.

Let h be any integer larger than or equal to 2 and $\mathcal{A}$ be a basis with

$$\mathrm{ord}^*(\mathcal{A}) = h.$$

By the same remark as in Section 2, we may assume that $\mathcal{A}$ contains 0.

In the sequel, we define

$$a_1 = \min(\mathcal{A} \cap \{1, 2, \dots\}).$$

By definition, there is an integer $n_0 \geq 1$ such that any $n \geq n_0$ belongs to $h\mathcal{A}$. For any positive $a \in \mathcal{A}$ and $x \in \mathbb{N}$, we define

$$w_a(x) = \min\{k \in \mathbb{N} \text{ such that } x \in k(\mathcal{A} \setminus \{a\})\}.$$

This function is well defined at least if $a \geq n_0 + a_1$ and $x \geq n_0$ in the sense that

$$w_a(x) < +\infty.$$

Under these conditions, the function w_a is subadditive: for any $x, y \geq n_0$,

$$w_a(x + y) \leq w_a(x) + w_a(y).$$

For $a \neq 0$, define now $N_a(n)$ to be the number of integers up to n that need a in any of their representations as a sum of h elements of $\mathcal{A}$; in other words,

$$N_a(n) = |\{x \in \mathbb{N} \text{ such that } n_0 \leq x \leq n \text{ and } w_a(x) > h\}|.$$

Define also

$$\alpha(a) = \liminf_{n \to +\infty} \frac{N_a(n)}{n}.$$

Lemma 4. *The series $\sum_{a \in \mathcal{A} \setminus \{0\}} \alpha(a)$ is convergent, and its sum is less than or equal to h.*

Proof. For any $x \geq n_0$, the cardinality of the set of positive a's needed in any representation of x as an element of $h\mathcal{A}$, namely

$$\{a \in \mathcal{A} \text{ such that } a > 0 \text{ and } w_a(x) > h\},$$

is clearly less than or equal to h. Therefore for any $n \geq n_0$,

$$\begin{aligned} \sum_{a \in \mathcal{A}, a>0} N_a(n) &= \sum_{a \in \mathcal{A}, a>0} \sum_{n_0 \leq x \leq n} 1_{w_a(x)>h} \\ &= \sum_{n_0 \leq x \leq n} \sum_{a \in \mathcal{A}, a>0} 1_{w_a(x)>h} \\ &\leq \sum_{n_0 \leq x \leq n} h = h(n - n_0 + 1) \leq hn. \end{aligned}$$

Since $N_a(n)/n \geq 0$, we thus infer

$$\sum_{a \in \mathcal{A}, a>0} \alpha(a) = \sum_{a \in \mathcal{A}, a>0} \liminf_{n \to +\infty} \frac{N_a(n)}{n} \leq \liminf_{n \to +\infty} \sum_{a \in \mathcal{A}, a>0} \frac{N_a(n)}{n} \leq h$$

and the result follows. □

Lemma 5. *Let a be a positive element of $\mathcal{A}$ such that $\alpha(a) < 1/h$. Then there is a value of $n \geq n_0$ such that*

$$w_a(n + ka) \leq h$$

for any k in $\{0, 1, \ldots, h-1\}$.

Proof. Suppose, on the contrary, that for any $n \geq n_0$, at least one of the $w_a(n+ka)$ is larger than h. This is tantamount to writing

$$\sum_{k=0}^{h-1} 1_{w_a(n+ka)>h} \geq 1. \tag{3.1}$$

If m is a sufficiently large integer, summing (3.1) for $n \in \{n_0, \ldots, m - (h-1)a\}$ yields

$$\begin{aligned} m - (h-1)a - n_0 + 1 &\leq \sum_{n=n_0}^{m-(h-1)a} \sum_{k=0}^{h-1} 1_{w_a(n+ka)>h} \\ &\leq h \sum_{n=n_0}^{m} 1_{w_a(n)>h} = hN_a(m). \end{aligned}$$

Dividing both sides by hm and letting m tend toward infinity, we get

$$\frac{1}{h} \leq \alpha(a),$$

contrary to the assumption. □

We now come to the very proof of Theorem 2. By Lemma 4, the sequence $\alpha(a)$ converges toward 0 as a tends to infinity. Thus for any $a \in \mathcal{A}$ large enough,

$$\alpha(a) < \frac{1}{h}.$$

Applying Lemma 5 yields an integer $n \geq n_0$ such that

$$w_a(n + ka) \leq h \tag{3.2}$$

for any k in $\{0, 1, \ldots, h-1\}$.

Now let x be any integer large enough ($x \geq n + n_0 + ha$ will do). The integer $y = x - n$ is larger than n_0 and therefore belongs to $h\mathcal{A}$: thus there exists an integer k, between 0 and $h-1$, such that

$$(y - ka) \in (h-k)(\mathcal{A} \setminus \{a\}). \tag{3.3}$$

Since

$$x = n + y = (n + ka) + (y - ka),$$

by (3.2) and (3.3), we get

$$x \in h(\mathcal{A} \setminus \{a\}) + (h-k)(\mathcal{A} \setminus \{a\}) = (2h-k)(\mathcal{A} \setminus \{a\}) \subseteq 2h(\mathcal{A} \setminus \{a\}).$$

Theorem 2 follows.

