E(2)-INVERTIBLE SPECTRA SMASHING WITH THE SMITH-TODA SPECTRUM V(1) AT THE PRIME 3

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Abstract. Let $L_2$ denote the Bousfield localization functor with respect to the Johnson-Wilson spectrum $E(2)$. A spectrum $L_2X$ is called invertible if there is a spectrum $Y$ such that $L_2X \wedge Y = L_2S^0$. Hovey and Sadofsky, invertible spectra in the $E(n)$-local stable homotopy category, showed that every invertible spectrum is homotopy equivalent to a suspension of the $E(2)$-local sphere $L_2S^0$ at a prime $p > 3$. At the prime 3, it is shown, A relation between the Picard group of the $E(n)$-local homotopy category and $E(n)$-based Adams spectral sequence, that there exists an invertible spectrum $X$ that is not homotopy equivalent to a suspension of $L_2S^0$. In this paper, we show the homotopy equivalence $v_2^3: \Sigma^{48} L_2 V(1) \simeq V(1) \wedge X$ for the Smith-Toda spectrum $V(1)$. In the same manner as this, we also show the existence of the self-map $\beta: \Sigma^{144} L_2 V(1) \to L_2 V(1)$ that induces $v_2^3$ on the $E(2)_*\text{-}\text{homology.}$

Introduction

Let $S_p$ denote the stable homotopy category of spectra localized away from the prime number $p$, and $E(n)$, the Johnson-Wilson spectrum such that $\pi_*(E(n)) = E(n)_* = v_1^{-1} \mathbb{Z}_p[v_1, \ldots, v_n]$. We denote by $L_n$, the full subcategory of $E(n)$-local spectra, and we have the Bousfield localization functor $L_n: S_p \to L_n \subset S_p$ with respect to $E(n)$. We call a spectrum $X \in L_n$ ($E(n)$)-invertible if there exists a spectrum $Y \in L_n$ such that $X \wedge Y = L_n S^0$. In [1], Hovey and Sadofsky showed that every $E(n)$-invertible spectrum is homotopy equivalent to a suspension of $L_n S^0$ if $n^2 + n < 2p - 2$, and that every $E(1)$-invertible spectrum is homotopy equivalent to a suspension of $L_1 S^0$ or $L_1 QM$ if $p = 2$. Here $QM$ denotes the so-called question mark complex $S^0 \cup e_1 e_1 \cup e_3$. In [3], Kamiya and the second author constructed an $E(2)$-invertible spectrum $X$ such that $X \not\simeq \Sigma^k L_2 S^0$ for any $k \in \mathbb{Z}$ and $X \wedge X \wedge X = L_2 S^0$ at the prime 3. Unfortunately, we do not know whether $X$ is an $E(2)$-localization of a finite spectrum. This case is different from the case where $p = 2$ and $n = 1$. These spectra $L_1QM$ and $X$ are, so far, the only known examples of $E(n)$-invertible spectra other than the sphere spectrum. For $QM$ at the prime 2, there is a homotopy equivalence $v_1^{-2}: L_1 V(0) \to \Sigma^4 L_1 V(0) \wedge QM$ (see Proposition [8]), where $V(0)$ denotes the mod 2 Moore spectrum. This is deduced from the structure of the homotopy groups $\pi_*(L_1 QM)$. We study here
the invertible spectrum $X$ in $L_2$ at the prime 3. Consider the $E(2)$-based Adams spectral sequence $E_2^{r,s}(W)$ for a spectrum $W$ converging to $\pi_*(L_2W)$. Then it is shown \[5\] that $d_2(g) = v_2^{-2}h_1b_0v_2g \in E_2^{0,4}(X)$ for the generator $g$ of $E_2^{0,0}(X) = \mathbb{Z}_2$. We compute the $E_\infty$-term $E_\infty^*(V(1) \wedge X)$, and then determine the homotopy groups $\pi_*(V(1) \wedge X)$ (see Corollary 2.12). This shows that $v_2^3g$ detects a homotopy element $v_2^3 \in \pi_*(V(1) \wedge X)$. Furthermore, this extends to the map $\Sigma^{48}L_2V(1) \to V(1) \wedge X$.

**Theorem A.** Here is a homotopy equivalence $v_2^3 : \Sigma^{48}L_2V(1) \simeq V(1) \wedge X$.

Here, the map $v_2^3$ denotes a map $f$ such that $E(2)_*(f) = v_2^3$. In the same manner as we obtain the map $v_2^3$, we also obtain the self-map $v_2^3$.

**Theorem B.** There is a homotopy equivalence $v_2^3 : \Sigma^{144}L_2V(1) \to L_2V(1)$.

Recently, M. Behrens and S. Pemmaraju show the existence of the self-map $v_2^3 : \Sigma^{144}V(1) \to V(1)$ \[11\].

