COHOMOLOGY OF SYMPLECTIC REDUCTIONS OF GENERIC COADJOINT ORBITS

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Abstract. Let $O_\lambda$ be a generic coadjoint orbit of a compact semi-simple Lie group $K$. Weight varieties are the symplectic reductions of $O_\lambda$ by the maximal torus $T$ in $K$. We use a theorem of Tolman and Weitsman to compute the cohomology ring of these varieties. Our formula relies on a Schubert basis of the equivariant cohomology of $O_\lambda$, and it makes explicit the dependence on $\lambda$ and a parameter in $\text{Lie}(T)^* =: t^*$.

1. Introduction

Let $K$ be a compact semisimple Lie group, $T \subset K$ a maximal torus and $t \subset \mathfrak{t}$ their Lie algebras. Pick a fundamental chamber in $t^*$ and a point $\lambda$ in the interior. Let $O_\lambda$ be the orbit of $\lambda$ under the coadjoint representation of $K$ on $\mathfrak{t}^*$. $O_\lambda$ is diffeomorphic to the flag variety $K/T$, and it has a naturally occurring symplectic form $\omega$ known as the Kirillov-Kostant-Souriau form. The action of $T$ on $O_\lambda$ is Hamiltonian, which means that there is an invariant map

$$\Phi : O_\lambda \to t^*$$

satisfying $\omega(X_\eta, \cdot) = d\Phi^\eta$, where $\eta \in t$, $X_\eta$ is the vector field on $O_\lambda$ generated by $\eta$, and $\Phi^\eta(m) = \Phi(m)(\eta)$ is defined by the natural pairing between $t$ and $t^*$. We call $\Phi$ a moment map for this action.

The image of $\Phi$ is the convex hull of $W \cdot \lambda$, the Weyl group orbit of $\lambda$. Let $\mu \in \Phi(O_\lambda)$ be a regular value of $\Phi$. We define the symplectic reduction at $\mu$ by

$$\Phi^{-1}(\mu)/T = O_\lambda//T(\mu).$$

The goal of this note is to give a presentation of the cohomology ring of $O_\lambda//T(\mu)$ in terms of the root system of $K$. We present $H^*(O_\lambda//T(\mu))$ as a quotient of the $T$-equivariant cohomology ring $H^*_T(O_\lambda)$ by a certain ideal. We rely on the following fundamental result.

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1 Only cohomology with coefficients in the field $\mathbb{Q}$ of rational numbers will be considered throughout this paper.
Theorem 1.1 (Kirwan). Let $M$ be a compact symplectic manifold with a Hamiltonian $T$ action, where $T$ is a compact torus. If $\mu \in \mathfrak{t}^*$ is a regular value of $\Phi$, then the restriction map in equivariant cohomology,

$$ \kappa : H_T^*(M) \to H_T^*(\Phi^{-1}(\mu)), $$

is surjective.

Since the $T$ action is locally free on level sets of the moment map at regular values, $H_T^*(\Phi^{-1}(\mu)) = H^*(M/T(\mu))$. The resulting map $\kappa : H_T^*(M) \to H^*(M/T(\mu))$ is called the Kirwan map. Kirwan’s result is of particular importance because the equivariant cohomology can be described in terms of the equivariant cohomology of the fixed point sets of the $T$ action. In the case of isolated fixed points, this is just a sum of polynomial rings.

Theorem 1.2 (Kirwan). Let $M$ be a compact Hamiltonian $T$-space. Let $M^T$ denote the fixed point set of the $T$ action. The restriction map

$$ i^* : H_T^*(M) \to H_T^*(M^T) $$

is injective. In the case that $M^T$ is a finite set of points,

$$ H_T^*(M^T) = \bigoplus_{p \in M^T} \mathbb{Q}[x_1, \ldots, x_n] $$

where $n = \dim T$.

A presentation of the cohomology ring of the reduced space $M/T(\mu)$ can be obtained by using the following description of the kernel of the Kirwan map, which is due to Tolman and Weitsman [TW]. If $\alpha$ is in $H_T^*(M)$ we denote

$$ \text{supp}(\alpha) = \{ p \in M^T : \alpha|_p \neq 0 \}. $$

Fix an arbitrary inner product $\langle \ , \ \rangle$ on $\mathfrak{t}^*$.

Theorem 1.3 (Tolman-Weitsman). The kernel of the Kirwan map $\kappa$ is the ideal of $H_T^*(M)$ generated by all $\alpha \in H_T^*(M)$ with the property that there exists $\xi \in \mathfrak{t}^*$ such that

$$ \Phi(\text{supp}(\alpha)) \subset \{ x \in \mathfrak{t}^* | \langle \xi, x \rangle \leq \langle \xi, \mu \rangle \}. $$

In other words, $\ker \kappa$ consists of sums of equivariant cohomology classes $\alpha$ with the property that all points of $\text{supp}(\alpha)$ are mapped by $\Phi$ to the same side of an affine hyperplane in $\mathfrak{t}^*$ that passes through $\mu$.

