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BASE-COVER PARACOMPACTNESS

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Abstract. Call a topological space X *base*-cover paracompact if X has an open *base* $\mathcal{B}$ such that every cover $\mathcal{C} \subset \mathcal{B}$ of X contains a locally finite subcover. A subspace of the Sorgenfrey line is base-cover paracompact if and only if it is F_σ. The countable sequential fan is not base-cover paracompact. A paracompact space is locally compact if and only if its product with every compact space is base-cover paracompact.

Introduction

It was shown in [10] that the irrationals as a subspace of the Sorgenfrey line S are not generalized left separated, although every F_σ subspace of S is. In [5] the F_σ subspaces of S were characterized as those subspaces that are continuous images of S. We define base-cover paracompactness and in Section 1 show that a subspace of S is F_σ if and only if it is base-cover paracompact, if and only if it is generalized left separated. In Section 2 we prove that the countable sequential fan is not base-cover metacompact, and base-cover paracompactness is not F_σ hereditary in general, although it is hereditary with respect to open F_σ subsets. Taking the product with a compact factor easily destroys base-cover paracompactness, and in Section 3 we show that a paracompact space is locally compact if and only if its product with every compact space is base-cover paracompact. In Section 4 related known properties including total paracompactness [13], base-base paracompactness and base paracompactness [23], [24], and the D-space property [10] are discussed.

Definition 0.1. A space X is *base*-cover paracompact [21] (respectively *base*-cover hypocompact, *base*-cover metacompact [21]) if it has an open *base* $\mathcal{B}$ such that every cover $\mathcal{C} \subset \mathcal{B}$ of X contains a locally finite (respectively star finite, point finite) subcover.

1. Base-cover paracompactness and subsets of the Sorgenfrey line

Recall that a *LOTS* is a *linearly ordered topological space* $(X, \leq)$ for which the family of all open intervals forms a base. A space $(X, \leq)$ is a *GO-space* (i.e., a *generalized ordered* space) if it has a base of order-convex sets, [17].

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Theorem 1.1. *Suppose* $(X, \leq)$ *is a GO-space such that:*
(a) *the set* $[x, \rightarrow)$ *is open for each* $x \in X$*, and*
(b) $X = \bigcup_{i<\omega} X_i$ *where every nonempty relatively closed subset of each* X_i *has a minimal element.*
Then every F_σ *subspace of* X *is base-cover hypocompact.*

Proof. Since every F_σ-set satisfies (a) and (b), it is enough to prove that X is base-cover hypocompact. Let $D = \{d_\alpha : \alpha < \kappa\}$ be a dense subset of X, and for each $\alpha \leq \kappa$, let $D_\alpha = \{d_\beta : \beta < \alpha\}$. For each non-isolated x, let $r(x, \alpha) = \inf((x, \rightarrow) \cap D_\alpha)$ (possibly a gap), and $B(x, \alpha) = [x, r(x, \alpha))$. Let $\alpha(x) = \min\{\alpha \leq \kappa : r(x, \alpha) = x\}$, and if $x \in X_i \setminus \bigcup_{k<i} X_k$, let $\mathcal{B}_x = \{B(x, \alpha) : \alpha < \alpha(x),\ B(x, \alpha) \cap (\bigcup_{k<i} X_k) = \emptyset\}$. Conditions (a) and (b) imply that each X_i is closed and hence $\mathcal{B}_x$ is a base at x.

Suppose $\mathcal{C}$ is a subcover of the base $(\bigcup\{\mathcal{B}_x : x \text{ is not isolated}\}) \cup \{\{x\} : x \text{ is isolated}\}$. For each x let $\mathcal{C}_x = \{C \in \mathcal{C} : x \in C\}$. Call an element C of $\mathcal{C}_x$ nice (for x and $\mathcal{C}$) if $C \cap [x, \rightarrow)$ is maximal possible, that is, if $C' \in \mathcal{C}_x$, then $C' \cap [x, \rightarrow) \subset C \cap [x, \rightarrow)$. If $C \subset (\leftarrow, x]$ for every $C \in \mathcal{C}_x$, then every element of $\mathcal{C}_x$ is nice. Else let $\mathcal{C}_x^{\rightarrow} = \{C \in \mathcal{C}_x : C \cap (x, \rightarrow) \neq \emptyset\}$. Each element of $\mathcal{C}_x^{\rightarrow}$ is of the form $B(z, \alpha)$ for some $z \leq x$ and some α. Let $\delta(x) = \min\{\alpha : B(z, \alpha) \in \mathcal{C}_x^{\rightarrow} \text{ for some } z\}$. Then every $B(z, \alpha)$ in $\mathcal{C}_x^{\rightarrow}$ with $\alpha = \delta(x)$ is nice, and $B(z, \alpha) \cap [x, \rightarrow) = [x, r(x, \delta(x)))$.