The $\beta$-element $\beta_s \in \text{AE}_2^*(S^0)$ for an integer $s > 0$ is defined to be the image of $v_2^s \in \text{AE}_2^*(V(1))$ under the composite of the connecting homomorphisms $\text{AE}_2^*(V(1)) \to \text{AE}_2^*(V(0))$ and $\text{AE}_2^*(V(0)) \to \text{AE}_2^*(S^0)$. Here $\text{AE}_r^*(W)$ denotes the $E_r$-term of the Adams-Novikov spectral sequence converging to $\pi_*(W)$. In \[9\], Oka showed how to prove the “if” part of the conjecture of Ravenel’s: The element $\beta_s \in \text{AE}_2^*(S^0)$ for an integer $s$ survives to $\pi_*(S^0)$ if and only if $s = 0, 1, 2, 3, 5, 6$ mod 9. The “only if” part is shown in \[11\] Th. F. In Oka’s arguments, the self-map $v_2^3 : \Sigma^{144}V(1) \to V(1)$ plays the principal role. Here we play the same game in $\pi_*(L_2S^0)$.

**Corollary C** (\[13\]). The element $\beta_s \in \text{AE}_2^*(S^0)$ for an integer $s$ survives to $\pi_*(L_2S^0)$ if and only if $s \equiv 0, 1, 2, 3, 5, 6$ mod 9.

Note that it is, so far, not known whether $\beta_s \in \text{AE}_2^*(S^0)$ for $s \equiv 3$ mod 9 survive to $\pi_*(S^0)$, even if there is the self-map $v_2^3 : \Sigma^{144}V(1) \to V(1)$ in \[1\].

By the definition of $\beta_s$, Theorem A seems to indicate that if the element $\beta_s$ survives to $\pi_*(L_2S^0)$, then $\beta_{s+3}$ survives to $\pi_*(X)$.

**Corollary D.** The element $\beta_{s+3} \in \text{AE}_2^*(X)$ for an integer $s$ survives to $\pi_*(X)$ if $s \equiv 0, 1, 2, 3, 5, 6$ mod 9.

Note that the non-existence of $\beta_{s+3}$ for $s \equiv 4, 7, 8$ mod 9 follows from \[11\] Th. F] together with the equivalence $v_2^6$ in Theorem A. Here we do not conclude the case where $s \equiv 3$ mod 9 (see Remark 3.6).

In the next section, we consider the self-maps on $L_2V(1)$ using ring spectra $V(1)_k$ with $k > 1$ and show Theorem B and Corollary C. In section 2, we recall some facts on invertible spectra and show that $v_2^3g : E_2^*(V(1)) \to E_2^*(V(1) \wedge X)$ is an isomorphism of spectral sequences, which induces an isomorphism of homotopy groups $\pi_*(L_2V(1)) \cong \pi_*(V(1) \wedge X)$. In the last section, we verify the equivalences $\Sigma^{-1}L_1V(0) \wedge QM = L_1V(0)$ for the invertible spectra $QM$ at the prime 2. Then we construct a map $v_2^3g : L_2V(1) \to V(1) \wedge X$ by the use of the result of the previous sections, which shows Theorem A. In the last section, we show Corollary D.

1. **The self-maps on the spectrum $L_2V(1)_k$**

Let $V(0)$ denote the mod 3 Moore spectrum and $V(1)_k$ be a cofiber of $\alpha^k : \Sigma^{4k}V(0) \to V(0)$ for the Adams map $\alpha : \Sigma^4V(0) \to V(0)$. Here $V(1)_1 = V(1)$ is
the Smith-Toda spectrum. Then we have the cofiber sequences
\begin{equation}
S^0 \xrightarrow{\beta} S^0 \xrightarrow{i} V(0) \xrightarrow{j} V(1) \quad \text{and} \quad \Sigma^{4k}V(0) \xrightarrow{\alpha^k} V(0) \xrightarrow{i_k} V(1)_k \xrightarrow{j_k} \Sigma^{4k+1}V(0).
\end{equation}

In [8 Th. 5.6], Oka showed that $V(1)_k$ is a ring spectrum for $k > 1$, in which a ring spectrum means a spectrum $V$ equipped with a unit map $i: S^0 \to V$ and a multiplication $\mu: V \wedge V \to V$ such that $\mu(i_1 \wedge 1_V) = 1_V = \mu(1_V \wedge i)$. In other words, a ring spectrum here is not assumed to satisfy the associative law. By a module spectrum, we also do not assume the associative law.

