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ON FINITENESS OF THE SET OF INTERMEDIATE SUBFACTORS

M. KHOSHKAM AND B. MASHOOD

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Abstract. For type II_1 factors $N \subset L$ with $[L : N] < \infty$, we show that the sets $\mathcal{L}_1 = \{M \in \mathcal{L}(N \subset L) \colon N' \cap L \subset M\}$ and $\mathcal{L}_2 = \{M \in \mathcal{L}(N \subset L) \colon N' \cap L = M' \cap L\}$ are finite. Moreover, $\mathcal{L}(N \subset L)$, the set of intermediate subfactors, is finite if and only if it is equal to $\mathcal{L}_1 \cup \mathcal{L}_2$. If N is an irreducible subfactor, then we recover a result of Y. Watatani.

0. Introduction

Let $N \subset L$ be an inclusion of II_1 factors. A subfactor M of L such that $N \subset M \subset L$ is called an intermediate subfactor. Intermediate subfactors inherit interesting rigidity properties. In ([B]), D. Bisch proves that if $N \subset L$ is a finite depth inclusion, then so are the two inclusions $N \subset M$ and $M \subset L$. Furthermore, he gives an abstract characterization of intermediate subfactors in terms of projections of $\langle L, e_N \rangle$. In general, the set of intermediate subfactors for an inclusion $N \subset L$ may be trivial, consisting of N and L. In this case, N is said to be a maximal subfactor of L. For example, for the inclusion $N \subset N \otimes M_p(\mathbb{C})$ where p is a prime number, there is no nontrivial intermediate subfactor. If p is not a prime, then there are an infinite number of intermediate subfactors. If G is a countable discrete group of outer automorphisms of a II_1 factor N, then any intermediate subfactor of the inclusion $N \subset N \rtimes G$ is of the form $N \rtimes H$ for some subgroup H of G. Thus, N is maximal if and only if G is cyclic of prime order. If N is irreducible, then any M with $N \subset M \subset L$ is automatically a factor and, as a result, the set of intermediate subfactors forms a lattice. If in addition $[L : N] < \infty$, Y. Watatani proved that the lattice is finite. This article deals mainly with the question of finiteness of the set of intermediate subfactors for finite-index-inclusion intermediate subfactors. It is worth mentioning that even in the case that $N' \cap L$ is abelian, the set of intermediate subfactors may not be finite (cf. Theorem 5.4 [TW]). Our main result states a necessary and sufficient condition, formulated in terms of the relative commutant $N' \cap L$, for the set of intermediate subfactors to be finite.

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For an inclusion $N \subset L$ we let $\mathcal{L}(N \subset L)$ denote the set of intermediate subfactors, which in the case that N is an irreducible subfactor of L, but not in general, is a lattice under the operations $M_1 \wedge M_2 = M_1 \cap M_2$ and $M_1 \vee M_2 = (M_1 \cup M_2)''$ in $\mathcal{L}(N \subset L)$. In order to prove our main result (Theorem 1.7), we identify two finite subsets, $\mathcal{L}_1$ and $\mathcal{L}_2$, of $\mathcal{L}(N \subset L)$ such that: a) the two sets coincide and coincide with $\mathcal{L}(N \subset L)$ in the irreducible case, and hence recovering Watatani's theorem, we moreover prove that b) $\mathcal{L}(N \subset L)$ is finite if and only if $\mathcal{L}(N \subset L) = \mathcal{L}_1 \cup \mathcal{L}_2$. Intuitively, $\mathcal{L}_1$ consists of those intermediate subfactors that are close to N, and $\mathcal{L}_2$ consists of the ones close to L (see Theorem 1.3 for the definition). The perturbation techniques developed by E. Christensen ([C]) play an essential role in our analysis, much similar to the methods that we developed in ([KM]). The crucial observation is the fact that if M_1 and M_2 are intermediate subfactors for an inclusion $N \subset L$ of finite index, $[L : M_1] = [L : M_2]$, and the distance between M_1 and M_2 is sufficiently small, then they are unitarily equivalent, i.e., $M_2 = uM_1u^*$ for a unitary $u \in L$, and the unitary u can be chosen to be close to the identity of L.