At stage i, form a family $\mathcal{C}_i \subset \mathcal{C}$ that covers the points of X_i not covered by $\bigcup_{k<i} \mathcal{C}_k$. First cover with one of its nice sets the minimal element of X_i not yet covered, then cover the next minimal with one of its nice sets, etc. We will show that the cover $\mathcal{C}' = \bigcup_{i<\omega} \mathcal{C}_i$ is star finite. Each $\mathcal{C}_i$ is well-ordered by the order in which its elements were added, which, together with the order in which the $\mathcal{C}_i$ were formed, defines a well-order on $\mathcal{C}'$ denoted by $\prec$. Also define the following order: $C_1 \ll C_2$ in $\mathcal{C}'$ if there is a $c_1 \in C_1$ such that $c_1 < C_2$ (i.e., $c_1 < c$ for every $c \in C_2$), but there is no $c' \in C_1$ such that $c' > C_2$. Every two distinct elements of $\mathcal{C}'$ are $\ll$-comparable, for if $C_1 \prec C_2$, then C_2 was added to cover some $t \notin C_1$, and either $t < C_1$ and hence $C_2 \ll C_1$ (for otherwise C_1 would not be nice), or $t > C_1$ and $C_2 \gg C_1$ (since if t_1 justified the addition of C_1 into $\mathcal{C}'$, then $t_1 < C_2$, for otherwise C_1 would not be nice). There could be no $C_1 \ll C_2 \ll C_3$ in $\mathcal{C}'$ with $C_1 \prec C_2$ and $C_1 \cap C_3 \neq \emptyset$, for then $C_2 \setminus C_1 \subset C_3$; hence $C_2 \prec C_3$ and C_3 witnesses that C_2 was not nice. Fix a $C \in \mathcal{C}'$. If $\prec$ and $\ll$ agree on a subfamily $\mathcal{C}''$ of $\mathcal{C}'$, then C can meet at most one of its $\ll$-predecessors and at most two of its $\ll$-successors contained in $\mathcal{C}''$. Thus, if C meets infinitely many elements of $\mathcal{C}'$, we may assume that there is a $\prec$-increasing, $\ll$-decreasing sequence $\{C_n : n < \omega\} \subset \mathcal{C}'$ with $C \cap C_n \neq \emptyset$ for all n. Since C meets at most three elements of each $\mathcal{C}_i$, we may assume that $C \prec C_n \ll C$, $(\bigcap_{k\leq n} C_k) \cap C = C \cap C_n \neq \emptyset$ and $C_n = B(z_n, \alpha_n)$ with $z_{n+1} < z_n$ for all n.

Case 1. Infinitely many z_n come from the same X_k.

Then let $z = \inf\{z_n : n < \omega\}$. Condition (b) implies that $z \in X_k$ and z is not isolated from the right. Fix a $C' \in \mathcal{C}'$ with $z \in C'$ and an n with $z_n \in C' \prec B(z_n, \alpha_n)$. But then $B(z_{n+1}, \alpha_{n+1}) \subset C' \cup B(z_n, \alpha_n)$, a contradiction.

Case 2. Each X_i contains only finitely many z_n.