**Lemma 1.2.** $V(1)$ is a $V(1)_k$-module spectrum with $\nu_k: V(1) \wedge V(1)_k \to V(1)$ for $k = 2, 4$.

**Proof.** Consider the exact sequence

$$[V(1) \wedge V(1)_k, V(1)]_0 \xrightarrow{i^*_k} [V(1) \wedge V(0), V(1)]_0 \xrightarrow{(\alpha^k)^*} [V(1) \wedge V(0), V(1)]_{4k}$$

associated to the cofiber sequence $\Sigma^{4k}V(0) \xrightarrow{\alpha^k} V(0) \xrightarrow{i_k} V(1)_k$. It is shown in [10 Th. 6.11] that $[V(1)_k, V(1)]_0 = 0$ if $l = 8, 9, 16, 17$. Therefore, $[V(1) \wedge V(0), V(1)]_{4k} = 0$ if $k = 2, 4$, and so the $V(0)$-module structure $\nu \in [V(1) \wedge V(0), V(1)]_0$ is pulled back to a $V(1)_k$-module structure $\nu_k \in [V(1) \wedge V(1)_k, V(1)]_0$.

Let $i_k$ and $j_k$ be the maps in the cofiber sequence
\begin{equation}
\Sigma^{4k}V(1) \xrightarrow{\alpha^k} V(1)_{k+1} \xrightarrow{i_k} V(1)_k \xrightarrow{j_k} \Sigma^{4k+1}V(1),
\end{equation}
obtained from the $3 \times 3$ Lemma (Verdier’s axiom), and let $i_0 = i_1 i: S^0 \to V(1)$ denote the inclusion to the bottom cell. Then

**Lemma 1.4.** $\nu_2(i_0 \wedge 1) \equiv \tilde{i}_1 + k\beta i_0j_2$ for some $k \in \mathbb{Z}/3$.

**Proof.** $\nu_2(i_0 \wedge 1)i_2 = \nu_2(i_1 \wedge j_2)(i_0 \wedge 1) = \nu(i_0 \wedge 1) = i_1$ and $\tilde{i}_1j_2 = i_1$. It follows that $\nu_2(i_0 \wedge 1) - \tilde{i}_1 \in [V(1)_2, V(1)]_0$ is in the image of $(j_2)^*: [V(0), V(1)]_0 \to [V(1)_2, V(1)]_0$. Since $[V(0), V(1)]_0 = \mathbb{Z}/3\{\beta i_0 j\}$ by [10 Prop. 6.9], we have the desired equation. \hfill \Box

**Lemma 1.5.** Let $U$ be an invertible spectrum with $E(2)_*(U) = E(2)_*$, and $\xi$, a homotopy element of $\pi_*(V(1)_2 \wedge U)$ that induces $v_2^* \in \pi_*(U)$ for $k \in \mathbb{Z}$ on $E(2)_*$-homology. Then $\xi$ induces the map $\xi: V(1) \to V(1) \wedge U$ such that $E(2)_*(\tilde{\xi}) = v_2^*$.

**Proof.** The map $\tilde{\xi}$ is defined as the composite $\Sigma^{4\ell}V(1) = V(1) \wedge S^{4\ell}1 \to V(1) \wedge V(1)_2 \wedge U \xrightarrow{\nu_2 \wedge 1} V(1) \wedge U$. Since $E(2)_*(j) = 0$, we have $E(2)_*(\nu_2(i_0 \wedge 1)) = E(2)_*(\tilde{i}_1)$ by Lemma 1.4. Then we compute $i_0^*(E(2)_*(\tilde{\xi})) = E(2)_*(\tilde{i}_0) = E(2)_*((\nu_2 \wedge 1)(1 \wedge \xi)i_0) = E(2)_*((i_0 \wedge 1)\xi) = E(2)_*((i_0 \wedge 1)\xi) = i_1^*(v_2^*) = v_2^*$. Noting that $i_0^*$ is a monomorphism, we see that the lemma is proved. \hfill \Box

For computing the homotopy groups $\pi_*(L_2W)$ for a spectrum $W$, we use the $E(2)$-based Adams spectral sequence $E^n_*(W)$ converging to $\pi_*(L_2W)$.

**Lemma 1.6.** $v_2^{gt} \in E^n_2(L_2V(1)_3)$ for $t \in \mathbb{Z}$ is a permanent cycle.
Proof. Recall [12] the spectrum $C$, which is defined to be a cofiber of the localization map $V(0) \to \alpha^{-1}V(0) = \text{colim}_a V(0)$. Then $E(2)_{\ast}(C) = E(2)_{\ast}/(3, \nu_{3})$. Since we have a commutative diagram

$$
\begin{array}{cccccc}
V(1)_3 & \xrightarrow{v_{1}^{-3}} & V(1)_j & \xrightarrow{\pi_j} & V(1)_{j-3} & \xrightarrow{v_{j}^{-1}} & V(1)_3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V(1)_3 & \xrightarrow{v_{j}^{-2}} & V(1)_{j+1} & \xrightarrow{\pi_{j+1}} & V(1)_{j-2} & \xrightarrow{v_{j+1}} & V(1)_3
\end{array}
$$

of cofiber sequences for $j > 3$, we obtain a cofiber sequence $V(1)_3 \xrightarrow{f} C \xrightarrow{v_{3}} C$ by taking homotopy colimits. It is shown in [12] that $v_{3}^0/\nu_{3} \in E_2^1(\nu_{3})$ is a permanent cycle. Furthermore, we read off from [12] that $v_{1}(v_{2}^0/v_{1}^0) = 0 \in \pi_{144}(L_2C)$, since $E_2^{0,144}(S^0)$ is zero for $s > 3$. Therefore, $v_{3}^0/\nu_{3}^0$ is pulled back to $v_{2}^0 \in \pi_{144}(V(1)_3).