1. The cardinality of the set of intermediate subfactors

Throughout, $N \subset L$ are fixed II_1 factors, and $[L : N] < \infty$. Recall that using the trace on L, E. Christensen ([C]) defined a metric d on the set of von Neumann subalgebras of L. In many interesting situations, Christensen proved that if $d(M_1, M_2)$ is sufficiently small, then M_1 and M_2 are $*$-isomorphic via a unitary operator close to the identity of L. We are going to rely on the ideas, notation, and results of ([C]). The following function, γ, appears frequently in perturbation calculations:

$$\gamma(x) = 2^{1/4}x^{1/2}(1 - 2^{1/4}x^{1/2})^{-1}.$$

Recall that if $[L : N] < \infty$, then $N' \cap L$ is a finite-dimensional C^*-algebra. Thus, $\{\mathrm{tr}(p) \colon p \in N' \cap L, \text{projection}\}$ is a finite set. This fact is often used in subsequent arguments. Given $M \subset L$, $\langle L, e_M \rangle$ denotes Jones' basic construction and e_M the corresponding Jones' projection. We refer to ([J]) for basic notation and facts on index theory. Finally, we let $|A|$ denote the cardinality of the set A.

1.1. **Lemma.** *Let $[L : N] < \infty$, $M_1, M_2 \in \mathcal{L}(N \subset L)$, and $[L : M_1] = [L : M_2]$. Then there exists $\epsilon > 0$ such that if $d(M_1, M_2) < \epsilon$, then $M_2 = uM_1u^*$ for a unitary $u \in L$ with $\|u - 1\|_2 < 2\epsilon + 52\gamma(\epsilon)$.*

Proof. Let $[L : N] = c$. Then, for any $M \in \mathcal{L}(N \subset L)$, $[L : M] < c$. Let $\delta = \max\{10^{-6}, 10^{-4}c^{-1}\}$, such that $\min\{\gamma(\delta)^2, 26\gamma(\delta) + \delta\} < c^{-1}$. First note that if $d(M, N) < \delta$, then $[L : M] = [L : N]$ (see the last paragraph of the proof of Theorem 6 of [C]). By (Lemma 2.1, [C]) there exists a projection $e \in M' \cap \langle L, e_M \rangle''$. We have that $\mathrm{tr}(e) > [L : M]^{-1} > c^{-1}$. $|\,\mathrm{tr}(e_M) - \mathrm{tr}(q)| < \gamma(\delta)^2 < c^{-1}$. Note that the projection p in the discussion preceding (Lemma 4.1, [C]) is e_M and $\pi = E_M$ in our context. But for each projection e in $M' \cap \langle L, e_M \rangle''$ we have that $\mathrm{tr}(e) > [L : M]^{-1} > c^{-1}$. Whence, $\mathrm{tr}(e_M) = \mathrm{tr}(q)$ and hence $q \sim e_M$. Now by (Lemma 4.1, [C]) there exists a homomorphism Φ of N into M such that $\|\Phi(x) - E_M(x)\| < 26\gamma(\delta_1)$. Since $d(M, N) < \delta$,

$$\|\Phi(x) - x\|_2 \leq \|\Phi(x) - E_M(x)\|_2 + \|E_M(x) - x\|_2 < 26\gamma(\delta) + \delta.$$

Then, by (Theorem 3.1, [C]) there exists $v \in \langle N, \Phi(N) \rangle''$ such that $q = v^*v \in \Phi(N)'$, $r = vv^* \in N'$, and such that $\|1 - v\|_2$, $\|1 - q\|_2$, $\|1 - r\|_2$ are all less than