The $T$-equivariant cohomology ring of the coadjoint orbit $O_\lambda = K/T$ is well understood. Kostant and Kumar constructed in [KK] a basis $\{ x_w \}_{w \in W}$ of $H_T^*(K/T)$ as a $H_T^*(pt)$-module, which we refer to as the Schubert basis. Let $B$ be a Borel subgroup in $G := K^C$, and let $B_- = K^C$ be the opposite Borel subgroup. For any $v \in W$, let $X_v = B_- v B / B$, where $\hat{v}$ is any choice of lift of $v \in W$ in the normalizer of the torus. These opposite Schubert varieties are $T$-invariant subvarieties of $G/B \cong K/T$. The basis $\{ x_w \}$ is uniquely defined by the property that

$$ \int_{X_v} x_w = \delta_{v,w}. $$

Theorem [12] suggests the importance of knowing how to restrict the classes $x_w$ to fixed points $W \cdot \lambda$. This formula was worked out for general $K$ by S. Billey [B]. In
particular, it is easy to show that $x_w|_v = 0$ if $v \not\leq w$ in the Bruhat order.\footnote{The class $x_w$ differs from the $\xi^w$ constructed in \cite{KK} by the relationship $x_w := w_0 \cdot \xi^{w_0 w}$, where $w_0$ is the longest element of $W$.} In other words,

$$\text{supp}(x_w) = \{v \lambda : v \leq w\}.$$ 

To each $\tau \in W$ we can associate the new basis

$$\{x^\tau_w = \tau \cdot x_{\tau^{-1}w}\}_{w \in W},$$

whose elements have the property

$$\text{supp}(x^\tau_w) = \{v \lambda : \tau^{-1}v \leq \tau^{-1}w\}.$$ 

Let $\lambda_1, \ldots, \lambda_l \in \mathfrak{t}^*$ denote the fundamental weights associated to the chosen fundamental chamber of $\mathfrak{t}^*$. Let $\langle \cdot, \cdot \rangle$ be the restriction to $\mathfrak{t}^*$ of a $K$-invariant product on $\mathfrak{t}^*$. Our main result is:

**Theorem 1.4.** The cohomology ring $H^*(\mathcal{O}_\lambda//T(\mu))$ is isomorphic to the quotient of $H^*_T(K/T)$ by the ideal generated by

$$\{x^\tau_w : \text{there exists } j \text{ such that } \langle \lambda_j, \tau^{-1}v \lambda \rangle \leq \langle \lambda_j, \tau^{-1}w \lambda \rangle\}.$$ 

**Remark 1.** One can take the description of $H^*_T(K/T)$ (see, for instance, \cite{Br}) and deduce a precise presentation of the cohomology ring $H^*(\mathcal{O}_\lambda//T(\mu))$ in terms of generators and relations.

**Remark 2.** For $K = \text{SU}(n)$ this result was proven by the first author in \cite{Go1}.

## 2. Primary description of $\ker \kappa$

For any $\xi \in \mathfrak{t}^*$ we denote by $f_\xi$ the corresponding height function on $\mathcal{O}_\lambda$,

$$f_\xi(x) = \langle \xi, x \rangle.$$ 

Under the pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$, the function $f_\xi$ is a component of the moment map. In fact, it is well known that $f_\xi$ is Morse-Bott for all $\xi \in \mathfrak{t}^*$. Denote by $C \subset \mathfrak{t}^*$ the fundamental (positive) Weyl chamber, which can be described by

$$C = \{r_1 \lambda_1 + \cdots + r_l \lambda_l : \text{ all } r_j > 0\},$$

and let $\overline{C}$ be its closure.

**Lemma 2.1.** Let $\tau$ be in $W$ and $\xi$ in $\tau \overline{C}$. If $\tau^{-1}v < \tau^{-1}w$ in the Bruhat order, then $f_\xi(v \lambda) \leq f_\xi(w \lambda)$.

**Proof.** The result follows immediately from the fact that if $\xi \in C$, then the unstable manifold of $f_\xi$ through $v \lambda$ with respect to the Kähler metric on $\mathcal{O}_\lambda = K/T = G/B$ is just the Bruhat cell $B \cdot vB/B$ (see, for instance, \cite{Ko}). \hfill $\square$
The main result of this section is:

**Theorem 2.1.** Suppose that $H^*_T(O_\lambda)$ has the property that
\[ \Phi(\text{supp}(\alpha)) \subset \{ x \in \mathfrak{t}^* : \langle \xi, x \rangle \leq \langle \xi, \mu \rangle \}. \]
Then $\alpha$ can be decomposed as
\[ \alpha = \sum_{w \in W} a^\tau_w x^\tau_w \]
with $a^\tau_w \in H^*_T(pt)$, such that if $a^\tau_w \neq 0$, then
\[ \Phi(\text{supp}(x^\tau_w)) \subset \{ x \in \mathfrak{t}^* : \langle \xi, x \rangle \leq \langle \xi, \mu \rangle \}. \]