Proof of Theorem B. By Lemma 1.6 we have a homotopy element $v_{0}^1 \in \pi_{144}(V(1)_1)$ as the image of $v_{2}^0 \in \pi_{\ast}(L_2V(1)_3)$ under the map $i_2: L_2V(1)_3 \to L_2V(1)_2$ of (1.3). Then this induces the desired self-map by Lemma 1.5 which induces an isomorphism on $E_{2\ast}$-homology.

As an application, we consider the $\beta$-elements in the homotopy groups $\pi_{\ast}(L_2S^0)$. In [7], the $\beta$-element $\beta_0$ of the $E_2$-term $E_2^{1,16s-4}(S^0)$ is defined as the image of $v_0^1 \in E_2^{0,16s}(V(1))$ under the composite of the connecting homomorphisms $E_2^{0,16s}(V(1)) \to E_2^{1,16s-4}(V(0))$ and $E_2^{1,16s-4}(V(0)) \to E_2^{2,16s-4}(S^0)$ associated to the cofiber sequences of (1.1). It is shown [13] that the $\beta$-element $\beta_0 \in E_2^{1,16s-4}(S^0)$ survives to a homotopy element of $\pi_{16s-6}(L_2S^0)$ if $s \equiv 0, 1, 2, 3, 5, 6 \mod 9$, which corresponds to one of Ravenel’s conjectures in $\pi_{\ast}(S^0)$. Here we give another proof, which is what Oka showed in [9].

Proof of Corollary C. Since $v_2$ and $v_2^5$ are homotopy elements of $\pi_{\ast}(V(1))$ [9], we define $\beta$-elements as follows:

$$
\beta_0 = j_0(v_2^0)^t, \quad \beta_{0+1} = j_0(v_2^0)^t v_2, \quad \beta_{0+2} = v_2^0(v_2^0)^t v_2, \\
\beta_{0+5} = j_0(v_2^0)^t v_2^5 \quad \text{and} \quad \beta_{0+6} = v_2^0(v_2^0)^t v_2^5,
$$

where $v_2^0$ is the element in Theorem B, $j_0: V(1) \to S^6$ is the projection to the top cell, and $v_2^5: V_0^0V(2) \to S^0$ is the Spanier-Whitehead dual of $v_2$. It follows from the Geometric Boundary Theorem (cf. 10.2.3.4) that each $\beta$-element $\beta_0$ in the $E_2$-term survives to the homotopy element $\beta_0$ defined above.

Since $V(1)_3$ is a ring spectrum and there is a map $\nu_2^0: \prod_{144}^0 S^0 \rightarrow L_2V(1)_3$ by Lemma 1.6 we have the self-map $\nu_2^0$ as the composite $\Sigma_{144}(V(1)_3) = V(1)_3 \wedge \Sigma_{144}^0 S^0 \xrightarrow{1 \wedge \nu_2^0} V(1)_3 \wedge L_2V(1)_3 \xrightarrow{L_0} L_2V(1)_3$. Oka also showed $v_2^0 v_2^3 \in \pi_{50}(V(1)_3)$ in [9] Lemma 4. Then the composite $\Sigma_{144+144}^0 S^0 \xrightarrow{\nu_2^0} \Sigma_{144+144}^0 S^0 \xrightarrow{v_2^0} \Sigma_{144+144} S^0 \xrightarrow{\nu_2^0} \Sigma_{144+144} S^0$ gives a homotopy element, which is shown to be detected by the element $\beta_{0+3} \in E_3^{1,144+144}(S^0)$ by the Geometric Boundary Theorem. Here $j_0^0$ is the projection to the top cell.
2. The homotopy groups $\pi_*(V(1) \wedge X)$