Suppose $z_0 \in X_i$. Fix an n with $z_n \in X_j \setminus \bigcup_{k<j} X_k$ and $j > i$. Then $z_0 \notin B(z_n, \alpha_n)$ and $B(z_n, \alpha_n) \cap B(z_0, \alpha_0) = \emptyset$, a contradiction. □

Remark 1.2. The proof of the above theorem can be modified to show that X is base-base ultraparacompact, i.e., X has a base such that every subfamily which is a base has a disjoint subcover. The base defined above works (as does any base consisting of convex, closed-and-open sets). Now we assume in addition that $\mathcal{C}$ is a base, not only a cover. In the construction of $\mathcal{C}_i$, if x is the minimal point of X_i not yet covered, cover x with a basic set disjoint from the elements of $\mathcal{C}'$ already defined. This is possible, since each $\mathcal{C}_i$ is discrete and hence with a closed-and-open union. See also [23], where it is shown that the Sorgenfrey line is base-base paracompact, and Remark 1.3 in [10], where it is shown that spaces satisfying somewhat stronger conditions than (a) and (b) above (the Sorgenfrey line among them) are ultraparacompact (i.e., every open cover has a disjoint open refinement).

Remark 1.3. We cannot drop hypothesis (a) or (b) in the above results, since ω_1 satisfies (b) and, with its order reversed, ω_1 satisfies (a). We cannot replace (a) above with the weaker condition (a') from Theorem 1.4 below; thus we cannot drop (c) in the latter theorem, since if we start with ω_1 and for each limit $\beta < \omega_1$ add a sequence between β and $\beta + 1$ decreasing to β, then the resulting $LOTS$ is not paracompact, satisfies (a') and every closed subset has a minimal element.

Theorem 1.4. *Let $(X, \leq)$ be a GO-space such that*

(a') *for each x, if $(\leftarrow, x]$ is open, then x is isolated,*

(b) *$X = \bigcup_{i<\omega} X_i$, where every nonempty relatively closed subset of each X_i has a minimal element, and*

(c) *X has a dense σ-discrete set (where "discrete" means "discrete in X").*

Then every F_σ subspace of X is base-cover hypocompact.

Proof. The proof is similar to the proof of the preceding theorem, though not completely analogous. We change the construction of the base and the proof that nice elements of $\mathcal{C}_x$ exist. Cases 1 and 2 are slightly different too.

Let $\bigcup_{m<\omega} D_m$ be a dense subset of X where, for each m, D_m is closed discrete and $D_m \subset D_{m+1}$. Let $I = \{x \in X : \{x\} \text{ is open}\}$ and $R = \{x \in X \setminus I : [x, \rightarrow) \text{ is open}\}$. By (a'), the set $\{x \in X \setminus I : (\leftarrow, x] \text{ is open}\}$ is empty. For each x and each m, let $l(x, m) = \sup((\leftarrow, x) \cap D_m)$ and $r(x, m) = \inf((x, \rightarrow) \cap D_m)$ (possibly gaps). Define $B(x, m)$ as follows. If $x \in R$, then $B(x, m) = [x, r(x, m))$. If $x \in X \setminus (I \cup R)$, then $B(x, m) = (l(x, m), r(x, m))$.

Suppose Y is an F_σ set. By (b) we may assume that $Y = \bigcup_{i<\omega} Y_i$, where every relatively closed subset of each Y_i has a minimal element. For each $x \in Y_i \setminus \bigcup_{k<i} Y_k$ with $x \notin I$, let $\mathcal{B}_x = \{B(x, m) : m \geq i\}$. Then the family $\mathcal{B}_Y = (\bigcup_{x \in Y \setminus I} \mathcal{B}_x) \cup \{\{x\} : x \in I\}$ is a base for Y in X (i.e., elements of $\mathcal{B}_Y$ are open in X and their intersections with Y form a base for Y). It is enough to show that if a family $\mathcal{C} \subset \mathcal{B}_Y$ covers Y, then there is a star finite subfamily of $\mathcal{C}$ that covers Y. Define $\mathcal{C}_x$ and $\mathcal{C}_x^{\rightarrow}$ as before. We need to show that if $\mathcal{C}_x^{\rightarrow} \neq \emptyset$, then it contains nice elements. Let $m(x) = \min\{m : B(z, m) \in \mathcal{C}_x^{\rightarrow} \text{ for some } z\}$. If $B(z, m) \in \mathcal{C}_x^{\rightarrow}$, then $z \leq r(x, m(x))$, for otherwise $r(x, m(x)) \leq y < z$ for some $y \in D_{m(x)}$, and $B(z, m) \subset B(z, m(x)) \subset (y, \rightarrow)$. If $z < r(x, m(x))$ and $B(z, m) \in \mathcal{C}_x^{\rightarrow}$, then $B(z, m) \cap [x, \rightarrow) \subset [x, r(x, m(x)))$. If $z < r(x, m(x))$ for every $B(z, m) \in \mathcal{C}_x^{\rightarrow}$, then any $B(z, m) \in \mathcal{C}_x^{\rightarrow}$ with $m = m(x)$ is nice, and $B(z, m) \cap [x, \rightarrow) = [x, r(x, m(x)))$. If $r(x, m(x)) \in X$, then there might be some elements of $\mathcal{C}_x^{\rightarrow}$ of the form $B(r(x, m(x)), j)$: then any $B(r(x, m(x)), j)$ in $\mathcal{C}_x^{\rightarrow}$ with minimal j is nice.

