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ON THE ELLIPTIC EQUATION $\Delta u + K(x)e^{2u} = 0$ ON B^2

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Abstract. In this paper we consider the existence problem for the elliptic equation $\Delta u + K(x)e^{2u} = 0$ on $B^2 = \{x \in R^2 \mid |x| < 1\}$, which arises in the study of conformal deformation of the hyperbolic disc. We prove an existence result for the above equation.

1. Introduction

Let $B^2 = \{x \in R^2 \mid |x| < 1\}$. We study the existence problem for the elliptic equation

$$\Delta u + K(x)e^{2u} = 0 \quad \text{on } B^2, \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the standard Laplacian and $K(x)$ is a locally Hölder continuous function.

Equation (1) arises in the study of conformal deformation of the hyperbolic disc H^2.

In general, given a two-dimensional Riemannian manifold (M, g), a function K on M is the Gaussian curvature of a conformal metric $\tilde{g} = e^{2u}g$ if and only if u is a solution of the following elliptic equation:

$$\Delta_g u - k_g + Ke^{2u} = 0, \tag{2}$$

where k_g and Δ_g are the Gaussian curvature function and the Laplace-Beltrami operator on M with respect to the metric g. This problem has been studied by many authors (cf. [1], [3], [9]).

In the special case where M is the hyperbolic disc H^2, equation (2) becomes

$$\Delta_h u + 1 + K(x)e^{2u} = 0, \tag{3}$$

where Δ_h is the hyperbolic Laplacian. Since the hyperbolic metric

$$g_h = \frac{4}{(1 - |x|^2)^2}\eta \tag{4}$$

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is conformal to the Euclidean metric η on B^2, a solution of (1) provides a conformal metric of g_h:

$$e^{2u}\eta = e^{2u+2\ln(1-|x|^2)-\ln 4}\cdot g_h. \tag{5}$$

Now we can state our main result as follows:

Theorem 1. *Suppose $K(x)$ is a locally Hölder continuous function on B^2 that is positive somewhere and for some positive constants C and σ, $0<\sigma<1$, the inequality*

$$|K(x)| \le \frac{C}{|1-|x||^{\sigma}} \tag{6}$$

holds for $|x|<1$. Then equation (1) admits a C^2 solution.

Remark. J. Bland and M. Kalka [1] and A. Ratto, M. Rigoli, and L. Véron [9] use the technique of supsolution and subsolution to study the equation (1), and get good results for the case that $K(x)$ is negative near the boundary of B^2. Our Theorem 1 allows $K(x)$ to go to positive infinity near the boundary of B^2.

2. Preliminaries

For $-\frac{1}{2}<\alpha<0$, consider the conformal metric

$$g_\alpha = (1-r^2)^{2\alpha}\eta, \tag{7}$$

where $r=|x|$. Since the Gaussian curvature of g_α is $\frac{4\alpha}{(1-r^2)^{2+2\alpha}}$, the Gaussian curvature $K(x)$ of the conformal metric $e^{2u}g_\alpha$ satisfies the following equation:

$$\Delta_\alpha u - \frac{4\alpha}{(1-r^2)^{2+2\alpha}} + K(x)e^{2u} = 0, \tag{8}$$

where Δ_α is the Laplace-Beltrami operator on (B^2, g_α).

It is easy to verify that

$$\Delta_\alpha(-\alpha\ln(1-r^2)) = \frac{4\alpha}{(1-r^2)^{2+2\alpha}}; \tag{9}$$

then make the substitution $u = w-\alpha\ln(1-r^2)+lr^2$ for a constant $l<0$. We have (still denote by u)

$$\Delta_\alpha u + \frac{4l}{(1-r^2)^{2\alpha}} + K_l(x)e^{2u} = 0, \tag{10}$$

where $K_l(x) = K(x)(1-r^2)^{-2\alpha}e^{2lr^2}$ and

$$\int_{B^2}\frac{4l}{(1-r^2)^{2\alpha}}dA = 4\pi l, \tag{11}$$

where dA is the area element of (B^2, g_α).