Let $E(n)$ denote the Johnson-Wilson spectrum, and $\mathcal{L}_n$ the category of the $E(n)$-local spectra. Then we have the Bousfield localization functor $L_n : \mathcal{S}_p \to \mathcal{L}_n$ with respect to $E(n)$, where $\mathcal{S}_p$ denotes the category of $(p)$-local spectra. We call a spectrum $U \in \mathcal{L}_n$ invertible if there exists a spectrum $U' \in \mathcal{L}_n$ such that $U \wedge U' = L_n S^0$. Let $\text{Pic}(\mathcal{L}_n)$ denote the collection of isomorphism classes of invertible spectra. Then in [4], Hovey and Sadofsky showed that $\text{Pic}(\mathcal{L}_n)$ is a group with multiplication given by $[U] \cdot [V] = [U \wedge V]$ for invertible spectra $U$ and $V$. Here $[U]$ denotes the isomorphism class of $U$. Since $[L_n S^k] \in \text{Pic}(\mathcal{L}_n)$, $\text{Pic}(\mathcal{L}_n)$ is the direct sum of $\mathbb{Z} = \{L_n S^k | k \in \mathbb{Z}\}$ and a subgroup $\text{Pic}(\mathcal{L}_n)^0$. We write $E^r_{p,t}(W)$ for a spectrum $W$ as the $E_r$-term of the $E(n)$-based Adams spectral sequence converging to $\pi_*(L_n W)$. It is further shown in [4] that $E(n)_*(U)$ for an invertible spectrum $U$ is isomorphic to $E(n)_*E(n)$ as an $E(n)_*E(n)$-comodule. It follows that the $E_2$-term $E^{s,t}_2(U)$ is isomorphic to the $E_2$-term $E^{s,t}_2(S^0)$. In [5] (cf. [6]) it is shown that there is a descending filtration $\{F_r\}$ of $\text{Pic}(\mathcal{L}_n)^0$ so that a monomorphism

$$(2.1) \quad \varphi_r : F_r/F_{r+1} \to E^{r,r-1}_r(S^0)$$

is defined for each $r > 1$ by assigning an isomorphism class $[U]$ to $d_r(g) \in E^{r,r-1}_r(U)$ where $g$ is the generator of $E_2^{0,0}(U) = E_2^{0,0}(S^0) = \mathbb{Z}$.

Now turn to the case where $p = 3$ and $n = 2$. Consider the chromatic comodules $N^0_0 = E(2)_*$, $M^0_2 = 3^{-1}E(2)_*$, $N^1_0 = E(2)_*/(3^\infty)$, $M^1_0 = v_1^{-1}N^1_0$ and $N^2_2 = M^2_2 = E(2)_*/(3^\infty, v_1^\infty)$ that fit in the exact sequences

$$0 \to N^0_0 \to M^0_0 \to N^1_0 \to 0 \quad \text{and} \quad 0 \to N^1_1 \to M^1_0 \to M^2_0 \to 0.$$ 

These associate the long exact sequences

$$(2.2) \quad 0 \to H^0 N^0_0 \to H^0 M^0_0 \to H^0 N^1_0 \to H^1 N^0_0 \to \cdots \quad \text{and} \quad 0 \to H^0 N^1_1 \to H^0 M^1_0 \to H^0 M^2_0 \to H^1 N^1_0 \to \cdots,$$

where $H^k M = \text{Ext}^k_{E(2), E(2)}(E(2)_*, M)$ for an $E(2)_*E(2)$-comodule $M$. Then the universal Greek letter map $\eta : H^k M^3_2 \to H^{k+2}N^0_0 = E^2_2(S^0)$ is defined as the composite $\eta = \delta \delta'$. The $E_2$-term $E^2_2(S^0) = H^* N^0_0$ is given in [14] (cf. [13]) by using (2.2). In particular, $E_2^{5,4}(S^0) = \mathbb{Z}/3\{\eta(v_2^{-1}h_1 b_{10}/3 v_1), \eta(v_2^{-1}x_2/3 v_1)\}$. In [6], we show that there is an invertible spectrum $X \in \mathcal{L}_2$ such that $\varphi_5([X]) = c\eta(v_2^{-1}h_1 b_{10}/3 v_1)$. In other words,

$$(2.3) \quad d_5(g) = c\eta(v_2^{-1}h_1 b_{10}/3 v_1)g \in E^{5,4}_5(X)$$

for the generator $g \in E^{0,0}_2(X)$. Here $c$ is the non-zero element of $\mathbb{Z}/3$ that appears in the Toda differential

$$(2.4) \quad d_5(\beta_{3/3}) = c\alpha_1 \beta_3^3,$$

where the elements $\beta_{3/3}$, $\alpha_1$ and $\beta_1$ are defined by

$$\beta_{3/3} = \eta(v_2^3/3 v_1^3), \quad \alpha_1 = \delta(v_1/3) \quad \text{and} \quad \beta_1 = \eta(v_2/3 v_1).$$

The $E_2$-term $E^{s,t}_2(V(1))$ of the $E(2)$-based Adams spectral sequence converging to the homotopy groups $\pi_*(L_2 V(1))$ is isomorphic to

$$(2.5) \quad (F \oplus F^*) \otimes K(2)_* b_{10} \otimes \Lambda(\zeta_2)$$

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as a $K(2)_*$-module. Here, $K(2)_* = \mathbb{Z}/3[v_2^{-1}]$, 

$$F = \mathbb{Z}/3\{h_{10}, h_{11}, b_{11}\}, \quad F^* = \mathbb{Z}/3\{\xi, \psi_0, \psi_1, b_{11}\xi\},$$

and the degrees of the generators are:

$$|1| = 0, \quad |h_{10}| = 3, \quad |h_{11}| = 11, \quad |b_{10}| = 10, \quad |b_{11}| = 34,$$

$$|\xi| = 6, \quad |\psi_0| = 13, \quad |\psi_1| = 21, \quad \text{and} \quad |\xi_2| = -1.$$