The family $\mathcal{C}'$ is constructed as before (using Y_i in place of X_i). Now $\bigcup \mathcal{C}'$ contains Y. We will show $\mathcal{C}'$ is star finite. It is enough to show that there is no $C \in \mathcal{C}'$ and a $\prec$-increasing, $\ll$-decreasing sequence $\{B(z_n, m_n) : n < \omega\} \subset \mathcal{C}'$ each element of which meets C and $\ll$-precedes C. We may assume **Case 1**: all m_n

coincide, say $m_n = M$, or **Case 2**: the m_n form an increasing sequence. Case 1 and Case 2 each imply that $z_{n+1} \notin B(z_n, m_n)$, and hence $B(z_n, m_n) \subset (z_{n+1}, \rightarrow)$ for all n. In Case 1, by the definition of $\mathcal{B}_{z_n}$ infinitely many z_n come from the same Y_k for some $k \leq M$, which leads to a contradiction as before. In Case 2, fix a large enough $N \geq 3$ such that there is a $d \in (z_3, z_1) \cap D_N$. Then $B(z_N, m_N) \subset B(z_N, N) \subset (\leftarrow, d) \subset (\leftarrow, z_1)$ and $B(z_0, m_0) \subset (z_1, \rightarrow)$, a contradiction. □

Remark 1.5. We do not know if condition (a') in the above theorem is essential. As we will see, any subset of the Sorgenfrey line that is not F_σ shows that condition (b) cannot be dropped in the two theorems above. Although the two proofs are similar, we do not see how to get one theorem to imply both theorems above. Condition (c), which has been extensively used (see [2], [3], [9], [11], [12], [17]), could not be a hypothesis of any theorem that would imply the first one, since $\omega_1 + 1$ with the reverse order satisfies the conditions of the first theorem, (a) and (b), but does not satisfy (c). We are left with conditions (a), (a') and (b), and we gave an example (in Remark 1.3) that (a') and (b) would not be enough. The Sorgenfrey rationals show that the above results would become weaker if we replace assumption (b) with the stronger one that every closed set has a minimal element. Although the Michael line is base-cover hypocompact [22], no *GO*-space homeomorphic to it satisfies (b).

Problem 1.6. Characterize the base-cover para(hypo,meta)-compact GO-spaces.

Definition 1.7 (E. van Douwen and W. Pfeffer [10])**.** If $\preceq$ is a *reflexive* binary relation on a set X and $F \subset X$, we call an $m \in F$ an $\preceq$-*minimal* element of F if $x = m$ for each $x \in F$ with $x \preceq m$. The space X is called a *GLS* (*Generalized Left Separated*) space if in addition X is a topological space and

(1) every nonempty closed subset of X has a $\preceq$-minimal element, and

(2) the set $\{y \in X : x \preceq y\}$ is open for each $x \in X$.

Then $\preceq$ is called a *GLS-relation* on the space X. Note that the topology of X is given without reference to $\preceq$, which is only required to be reflexive. For example, the countable sequential fan is a *GLS*-space witnessed by any well-founded relation in which the non-isolated point precedes all other points.