4. Proof of Theorem 3

For any positive h, take a basis $\mathcal{A}$ of order h such that

$$\limsup_{a \in \mathcal{A}^*} \quad \mathrm{ord}^*(\mathcal{A} \setminus \{a\})$$

attains $S(h)$. Again, by the same remark as above, we may assume that $\mathcal{A}$ contains 0.

We first notice that there is at least one odd element in $\mathcal{A}$, say b. We define

$$\mathcal{B} = \{b\} \cup \{2a \text{ such that } a \in \mathcal{A}\}$$

and observe that

$$(h+1)\mathcal{B} \sim \mathbb{N}.$$

To prove $S(h+1) \geq S(h) + 1$, we shall prove that for any $c > b$ in $\mathcal{A}^*$, $\mathcal{B} \setminus \{2c\}$ (which is clearly a basis) has an exact order satisfying

$$\mathrm{ord}^*(\mathcal{B} \setminus \{2c\}) \geq \mathrm{ord}^*(\mathcal{A} \setminus \{c\}) + 1. \tag{4.1}$$

Let us define $k = \mathrm{ord}^*(\mathcal{B} \setminus \{2c\})$, so that $k(\mathcal{B} \setminus \{2c\}) \sim \mathbb{N}$. If $x \in \mathbb{N}$ is large enough, then $2x + b$ can be written as the sum of k elements in $\mathcal{B} \setminus \{2c\}$. In other words, we can write $2x + b = jb + y$ with

$$y \in (k-j)(\mathcal{B} \setminus \{b, 2c\}) = (k-j)\{2a \text{ such that } a \in \mathcal{A} \setminus \{c\}\}.$$

Since y is even, it follows that j is odd. Also $y/2$ can be written as the sum of $k-j$ elements in $\mathcal{A} \setminus \{c\}$. Therefore, writing

$$x = \frac{j-1}{2} b + \frac{y}{2},$$

shows that x can be written as the sum of $(j-1)/2 + k - j \leq k - 1$ elements of $\mathcal{A} \setminus \{c\}$. Consequently, since $0 \in \mathcal{A}$, $(k-1)(\mathcal{A} \setminus \{c\}) \sim \mathbb{N}$. This shows that $\mathrm{ord}^*(\mathcal{A} \setminus \{c\}) \leq k - 1$ and proves (4.1).

Theorem 3 is proved.

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References

1. R. de la Bretèche, *Problèmes extrémaux pour les bases additives*, manuscript (2001).
2. B. Deschamps and G. Grekos, *Estimation du nombre d'exceptions à ce qu'un ensemble de base privé d'un point reste un ensemble de base*, J. reine angew. Math. **539** (2001), 45–53. MR 2002m:11008
3. P. Erdős, *Einige Bemerkungen zur Arbeit von A. Stöhr "Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe"*, J. reine angew. Math. **197** (1957), 216–219. MR 19:122b
4. P. Erdős and R. L. Graham, *On bases with an exact order*, Acta Arith. **37** (1980), 201–207. MR 82e:10093
5. P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*, Enseign. Math. **28** (1980), 128 pp. MR 82j:10001
6. G. Grekos, *Minimal additive bases and related problems*, in: "Journées Arithmétiques, Exeter (Grande-Bretagne), 1980", edited by J. V. Armitage, London Math. Soc. Lecture Note Series **56**, Cambridge Univ. Press, 1982, 300–305. MR 85a:11005
7. G. Grekos, *Sur l'ordre d'une base additive*, Séminaire de Théorie des Nombres de Bordeaux, Année 1987/88, exposé 31. MR 90d:11021
8. G. Grekos, *Extremal problems about additive bases*, Acta Math. Inform. Univ. Ostraviensis **6** (1998), 87–92. MR 2002d:11013
9. H. Halberstam and K. Roth, *Sequences*, The Clarendon Press, Oxford, 1966. MR 35:1565
10. E. Härtter, *Ein Beitrag zur Theorie der Minimalbasen*, J. reine angew. Math. **196** (1956), 170–204. MR 19:122a
11. J. C. M. Nash, *Some applications of a theorem of M. Kneser*, J. Number Theory **44** (1993), 1–8. MR 94k:11015
12. M. B. Nathanson, *Minimal bases and powers of 2*, Acta Arith. **49** (1988), 525–532. MR 89m:11015
13. A. Plagne, *Removing one element from an exact additive basis*, J. Number Theory **87** (2001), 306–314. MR 2002d:11014
14. A. Stöhr, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe I, II*, J. reine angew. Math. **194** (1955), 40–65 and 111–140. MR 17:713a

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