$26\gamma(\delta) + \delta$, and $q\Phi(x) = v^*xv$. Since q and r belong to the finite-dimensional algebra $M' \cap L$, our choice of the constant δ implies that $q = r = 1$. Thus, v is a unitary in L that implements Φ, i.e., $\Phi(x) = vxv^*$ for each $x \in N$ (cf. Lemma 4.1, [C]). Since $v \in L$, $[L : \Phi(N)] = [L : vNv^*] = [L : N] = [L : M]$, but $\Phi(N) \subset M$. Hence, $\Phi(N) = M$ (cf. [J]). □

1.2. **Corollary.** *Let $M_1, M_2 \in \mathcal{L}(N \subset L)$. Then there exists a unitary $u \in L$ such that $uM_2u^* = M_1$ and $uNu^* = N$.*

Proof. Let $\varphi\colon M_2 \mapsto M_1$ be the isomorphism of Lemma 1.1. Note that N and $\varphi(N)$ are included in M_1, and $d(N, \varphi(N)) < 2\epsilon + 52\gamma(\epsilon)$. Moreover, φ maps $N' \cap M_2$ onto $\varphi(N)' \cap M_1$. Extend φ to the algebra $\langle L, e_N\rangle$ by $\varphi(x) = vxv^*$ for all $x \in \langle L, e_N\rangle$. This is a trace-preserving isomorphism, and hence the minimal projections in $\varphi(N)' \cap M_1$ and those of $N' \cap M_2$ have the same set of trace values (up to permutations). Then, the argument of the preceding lemma can be applied to N and $\varphi(N)$ as subfactors of M_1 to get projections $q_1 \in \varphi(N) \cap M_1$, $r_1 \in N' \cap M_1$, and a partial isometry $v_1 \in M_1$ such that $v_1v_1^* = q_1$ and $v_1^*v_1 = r_1$, all of which are close to the identity in $\| \ \|_2$. Since $N' \cap M_i \subset N' \cap L$, a trace argument as before implies that $q_1 = r_1 = 1$. Hence, v_1 is a unitary and $\varphi(N) = v_1Nv_1^*$. If v is the unitary of Lemma 1.1, then $u = v_1^*v$ is the desired unitary, i.e., $M_1 = uM_2u^*$ and $uNu^* = N$. □

1.3. **Theorem.** *Let $N \subset L$ be II_1 factors such that $[L : N] < \infty$. Then,*

i) $\mathcal{L}_1 = \{M \in \mathcal{L}(N \subset L)\colon N' \cap L \subset M\}$,

and

ii) $\mathcal{L}_2 = \{M \in \mathcal{L}(N \subset L)\colon N' \cap L = M' \cap L\}$

are finite subsets of $\mathcal{L}(N \subset L)$.

Proof. The first part of the proof is the same for both i) and ii). For each $M \in \mathcal{L}(N \subset L)$, $e_M \in N' \cap \langle L, e_N\rangle$ with trace equal to $[L : M]^{-1}$. Since $N' \cap \langle L, e_N\rangle$ is finite dimensional, the set $\{[L : M] : M \in \mathcal{L}(N \subset L)\}$ must be finite. Hence, it suffices to show that for $c > 1$, the intersections of $\mathcal{L}_1$ and $\mathcal{L}_2$ with the set $\{M \in \mathcal{L}(N \subset L) : [L : M] = c\}$ is a finite set. If not there exists a sequence (K_n) in $\mathcal{L}_1$ (respectively in $\mathcal{L}_2$) such that $K_n \neq K_m$ if $n \neq m$ and $[L : K_n] = c$ for each n. Since $e_{K_n} \in N' \cap \langle L, e_N\rangle$, which is finite dimensional, the sequence (e_{K_n}) must have a limit point in $N' \cap \langle L, e_N\rangle$. Assume, without loss of generality, that the sequence (e_{K_n}) converges in the uniform topology to a projection $p \in N' \cap \langle L, e_N\rangle$, and such that $\mathrm{tr}(e_{K_n}) = \mathrm{tr}(p) = c$. Then, $d(K_n, K_m) < \|e_{K_n} - e_{K_m}\|_2$, which shows that $d(K_n, K_m)$ can be made arbitrarily small by choosing m and n sufficiently large. Hence, by Lemma 1.1, there exists a unitary $u \in L$ such that $K_m = uK_nu^*$ for sufficiently large n and m such that $\|u - 1\| < \epsilon$ for a given $\epsilon > 0$. Thus, for each $z \in N \subset K_n$, $uzu^* = k$ for some $k \in K_m$. Whence,