Let H_k denote the Hilbert space of L^2_{loc}-functions for which

$$\|u\|_k = [\sum_{j=0}^{k}\int_{B^2}|D^j u|^2 dA]^{\frac{1}{2}} < +\infty, \tag{12}$$

where $|D^j u|$ is the pointwise norm (with respect to g_α) of the jth covariant derivative of u. In particular,

$$\|u\|_1 = [\int_{B^2} u^2 dA + \int_{B^2} |\nabla u|^2 dA]^{\frac{1}{2}}, \tag{13}$$

where ∇u is the gradient of u with respect to g_α.

The following is a type of Trudinger inequality (cf. [7], which we also follow in the proof).

Proposition 2. *There exist positive constants β, γ such that*

$$\int_{B^2} e^{\beta u^2} dA \leq \gamma \tag{14}$$

for all $u \in H_1$ with $\int_{B^2} u\, dA = 0$ and $\|\nabla u\|_{L^2} \leq 1$.

Proof. We apply symmetrization which is based on the isoperimetric inequality that holds on (B^2, g_α) (cf. [6], [7]); to be specific, with $u(x)$ we associate a nonincreasing radial function $u^*(r)$ by the requirement

$$\mu\{x \mid u^* > \rho\} = \mu\{x \mid u > \rho\}$$

for every ρ, where μ denotes the measure on (B^2, g_α).

Since the Dirichlet norm is a conformal invariant and symmetrization decreases the Dirichlet norm,

$$\|\nabla u^*\|_{L^2} \leq \|\nabla u\|_{L^2}.$$

Thus we may assume that $u = u(r) = u^*(r)$. Now introduce $\omega(t) = \sqrt{4\pi} u(r)$, where $r^2 = e^t$; then

$$\|\dot{\omega}\|_{L^2(dt)} = \|\nabla u\|_{L^2},$$

where $\dot{\omega} = \frac{d\omega}{dt}$, and we must show that

$$\int_{-\infty}^{0} e^{\frac{\beta}{4\pi}\omega^2} \cdot (1 - e^t)^{2\alpha} e^t dt \leq \frac{\gamma}{\pi}. \tag{15}$$

Using the Schwarz inequality, we find that

$$(\omega(t) - \omega(s))^2 \leq |t - s| \int_s^t \dot{\omega}^2 dt \leq |t - s|,$$

or

$$-|t - s|^{\frac{1}{2}} \leq \omega(t) - \omega(s) \leq |t - s|^{\frac{1}{2}}. \tag{16}$$

Let $\rho(t) = C(1 - e^t)^{2\alpha} e^t$ be such that (in this paper we use C to denote different positive constants)

$$\int_{-\infty}^{0} \rho(t) dt = 1$$

and

$$\int_{-\infty}^{0} \omega(t)\rho(t) dt = 0.$$

Multiplying (16) by $\rho(s)$ and integrating ds, we find that

$$|\omega(t)|^2 \leq (\int_{-\infty}^{0} |t - s|^{\frac{1}{2}} \rho(s) ds)^2 \leq |t| + C. \tag{17}$$

Thus we obtain (15) provided $\beta < 4\pi$. □

Corollary 3. *If $\beta < 4\pi$, then*

$$\int_{B^2} e^{\delta|v|} dA \le C e^{(\delta^2/(4\beta))\|\nabla v\|_{L^2}^2} \tag{18}$$

for all $v \in H_1$ with $\int_{B^2} v\, dA = 0$.

Proof. Write $v = \|\nabla v\| \cdot u$, so that $\|\nabla u\| \le 1$. Apply Proposition 2 to u using $\delta|v| \le \beta u^2 + \delta^2 \|\nabla v\|^2/(4\beta)$. □

The next result is a type of Poincaré inequality, which we also need in the proof of Theorem 1.

Proposition 4. *There exists a positive constant C_1 such that*

$$\|v\|_{L^2}^2 \le C_1 \|\nabla v\|_{L^2}^2 \tag{19}$$

for all $v \in H_1$ with $\int_{B^2} v\, dA = 0$.

Proof. Let $v = \|\nabla v\| \cdot u$, so that $\|\nabla u\| \le 1$. It suffices to show that

$$\|u\|_{L^2}^2 \le C.$$

We may use symmetrization to assume that $u = u(r)$ and introduce $\omega(t)$ as in the proof of Proposition 2. Using (17), we find that

$$\|u\|_{L^2}^2 \le C \int_{-\infty}^0 |\omega|^2 (1 - e^t)^{2\alpha} e^t dt \le C.$$

□

3. Proof of Theorem 1

Now we can give the following.