Theorem 1.8. *For a subspace X of the Sorgenfrey line S, the following are equivalent:*

(a) X is F_σ,

(b) X is base-cover hypocompact,

(c) X is base-cover paracompact,

(d) X is base-cover metacompact,

(e) X is a continuous image of S,

(f) X is a GLS space.

Proof. $(a) \rightarrow (b)$ follows from each of Theorem 1.1 and 1.4. $(b) \rightarrow (c) \rightarrow (d)$ is trivial. $(a) \leftrightarrow (e)$ is a result of D. Burke and J.T. Moore [5]. E.K. van Douwen and W. Pfeffer proved in [10] that every finite power of S is a *GLS*-space, and that an F_σ subspace of a *GLS*-space is a *GLS*-space itself; thus $(a) \rightarrow (f)$. Although $(f) \rightarrow (a)$ was not proved in [10], it was proved that the Sorgenfrey irrationals are not a *GLS*-space, and the proof of $(f) \rightarrow (a)$ is an easy combination of that proof and the proof of $(d) \rightarrow (a)$ that follows.

Fix a base $\mathcal{B}$ for X. For each $x \in X$ fix a $B_x \in \mathcal{B}$ with $x \in B_x \subset [x, \rightarrow)$, and fix $n_x < \omega$ with $[x, x + \frac{1}{n_x}) \cap X \subset B_x$. Let $X_n = \{x \in X : n_x = n\}$. Clearly

$X = \bigcup_{n<\omega} X_n \subset \bigcup_{n<\omega} \overline{X_n}$ (where the closure is taken in S), and if X is not F_σ, then there is a k such that $\overline{X_k} \not\subset X$. Fix a $z \in \overline{X_k} \setminus X$ and a sequence x_i decreasing to z with $x_i \in X_k$ for each i and $x_0 < z + \frac{1}{k}$. The family $\mathcal{C}' = \{B_{x_i} : i < \omega\}$ covers $\big(z, z + \frac{1}{k}\big) \cap X$. There is a subfamily $\mathcal{C}''$ of $\mathcal{B}$ with $\bigcup \mathcal{C}'' = X \setminus \big[z, z + \frac{1}{k}\big)$. Then $\mathcal{C}' \cup \mathcal{C}''$ is a subcover of $\mathcal{B}$ with no point finite subcover, since to cover $\{x_i : i < \omega\}$ we need infinitely many B_{x_i} from $\mathcal{C}'$, but each B_{x_i} contains x_0. □

Remark 1.9. No base $\mathcal{B} = \{[x_\alpha, r_\alpha) : \alpha < 2^\omega\}$ with $|\{r_\alpha : \alpha < 2^\omega\}| \geq \omega_1$ can witness base-cover metacompactness of the Sorgenfrey line S, though ([23] and Remark 1.2 here) any base of half-open intervals does witness its base-base paracompactness. There is a $k < \omega$ such that the set $\{r_\alpha : r_\alpha - x_\alpha > \frac{1}{k}\}$ is uncountable, and hence has a two-sided limit point $x \in S$. Fix a sequence r_{α_i}, $i < \omega$, increasing to x, and such that $r_{\alpha_i} - x_{\alpha_i} > \frac{1}{k}$ for each i. The family $\mathcal{C}' = \{[x_{\alpha_i}, r_{\alpha_i}) : i < \omega\}$ covers the interval $[x - \frac{1}{k}, x)$. Let $\mathcal{C}''$ be a subfamily of $\mathcal{B}$ with $\bigcup \mathcal{C}'' = S \setminus [x - \frac{1}{k}, x)$. The family $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ is a subcover of $\mathcal{B}$. Any subcover of $\mathcal{C}$ must contain infinitely many members of $\mathcal{C}'$ and therefore cannot be point finite at $x - \frac{1}{k}$.

The above remark shows that we do need a special base to show that the Sorgenfrey line S is base-cover hypocompact. A special base for S was already used by de Caux [6] to show that every finite power of S is hereditarily a D-space, answering a question of van Douwen and Pfeffer from [10]. It was shown in [10] that each GLS-space (thus each finite power of S) is a D-space, and it was observed that S itself is hereditarily a D-space. A space X is a *D-space* if for every open neighborhood assignment $\{U_x : x \in X\}$ there is a closed discrete subspace D of X such that $\{U_x : x \in D\}$ covers X (see also [4], [9], [12]).