Let $(i_0)_*: E_2^*(S^0) \to E_2^*(V(1))$ be the induced map from the inclusion $i_0: S^0 \to V(1)$. Then

$$(i_0)_*(\beta_{3/3}) = h_{11}, \quad (i_0)_*(\alpha_1) = h_{10} \quad \text{and} \quad (i_0)_*(\beta_1) = b_{10}.$$

Since $E(2) \wedge X = E(2)$, the generator $g \in E_2^*(X)$ induces an isomorphism $g: E_2^*(V(1)) \cong E_2^*(V(1) \wedge X)$. By the structure (2.5), we see the following.

**Lemma 2.6.** $v_3^3g$ induces an isomorphism $E_2^{a,t}(V(1)) \cong E_2^{a,t+48}(V(1) \wedge X)$.

We will show that the map $v_3^3g$ induces an isomorphism of the differential modules on $E_2$ and $E_0$-terms. Here $E_2^*(W) = E_5^*(W)$ and $E_0^*(W) = E_5^*(W)$ for a spectrum $W$ are differential modules with differentials $d_5$ and $d_9$ of the $E(2)$-based Adams spectral sequence.

**Lemma 2.7.** $\beta_{3/3}^3g (\neq 0) \in E_0^*(X)$.

**Proof.** Recall [11, Prop. 5.9] the relations in the $E_2$-term $E_2^*(V(1))$:

$$v_2^2b_{11}b_{10} = -h_{10}b_{11} \quad \text{and} \quad b_{11}^2 = -v_2^2b_{10}^2. \tag{2.8}$$

Send $\beta_{3/3}^3$ to $E_2^*(V(1))$ under the map $(i_0)_*$, and we have $(i_0)_*(\beta_{3/3}^3) = b_{11}^2 = -v_2^2b_{10}^2 \neq 0 \in E_2^*(V(1))$. It follows that $b_{11}^2g \neq 0 \in E_2^*(V(1) \wedge X)$ and so $\beta_{3/3}^3g \neq 0 \in E_2^*(X)$.

From the Toda differential (2.4), the derivation formula on $d_5$ induces

$$d_5(\beta_{3/3}^3g) = -c\alpha_1\beta_{3/3}^3 g = \in E_5^*(S^0).$$

By the trivial pairing $S^0 \wedge X \to X$, we have the derivation formula on $d_5$. Furthermore, the universal Greek letter map $\eta$ is the map of $E_2^*(S^0)$-modules. Therefore, by (2.3),

$$d_5(\beta_{3/3}^3g) = -c\alpha_1\beta_{3/3}^3 g = \in E_5^*(S^0).$$

Here $v_2^{-1}h_{11}b_{10}b_{11}/3v_1 = v_2h_{10}b_{11}b_{10}^2/3v_1$ by (2.8). Thus,

$$d_5(\beta_{3/3}^3g) = -c\alpha_1\beta_{3/3}^3 g = \in E_5^*(S^0).$$

Since nothing hits $\beta_{3/3}^3g$ under the differential $d_5$ by reason of degree, we obtain $\beta_{3/3}^3g \neq 0 \in E_2^0(X)$. \hfill $\Box$

By the trivial pairing $V(1) \wedge X \to X$, we obtain the derivation formula:

$$d_r(xy) = d_r(x)y + (-1)^{1-s}xd_r(y) \tag{2.9}$$

for $x \in E_r^{a,t}(V(1))$ and $y \in E_r^*(X)$ (cf. [11, Th. 2.3.3]).

**Lemma 2.10.** $v_3^3g$ induces an isomorphism $E_9^{a,t}(V(1)) \cong E_9^{a,t+48}(V(1) \wedge X)$. 
Proof. By [24] and Lemma 2.7, we see that \( d_5(xb_{11}^2g) = d_5(x\beta_{3/3}^2g) = d_5(x)\beta_{3/3}^2g = d_5(x)b_{11}^2g \) for \( x \in E_2^s(V(1)) \). Since \( b_{11}^2 = -\omega_{3}b_{10}^2 \), we obtain \( d_5(xb_{11}^2\omega_{3}^2g) = d_5(x)b_{11}^2\omega_{3}g \). Since \( b_{10} \) is a polynomial generator, \( b_{10} \) acts monomorphically, and so we have
\[
d_5(xv_2^5g) = d_5(x)v_2^5g.
\]
This shows that \( v_2^5g \) is a map of differential modules and induces the desired isomorphism. \( \square \)

Lemma 2.11. \( v_2^5g \) induces an isomorphism \( E_\infty^{s,t}(V(1)) \cong E_\infty^{s,t+48}(V(1) \wedge X) \).