Proof of Theorem 1. Define E by

$$E = \left\{ u \in H_1(B^2) \mid \int_{B^2} K_l e^{2u} dA = -4\pi l \right\}. \tag{20}$$

Since $K(x)$ is positive somewhere and $l < 0$, it is easy to see that E is not empty.

We shall minimize the functional

$$J(u) = \int_{B^2} \left(\frac{1}{2} |\nabla u|^2 - \frac{4l}{(1-r^2)^{2\alpha}} \cdot u \right) dA \tag{21}$$

for $u \in E$. Writing $u = v + b$, so that $\int_{B^2} v\, dA = 0$, and solving for b in the constraint (20), one sees that J can be expressed as

$$J(u) = \frac{1}{2} \int_{B^2} |\nabla v|^2 dA + 2\pi l \ln \int_{B^2} K_l e^{2v} dA - 2\pi l \ln(-4\pi l) - \int_{B^2} \frac{4l}{(1-r^2)^{2\alpha}} \cdot v\, dA. \tag{22}$$

By assumption (6) (take $\alpha = -\frac{1}{2}\sigma$), K_l is bounded. Then use (18) and choose $-l$ small such that $\left| \frac{2\pi l}{\beta} \right| < \frac{1}{8}$. Also, we note that $\frac{4l}{(1-r^2)^{2\alpha}}$ is bounded on B^2. Then using (19) we have

$$\left| \int_{B^2} \frac{4l}{(1-r^2)^{2\alpha}} \cdot v\, dA \right| \le \epsilon \int_{B^2} v^2 dA + \frac{1}{4\epsilon} \int_{B^2} \frac{4l}{(1-r^2)^{2\alpha}} dA \le \epsilon C_1 \|\nabla v\|_{L^2}^2 + C.$$

Taking $\epsilon = \frac{1}{8C_1}$, we find that

$$J(u) \geq \frac{1}{4}\|\nabla v\|_{L^2}^2 - C. \tag{23}$$

Now let $u_j = v_j + c_j \in E$ be a minimal sequence, where $\int_{B^2} v_j dA = 0$. Since $J(u_j)$ is bounded, we have $\|\nabla v_j\|_{L^2} \leq C$ by (23). Then it follows from (19) that $\|v_j\|_1$ is bounded. Because the unit ball of any Hilbert space is weakly compact, we can extract a subsequence (which we again denote by v_j) converging weakly in H_1 to v, and this implies that $\int_{B^2} v\, dA = 0$.

The fact that $\|\nabla v_j\|_{L^2} \leq C$ and (19) hold for all v_j implies that $\{v_j\}$ is precompact in L^2 (cf. [7], [10]). Then $v_j \to v$ strongly in L^2. Using the inequality $|e^z - 1| \leq |z|e^{|z|}$ and (18), we find that $\int_{B^2} K_l e^{2v_j} dA \to \int_{B^2} K_l e^{2v} dA$:

$$|\int_{B^2} K_l(e^{2v_j} - e^{2v})dA| \leq C\int_{B^2} e^{2|v|}|v_j - v|e^{2|v_j - v|}dA \leq C\|v_j - v\|_{L^2}.$$

Since $e^{2c_j}\int_{B^2} K_l e^{2v_j} dA = -4\pi l$ for each j, we find that

$$c_j \to c = \frac{1}{2}\ln[-4\pi l/\int_{B^2} K_l e^{2v}\, dA],$$

and $u = v + c \in E$ gives the minimum for J on E. Then by standard Lagrange multiplier theory we find that there is a constant λ such that

$$\int_{B^2}[\nabla u \cdot \nabla\phi - \frac{4l}{(1-r^2)^{2\alpha}} \cdot \phi]dA = \lambda\int_{B^2} K_l e^{2u}\phi\, dA \tag{24}$$

for all $\phi \in H_1$. Taking $\phi \equiv 1$ we find that $\lambda = 1$. So u is a weak solution of (10). Local regularity implies that $u \in C^2(B^2)$. This completes the proof of Theorem 1. □

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