Remark 1.10. The Cantor middle-third set minus all points of the form $\frac{m}{3^i}$, where m is odd and $i < \omega$, is not F_σ in the real line but is closed in the Sorgenfrey line.

Remark 1.11. There is a perfectly normal Lindelöf $LOTS$ that is not base-cover metacompact: Take a non-F_σ subspace X of the Sorgenfrey line and split each $x \in X$ into x^- and x^+ with $x^- < x^+$.

2. Subspaces and unions of base-cover paracompact spaces

The *countable sequential fan* S_ω is the union of countably many sequences converging to the same point, say $S_\omega = \{0\} \cup \{x^i_j : i, j < \omega\}$, where all x^i_j are isolated, and neighborhoods of 0 have the form $\{0\} \cup (\bigcup_{i<\omega} T^i)$, where each T^i contains a tail of the sequence $S^i = \{x^i_j : j < \omega\}$. We need several lemmas before we prove that S_ω is not base-cover metacompact. Recall that ${}^\omega\omega$ is the set of all functions from ω to ω partially pre-ordered by: $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. A set $\mathcal{D} \subset {}^\omega\omega$ is *dominating* if it is cofinal in $\langle {}^\omega\omega, \leq^* \rangle$, i.e., for each $f \in {}^\omega\omega$ there is a $g \in \mathcal{D}$ with $f \leq^* g$.

Lemma 2.1. *If $\mathcal{D} = \bigcup_{n<\omega} \mathcal{D}_n \subset {}^\omega\omega$ is dominating, then some $\mathcal{D}_n$ is dominating.*

Proof. This is a well-known diagonalization argument. If for each n there is an $f_n \in {}^\omega\omega$ which witnesses that $\mathcal{D}_n$ is not dominating, then define $f \in {}^\omega\omega$ by $f(k) = \max\{f_n(k) : n \leq k\}$ for each k. Then $f_n(k) \leq f(k)$ for all $k \geq n$; hence $f_n \leq^* f$ for each n, and f witnesses that $\mathcal{D}$ is not dominating. □

Lemma 2.2. *If $\mathcal{D}$ is dominating, then there is a $g \in {}^\omega\omega$ such that every neighborhood of g (in the product topology of ${}^\omega\omega$) meets $\mathcal{D}$ in a dominating family.*

Proof. Find n_0 such that all $f \in \mathcal{D}$ with $f(0) = n_0$ form a dominating family (use the preceding lemma); then find n_1 such that all $f \in \mathcal{D}$ with $f(0) = n_0$ and $f(1) = n_1$ form a dominating family, and so on. Let $g(i) = n_i$ for each $i < \omega$. □

Lemma 2.3. *If $\mathcal{D}$ is dominating, then there is an infinite $\mathcal{F} \subset \mathcal{D}$ such that if $\mathcal{F}'$ is a finite subfamily of $\mathcal{F}$, then $\min \mathcal{F}'(i) > \min \mathcal{F}(i)$ for all but finitely many $i < \omega$ (where $\min \mathcal{F}(i) = \min \{f(i) : f \in \mathcal{F}\}$).*

Proof. Let g be as in the preceding lemma and for each $n < \omega$ pick an $f_n \in D$ with $f_n|[0, n] = g|[0, n]$ and $f_n(i) > g(i)$ for all but finitely many i. If $\mathcal{F} = \{f_n : n < \omega\}$, then $\min \mathcal{F}(i) \leq g(i)$ for all i. If $\mathcal{F}'$ is a finite subfamily of $\mathcal{F}$, then $\min \mathcal{F}'(i) > g(i)$ for all but finitely many i. □

Theorem 2.4. (a) *The countable sequential fan S_ω is not base-cover metacompact.*
(b) *Base-cover para(hypo,meta)-compactness is not F_σ hereditary in general.*