Proof. Since \( E_2^{13,80}(V(1) \wedge X) = \mathbb{Z}/3\{v_2^{-1}b_{11}b_{10}^3\zeta_2g\} \) and \( d_5(v_2^{-1}b_{11}b_{10}^3\zeta_2g) = d_5(v_2^{-1}b_{11}b_{10}^3\zeta_2v_2) = cv_2^{-4}h_{10}b_{10}^3\zeta_2(v_2) \) \( \neq 0 \) by [11] Prop. 9.9, Cor. 10.4, we obtain \( E_9^{13,80}(V(1) \wedge X) = 0 \). Therefore, \( (i_0)_*\left(d_5(\beta_{3/3}^2g)\right) = 0 \). If we show that
\[
d_9(xb_{10}) = yb_{10} \implies d_9(x) = y \text{ in } E_9^*(V(1) \wedge X),
\]
then we see that \( v_2^5g \) induces an isomorphism \( E_9^{s,t}(V(1)) \cong E_9^{s,t+48}(V(1) \wedge X) \) in the same way as Lemma 2.10. Since \( E_9^s(V(1)) = 0 \) if \( s > 12 \), \( d_{13} = 0 \) and so \( E_9^s(V(1) \wedge X) = E_\infty^s(V(1) \wedge X) \).

Turn to (2.12). If we assume that \( d_9(xb_{10}) = yb_{10} \), then there is an element \( z \in \text{Ker } (b_{10} : E_9^s(V(1) \wedge X) \to E_9^{s+2}(V(1) \wedge X)) \) such that \( d_9(x) = y + z \). Note that \( s \geq 9 \). Since \( xb_{10} = 0 \in E_9^{s+2}(V(1) \wedge X) \), there is an element \( w \in E_9^{s+3}(V(1) \wedge X) \) such that \( d_9(w) = zb_{10} \). By the structure \( (2.5) \) of the \( E_9(=E_2) \)-term, we see that there is an element \( w' \in E_9^{s+5}(V(1) \wedge X) \) such that \( w = w'b_{10} \), and \( d_9(w') = z \), since \( b_{10} \) is a monomorphism on \( E_5 \)-terms. It follows that \( z = 0 \) in the \( E_9 \)-term, and we have \( d_9(x) = y \) as desired. \( \square \)

Since \( \pi_*(V(1) \wedge X) \) is a \( \mathbb{Z}/3 \)-vector space, there is no extension problem in the spectral sequence.

Corollary 2.13. The homotopy groups \( \pi_*(V(1) \wedge X) \) are isomorphic to the \( E_\infty \)-terms for them.

3. Invertible spectra and the Smith-Toda spectra

First we consider the case \( p = 2 \) and \( n = 1 \). Then it is shown in [11] (cf. [23]) that \( \text{Pic}(L_1)^0 = \mathbb{Z}/2 \), whose generator is represented by the \( E(1) \)-localization of the question mark complex \( QM = V(0) \cup e_1^3 \) is the mod 2 Moore spectrum. Let \( E_2^*(W) \) for a spectrum \( W \) denote the \( E_\infty \)-term of the \( E(1) \)-based Adams spectral sequence for \( \pi_*(L_1W) \). Since \( L_1QM \) is invertible, the \( E_2 \)-term for \( \pi_*(L_1V(0) \wedge QM) \) is isomorphic to that for \( \pi_*(L_1V(0)) \):
\[
(3.1) \quad K(1)_*[h_{10}] \otimes \Lambda(\rho_1),
\]
where \( K(1)_* = \mathbb{Z}/2[v_1^{-1} \alpha_1^2] \). The map \( \varphi_3 : \text{Pic}(L_1)^0 = F_1 \to E_3^{3/2}(S^0) = \mathbb{Z}/2[\alpha_{-1} \alpha_1^2] \) of (2.1) is an isomorphism such that \( \varphi_3([L_1QM]) = \alpha_{-1} \alpha_1^2 \). Here \( \alpha_k \) for \( k \in \mathbb{Z} \) denotes \( \delta(v_k) \) for the connecting homomorphism \( \delta : E_2^*(V(0)) \to E_2^{*+1}(S^0) \) associated to the cofiber sequence \( S^0 \to S^0 \xrightarrow{i} V(0) \) and for \( v_k \in E_2^s(V(0)) = K(1)_* \).