$$E_{K_m}(u)z = kE_{K_m}(u),$$

which shows that $u^*E_{K_m}(u)z = zu^*E_{K_m}(u)$, i.e., $u^*E_{K_m}(u) \in N' \cap L$. At this point we consider the two cases separately.

i) Let $u^*E_{K_m}(u) = h$. Since $\|u - I\|_{\mathrm{tr}}$ can be made sufficiently small, and $\|h - I\| < 2\|u - I\|_{\mathrm{tr}}$, we can choose ϵ such that

$$\|h - I\| < 1.$$

Thus, the element h is invertible, and we have $u^*E_{K_m}(u)h^{-1} = I$. Since $K_m \in \mathcal{L}_1$, we have $N' \cap L \subset K_m$. Hence $h \in K_m$. Whence,

$$E_{K_m}(uh^{-1}) = u,$$

which shows that $u \in K_m$. Whence, $K_n = K_m$, which is a contradiction.

ii) Since $E_{K_m}(u) = uh$, the element $E_{K_m}(u)$ is invertible. Since $h \in K_m' \cap L = N' \cap L$, for any $x \in K_m$ we have

$$\begin{aligned} uxu^* &= E_{K_m}(u)h^{-1}xhE_{K_m}(u)^{-1} = E_{K_m}(u)h^{-1}hxE_{K_m}(u)^{-1} \\ &= E_{K_m}(u)xE_{K_m}(u)^{-1}. \end{aligned}$$

Thus $uxu^* \in K_m$, and it follows that $K_m = K_n$, which is in contradiction with the choice of K_n's. We conclude from this argument that $\mathcal{L}_1$ and $\mathcal{L}_2$ are not finite sets. □

The following corollary is a theorem of Y. Watatani ([W]).

1.4. **Corollary.** *Let $[L : N] < \infty$ and $N' \cap L = \mathbb{C}$. Then, $\mathcal{L}at(N \subset L)$ is finite.*

Proof. Observe that if $N' \cap L = \mathbb{C}$, then $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}at(N \subset L)$, and the corollary follows from Theorem 1.3. □

1.5. **Corollary.** *There exists an $\epsilon > 0$ such that if $M_1, M_2 \in \mathcal{L}(N \subset L)$ and $d(M_1, M_2) < \epsilon$, then $M_2 = xM_1x^{-1}$ for an invertible $x \in N' \cap L$.*

Proof. By (Theorem 5, [MT]), there exists a $\delta > 0$ such that $[L : M_1] = [L : M_2]$ when $d(M_1, M_2) < \delta$. Let ϵ be the minimum of δ and the constant given by Lemma 1.1. Then there exists $u \in L$ such that $M_1 = uM_2u^*$ and $\|u - I\| < 2\epsilon + 52\gamma(\epsilon)$. By the argument of Theorem 1.3, $uE_{M_2}(u) = x \in N' \cap L$. Now, choose ϵ sufficiently small such that $\|x - I\| < 1$. (In fact, $\|x - I\| < 2\epsilon + 104\gamma(\epsilon)$.) Then x is invertible, $x \in N' \cap L$, and

$$xM_2x^{-1} = uE_{M_2}(u)M_2E_{M_2}(u)^{-1}u^* = uM_2u^* = M_1. \qquad \square$$