**Proof.** Take $2W$ such that $2C$. Suppose that the decomposition of $\alpha$ with respect to the basis $\{ x^\tau_w \}_{w \in W}$ is of the form
\[ \alpha = \sum_{w \in W} a^\tau_w x^\tau_w + a^\tau_{v_1} x^\tau_{v_1} + \cdots + a^\tau_{v_r} x^\tau_{v_r}, \]
where the first sum contains only $w$ with
\[ \langle \xi, w\lambda \rangle \leq \langle \xi, \mu \rangle, \]
whereas
\[ \langle \xi, v_j \lambda \rangle > \langle \xi, \mu \rangle, \quad a^\tau_{v_j} \in S(\mathfrak{t}^*), a^\tau_{v_j} \neq 0, \]
for any $1 \leq j \leq l$. We may assume that $v_1$ has the property that there exists no $j > 1$ with $\tau^{-1}v_1 < \tau^{-1}v_j$. Now let us evaluate both sides of (1) at $v_1\lambda$. Since
\[ \langle \xi, w\lambda \rangle \leq \langle \xi, \mu \rangle < \langle \xi, v_1\lambda \rangle, \]
by Lemma 2.1 we must have
\[ x^\tau_{v_1}|_{v_1\lambda} = 0 \]
for any $w$ corresponding to a term in the first sum in (1). It follows that
\[ \alpha|_{v_1\lambda} = a^\tau_{v_1} x^\tau_{v_1}|_{v_1\lambda} \neq 0; \]
so $v_1\lambda$ is in $\text{supp}(\alpha)$ even though $\langle \xi, v_1\lambda \rangle > \langle \xi, \mu \rangle$. This is a contradiction. \qed

3. **Proof of the main result**

We now prove Theorem 1.4. Let $v$ and $\tau$ in $W$ be such that
\[ \langle \lambda_j, \tau^{-1}v\lambda \rangle \leq \langle \lambda_j, \tau^{-1}\mu \rangle, \]
for some $1 \leq j \leq l$. We show that $x^\tau_v$ is in the kernel of the Kirwan map
\[ \kappa : H^*_T(O_\lambda) \rightarrow H^*(O_\lambda//T(\mu)). \]
Let $\xi = \tau\lambda_j$ be in $\tau\mathcal{C}$. Note that if $w \in \text{supp}(x^\tau_v)$, then $\tau^{-1}w \leq \tau^{-1}v$ implies by Lemma 2.1 that
\[ \langle \xi, w\lambda \rangle \leq \langle \xi, v\lambda \rangle \leq \langle \xi, \mu \rangle. \]
Thus $x^\tau_v \in \ker \kappa$.

Now let us consider $\alpha \in H^*_T(K/T)$ with the property that there exists $\xi \in \mathfrak{t}^*$ with
\[ \text{supp}(\alpha) \subset \{ x \in \mathfrak{t}^* | \langle \xi, x \rangle \leq \langle \xi, \mu \rangle \}. \]
Take \( \tau \in W \) such that \( \xi \in \tau \mathcal{C} \). By Theorem 2.2, we can decompose \( \alpha \) as

\[
\alpha = \sum_{w \in W} a_w^\tau x_w^\tau
\]

where \( a_w^\tau \) can be nonzero only if

\[
\text{supp}(x_w^\tau) \subset \{ x \in t^* | \langle \xi, x \rangle \leq \langle \xi, \mu \rangle \}.
\]

In particular, if \( a_w^\tau \neq 0 \), then

\[
\langle \xi, w \lambda \rangle \leq \langle \xi, \mu \rangle.
\]

Since \( \xi \) is in \( \tau \mathcal{C} \), it can be written as

\[
\xi = \tau \sum_{j=1}^l r_j \lambda_j,
\]

where all \( r_j \) are nonnegative. So (4) and (5) imply that there exists \( j \in \{1, \ldots, l\} \) such that

\[
\langle \tau \lambda_j, w \lambda \rangle \leq \langle \tau \lambda_j, \mu \rangle.
\]

In other words, each nonzero term in the right-hand side of (3) is a multiple of a \( x_w^\tau \) of the type claimed in Theorem 1.4.

\[\square\]

**Remark.** It follows that, in the particular situation of generic coadjoint orbits, in order to cover the whole Tolman-Weitsman kernel of the Kirwan map it is sufficient to consider affine hyperplanes through \( \mu \) that are perpendicular to vectors of the type \( \tau \lambda_j \), with \( \tau \in W \) and \( j \in \{1, \ldots, l\} \). But these are just the hyperplanes parallel to the walls of the moment polytope. This result concerning a "sufficient set of hyperplanes" has been proved by the first author in [Go2], for an arbitrary Hamiltonian torus action on a compact manifold.

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**References**


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