Note that \( i_*\alpha_{-1} = v_1^{-1}h_{10} \) and \( i_*\alpha_1 = h_{10} \) for the induced map \( i_* : E_2^s(S^0) \to E_2(V(0)) \) from the inclusion \( i \). The \( E_2 \)-term for \( QM \) is isomorphic to that for the sphere \( L_1S^0 \), and \( d_3(g) = \alpha_{-1} \alpha_1^2 \in E_3^{3}(QM) = E_3^{3}(QM) \) by the definition of \( \varphi_3 \). Therefore, the Adams differentials on \( E_2^*(V(0) \wedge QM) \) are computed by \( d_3(i_*(g)) = v_1^{-1}h_{10}g \in E_3^{3/2}(V(0) \wedge QM) \) and the derivation formulas associated
to the trivial pairing \( L_1 V(0) \wedge QM \to L_1 V(0) \wedge QM \). In fact, \( v_{1}^{-2}g \) induces the isomorphism \( E_\infty^*(V(0)) \cong E_\infty^*(V(0) \wedge QM) \) of spectral sequences. Therefore, we have

\[
E_\infty^{*,*}(V(0) \wedge QM) = N \otimes \mathbb{Z}/2[v_1, v_1^{-4}] \otimes \Lambda(\rho_1),
\]

where \( N \) is the module given by

\[
(3.2)
\]

in which each dot denotes \( \mathbb{Z}/2 \) generated by the indicated element. Therefore, we see that \( v_1^{-2}g \in \pi_{-4}(L_1 V(0) \wedge QM) \) and \( 2v_1^{-2}g = 0 \). Thus we obtain the map \( v_1^{-2}: \Sigma^{-4}V(0) \to L_1 V(0) \wedge QM \), which induces an isomorphism on \( E(1)_* \)-homology.

**Proposition 3.3.** The element \( v_1^{-2}g \in \pi_{-4}(L_1 V(0) \wedge QM) \) induces an equivalence \( L_1 \Sigma^{-4}V(0) \simeq L_1 V(0) \wedge QM \).

We will play the same game for the case where \( p = 3 \) and \( n = 2 \).

**Lemma 3.4.** There is a homotopy element \( v_2^3g \in \pi_{48}(V(1) \wedge X) \) for \( i = 1, 2 \) such that \( E(2)_*(v_2^3g) = v_2^3 \).

**Proof.** We have seen that \( v_2^3g \in \pi_{48}(V(1) \wedge X) \) in the previous section. We also see that \( \pi_{43}(V(1) \wedge X) = \mathbb{Z}/3\{v_2^3h_{11}g\} \). Therefore, we obtain \( v_2^3g \in \pi_{48}(V(1)_2 \wedge X) \) from the exact sequence \( \pi_{48}(V(1)_2 \wedge X) \xrightarrow{t_2^1} \pi_{48}(V(1) \wedge X) \xrightarrow{\delta} \pi_{43}(V(1)_2 \wedge X) \) associated to the cofiber sequence \( (1.3) \) with \( k = 1 \). Indeed, \( v_2^3h_{11}g \) has the Adams filtration 1, while \( \delta(v_2^3g) = 0 \) for the connecting homomorphism \( \delta \) corresponding to \( t_1^1 \).

Now we have the similar results to Proposition 3.3.

**Theorem 3.5.** The element \( v_2^3g \in \pi_{48}(V(1) \wedge X) \) induces an equivalence \( \Sigma^{48}L_2 V(1)_i \simeq V(1)_i \wedge X \) for \( i = 1, 2 \).

**Proof.** Since \( V(1)_2 \) is a ring spectrum, the element \( v_2^3g \) yields the self-map \( \overline{v_2^3g}: V(1)_2 \to V(1)_2 \wedge X \), which induces an isomorphism on \( E(2)_* \)-homology. Therefore, the proposition for \( i = 2 \) follows. For \( i = 1 \), Lemmas 3.4 and 1.5 show the existence of the map \( v_2^3g: \Sigma^{48}L_2 V(1)_1 \to V(1) \wedge X \), which is also an \( E(2)_* \)-equivalence.

**Proof of Corollary D.** In the same manner as the proof of Corollary C, the \( \beta \)-elements are defined as follows:

\[
\begin{align*}
\beta_{9t+3g} &= (j_0 \wedge 1_X)(v_2^3g)(v_2^3)^t, & \beta_{9t+4g} &= (j_0 \wedge 1_X)(v_2^3g)(v_2^3)^tv_2, \\
\beta_{9t+5g} &= (v_2^3 \wedge 1_X)(v_2^3g)(v_2^3)^tv_2, & \beta_{9t+8g} &= (j_0 \wedge 1_X)(v_2^3g)(v_2^3)^tv_2^2, & \beta_{9t+9g} &= (v_2^3 \wedge 1_X)(v_2^3g)(v_2^3)^tv_2^2.
\end{align*}
\]

\( \square \)
Remark 3.6. $v_2^3g \in E_2^{0,48}(V(1)_3 \wedge X)$ cannot be a permanent cycle. In fact, we see that $d_9(j_{3*}(v_2^3 g)) = d_9(\delta(v_2^3 g)) = d_9(v_2^3 h_{10}g) = v_2^{1-1}b_{10}^1g \neq 0$, where $j_3$ is the map of $\Sigma_3$ with $k = 3$. Therefore, we cannot tell whether or not $\beta_{9+6}g$ survives to a homotopy element of $\pi_*(X)$ different from Corollary C. Indeed, we do not know if $v_2^4v_2^3g \in \pi_*(V(1)_3 \wedge X)$.

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