1.6. **Proposition.** *Let $N \subset L$ be an inclusion of II_1 factors such that $[L : N] < \infty$. Suppose that $N' \cap L \neq \langle N' \cap M, M' \cap L\rangle''$. Then $|\mathcal{L}(N \subset L)|$ is infinite.*

Proof. Choose a minimal projection $f \in N' \cap L$ that is not in $\langle N' \cap M,\, M' \cap L\rangle''$. Let p_1 and p_2 be distinct prime numbers larger than $[L : N]$. Consider the unitary

$$u = \exp(2\pi i/p_1)f + \exp(2\pi i/p_2)(1 - f) \in L.$$

Then u generates a group G of unitary elements of order p_1p_2.

Claim. The set $\{vMv^* : v \in G\}$ consists of distinct subfactors.

If not, there will be an element $v \in G$ such that $vMv^* = M$. Now $\operatorname{Ad} v$ is an outer automorphism of M. For otherwise, there exists a unitary $w \in M$ such that $\operatorname{Ad} v = \operatorname{Ad} w$. Then, $v^*w \in M' \cap L$ and since $w \in N' \cap M$ it follows that v and hence f, being a spectral projection of v, belongs to $\langle N' \cap M, M' \cap L\rangle''$, which is in contradiction with the choice of f. Let H be the subgroup of G generated by v. Then, $|H| > [L : N]$ by the choice of p_1 and p_2. Now the fixed point algebra M^H contains N, and $[M : M^H] = |H|$ (see [J], Example 2.3.3). Then, from the inclusions $N \subset M^H \subset M \subset L$ we have

$$[L : N] = [M^H : N][M : M^H][L : M] > |H| > [L : N],$$

which is a contradiction, and our claim is established.

We conclude that

$$|\mathcal{L}_L(N \subset L)| \geq p_1 p_2.$$

Since p_1 and p_2 can be chosen as large as we want, it follows that $|\mathcal{L}_L(N \subset L)| = \infty$. $\square$

We are now ready to state our main result.

1.7. **Theorem.** *Let $N \subset L$ be an inclusion of II_1 factors such that $[L : N] < \infty$. Then $\mathcal{L}(N \subset L)$ is finite if and only if $\mathcal{L}(N \subset L) = \mathcal{L}_1 \cup \mathcal{L}_2$.*

Proof. The if part is just Theorem 1.3. Suppose there exists $M \in \mathcal{L}(N \subset L) \setminus \mathcal{L}_1 \cup \mathcal{L}_2$. Then we claim that $\mathcal{L}(N \subset L)$ is an infinite set. If

$$N' \cap L \neq \langle N' \cap M, M' \cap L \rangle'',$$

then the claim holds by Theorem 1.7. Hence, assume that

$$(*) \qquad N' \cap L = \langle N' \cap M, M' \cap L \rangle''.$$

Let $\{p_1, p_2, \ldots, p_n\}$ and $\{q_1, q_2, \ldots, q_m\}$ be, respectively, the sets of minimal central projections of $N' \cap M$ and $M' \cap L$. Then n and m are larger than one. For if either n or m equals one, then $(*)$ implies that $M \in \mathcal{L}_1 \cup \mathcal{L}_2$, which is contrary to our assumption. Choose prime numbers $\{p_{jl} \colon 1 \leq j \leq n,\ 1 \leq l \leq m\}$ such that

i) $p_{11} > [L : N]$;
ii) $p_{jl} < p_{j+1l}$ and $p_{jl} < p_{jl+1}$ for all j and l.