Proof. (a) Let $\mathcal{B}$ be a base for S_ω. The family $\mathcal{B}_0 = \{B \in \mathcal{B} : 0 \in B\}$ is a base at 0. For each $B \in \mathcal{B}_0$ and $i < \omega$, let $f_B(i) = \min \{j : x^i_j \in B\}$. Since $\mathcal{B}_0$ is a base at 0, the family $\mathcal{D} = \{f_B : B \in \mathcal{B}_0\}$ is dominating in ${}^\omega\omega$ (it is cofinal in $\langle {}^\omega\omega, \leq \rangle$ too). There is a subfamily $\mathcal{C}_0$ of $\mathcal{B}_0$ such that the set $\mathcal{F} = \{f_B : B \in \mathcal{C}_0\}$ satisfies the conclusion of the preceding lemma. Let $f(i) = \min \mathcal{F}(i)$ for each i. The family $\mathcal{C} = \mathcal{C}_0 \cup \{\{x^i_j\} : f(i) \neq j\}$ is a subcover of $\mathcal{B}$, since if $f(i) = j$, then $f_B(i) = j$ for some $B \in \mathcal{C}_0$ and $x^i_j \in B$. If $\mathcal{C}'$ is a point finite subfamily of $\mathcal{C}$, then $\mathcal{C}' \cap \mathcal{C}_0$ is finite and there is an i such that $\min \{f_B(i) : B \in \mathcal{C}' \cap \mathcal{C}_0\} > f(i)$; hence $x^i_{f(i)}$ is not covered by $\mathcal{C}' \cap \mathcal{C}_0$. It follows that $x^i_{f(i)}$ is not covered by $\mathcal{C}'$, since $\mathcal{C}' \setminus \mathcal{C}_0 \subset \{\{x^i_j\} : f(i) \neq j\}$.

(b) The Stone-Čech compactification of S_ω is base-cover hypocompact. □

Theorem 2.5. (a) *A space is base-cover para(meta)-compact if it has an open cover the closures of the elements of which are base-cover para(meta)-compact and form a locally finite (point finite) cover.*

(b) *Every (normal) para(meta)-compact, locally base-cover para(meta)-compact space is base-cover para(meta)-compact.*

(c) *Every open F_σ subspace of a (normal) base-cover para(meta)-compact space is base-cover para(meta)-compact.*

(d) *Every open subspace of a perfectly normal, base-cover para(meta)-compact space is base-cover para(meta)-compact.*

Proof. (a) Fix an open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of a space X such that $\{\overline{U_\alpha} : \alpha < \kappa\}$ is locally finite and each $\overline{U_\alpha}$ is base-cover paracompact. Let $\mathcal{B}^+_\alpha$ be an open (relative to $\overline{U_\alpha}$) base of $\overline{U_\alpha}$, every subcover of which has a locally finite subcover. Let $\mathcal{B}_\alpha = \{B \in \mathcal{B}^+_\alpha : B \subset U_\alpha\}$. The family $\mathcal{B} = \bigcup_{\alpha<\kappa} \mathcal{B}_\alpha$ is an open base for X.

Suppose $\mathcal{C}$ is a subcover of $\mathcal{B}$. For each $\alpha < \kappa$, let $\mathcal{C}_\alpha = \mathcal{C} \cap \mathcal{B}_\alpha$. We describe how to define families $\mathcal{C}^-_\alpha \subset \mathcal{C}_\alpha$ and closed sets H_α such that for all $\alpha < \kappa$:

(i) $H_\alpha = (\bigcup \mathcal{C}_\alpha) \setminus [(\bigcup_{\beta<\alpha} (\bigcup \mathcal{C}^-_\beta)) \cup (\bigcup_{\alpha<\beta<\kappa} (\bigcup \mathcal{C}_\beta))] \subset \bigcup \mathcal{C}^-_\alpha$, and

(ii) $\mathcal{C}^-_\alpha$ is locally finite at each point of $\overline{U_\alpha}$.