Let,

$$u = \sum_{j,l} e^{(\frac{2\pi}{p_{jl}})i} p_j q_l.$$

Then, u is a unitary of order $\prod p_{jl}$ in $N' \cap L$. Hence, $N \subset u^k M u^{*^k} \subset L$. If the $u^k M u^{*^k}$'s, $1 \leq k \leq \prod p_{jl}$, were distinct, then $|\mathcal{L}(N \subset L)|$ must be infinite (for otherwise we can choose p_{ij}'s such that $\prod p_{ij}$ is larger than the cardinality of $\mathcal{L}(N \subset L)$). If not, there exists a positive integer k such that $\operatorname{Ad} u^k$ is an automorphism of M. We may assume that $k < p_{11}$ (for otherwise, by increasing p_{11} large enough we obtain a set of distinct intermediate subfactors whose cardinality is arbitrarily large, which is what we want). Moreover, $\operatorname{Ad} u^k$ is an outer automorphism of M. To see this suppose that $\operatorname{Ad} u^k = \operatorname{Ad} v$ for a unitary $v \in M$. If so, $v \in N' \cap M$ and $u^k = vw$ for some unitary $w \in M' \cap L$. Let $v = \sum_j x_j p_j$ with each $x_j \in N' \cap M$ and $w = \sum_l y_l q_l$ with each $y_l \in M' \cap L$. Then from $u^k = vw$, we obtain $x_j y_l = e^{(\frac{2k\pi}{p_{jl}})i}$. From this equation it follows that each x_j and each y_l are invertible with inverses respectively in $M' \cap L$ and $N' \cap M$. It is then easy to see that the elements x_j' and y_l' belong to the intersection of $N' \cap M$ with $M' \cap L$, which is trivial. Whence, x_j and y_l must be scalers. Whence, $x_j = e^{\theta_j i}$ and $y_l = e^{\beta_j i}$ for $1 \leq j \leq n$, $1 \leq l \leq m$, and $e^{(\theta_j + \beta_l)i} = e^{\frac{2k\pi}{p_{jl}} i}$. Hence,

$$\theta_j + \beta_l + 2r\pi = \frac{2k\pi}{p_{jl}},$$

which shows that $(\theta_j + \beta_l + 2r\pi)p_{jl} = 2k\pi$. By the choice of p_{jl}'s we must have $k > p_{1,1}$, which is a contradiction. Next, let r be the smallest power of u^k such that $\operatorname{Ad} u^{kr}$ is an inner automorphism M. Then $1 < r \leq \prod p_{jl}$ and the group H generated by $\operatorname{Ad} u^{k(r+1)}$ is a group of outer automorphisms of M. Moreover, the order of H divides $\prod p_{jl}$, and hence $|H| > [L : N]$. Now, the fixed point algebra

M^H contains N, and the argument of Proposition 1.6 can be repeated to arrive at a contradiction. The contradiction shows that $\{u^k M u^{*^k}\}$ consists of distinct intermediate subfactors, and since its cardinality can be made arbitrarily large, we conclude that $\mathcal{L}(N \subset L)$ is an infinite set, which is what we wanted to show. $\square$

Define an equivalence relation on $\mathcal{L}(N \subset L)$ by $M_1 \sim M_2$ if there exists a unitary $u \in L$ such that $M_1 = uM_2u^*$ and $N = uNu^*$. Let $\mathcal{L}_N(N \subset L)$ be the corresponding quotient space. A second equivalence relation can be defined by $M_1 = uM_2u^*$, but u need not leave N invariant. Denote by $\mathcal{L}_L(N \subset L)$ the subsequent quotient. Let $|A|$ be the cardinality of the set A. Then we have the following theorem.

1.8. **Theorem.** *Let $N \subset L$ be an inclusion of II_1 factors such that $[L : N] < \infty$. Then $\mathcal{L}_N(N \subset L)$ and $\mathcal{L}_L(N \subset L)$ are finite sets. Moreover,*

$$|\mathcal{L}_L(N \subset L)| \leq |\mathcal{L}_N(N \subset L)| \leq |\mathcal{L}(N \subset L)|.$$

Proof. The finiteness of $\mathcal{L}_L(N \subset L)$ follows by using the type of argument given in Theorem 1.3 and by Lemma 1.1. Also, Corollary 1.2 shows that $\mathcal{L}_N(N \subset L)$ is finite. $\square$

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Department of Mathematics and Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6

Department of Mathematics and Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6