Suppose $\alpha < \kappa$ and $\mathcal{C}^-_\beta$ were defined for all $\beta < \alpha$. A standard argument (using for limit α that $\mathcal{U}$ is point-finite) shows that $X \setminus [(\bigcup_{\beta<\alpha} (\bigcup \mathcal{C}^-_\beta)) \cup (\bigcup_{\alpha<\beta<\kappa} (\bigcup \mathcal{C}_\beta))] \subset \bigcup \mathcal{C}_\alpha$; hence H_α is closed. The set $V_\alpha = \overline{U_\alpha} \setminus H_\alpha$ is open in $\overline{U_\alpha}$. Fix $\mathcal{C}(V_\alpha) \subset \mathcal{B}^+_\alpha$ with $\bigcup \mathcal{C}(V_\alpha) = V_\alpha$, and let $\mathcal{C}^+_\alpha = \mathcal{C}_\alpha \cup \mathcal{C}(V_\alpha)$. Since $\mathcal{C}^+_\alpha$ is a subcover of $\mathcal{B}^+_\alpha$, there is a locally finite subcover $\mathcal{C}'_\alpha$ of $\mathcal{C}^+_\alpha$. The above conditions are satisfied with $\mathcal{C}^-_\alpha = \mathcal{C}'_\alpha \setminus \mathcal{C}(V_\alpha)$. Now, the family $\mathcal{C}' = \bigcup_{\alpha<\kappa} \mathcal{C}^-_\alpha$ is a locally finite subcover of $\mathcal{C}$.

The other statement in part (a) has a similar proof.

The proof of $(a) \to (b)$ is easy (use also that every point finite open cover of a normal space has a shrinking). To prove $(a) \to (c)$, suppose Y is an open F_σ subspace of a base-cover paracompact space X. Then $Y = \bigcup_{i<\omega} F_i$, where F_i is closed and $F_i \subset \operatorname{int} F_{i+1}$ for each i. Let $U_i = (\operatorname{int} F_i) \setminus F_{i-2}$ (where $F_{-2} = F_{-1} = \emptyset$). It is easy to see that base-cover para(hypo,meta)-compactness is hereditary with respect to closed sets, and that $\{U_i : i < \omega\}$ is a cover of Y that satisfies the premise in (a). □

Remark 2.6. Unlike most covering properties, for base-cover para(hypo,meta)-compactness "open hereditary" is not equivalent to "hereditary" even in perfectly normal spaces: the Sorgenfrey line is an example, as we see from Theorem 1.8.

Question 2.7. Is every F_σ subspace of a perfectly normal, base-cover paracompact space base-cover paracompact? What if, in addition, the space is Lindelöf?

3. Products of base-cover paracompact and compact spaces

Example 3.1. The product $S\times(\omega+1)$ of the Sorgenfrey line S and the converging sequence $\omega+1$ is not base-cover paracompact (even though S is).

Proof. Suppose $\mathcal{B}$ is a base for $S \times (\omega+1)$. For each $n < \omega$ fix x_n and r_n in S and $B_n \in \mathcal{B}$ such that $[x_n, r_n) \times \{n\} \subset B_n \subset S \times \{n\}$ and $x_n < x_{n+1} < r_{n+1} < r_n$. If $F = \{\langle x_n, n\rangle : n < \omega\})$, then F is closed and covered by $\mathcal{C}' = \{B_n : n < \omega\}$. There is a subfamily $\mathcal{C}''$ of $\mathcal{B}$ with $\bigcup \mathcal{C}'' = X \setminus F$. Then $\mathcal{C}' \cup \mathcal{C}''$ is a subcover of $\mathcal{B}$. For each n the only element of $\mathcal{C}' \cup \mathcal{C}''$ that contains $\langle x_n, n\rangle$ is B_n, and hence every subcover of $\mathcal{C}' \cup \mathcal{C}''$ must contain $\mathcal{C}'$ — but $\mathcal{C}'$ is not locally finite at $\langle \sup_{n<\omega} x_n, \omega\rangle$. □

In a similar way one can show that $M \times (\omega+1)$ and $L(\omega_1) \times (\omega+1)$ are not base-cover paracompact, where M is the Michael line and $L(\omega_1)$ is the one-point Lindelöfication of a discrete space of cardinality ω_1, although M and $L(\omega_1)$ are base-cover hypocompact, and $M\times(\omega+1)$ is base-cover metacompact [22]. It follows that perfect preimages of base-cover para(hypo)compact spaces need not be base-cover paracompact (recall that a perfect map is a continuous, closed map such that the preimage of every point is compact). Base-cover hypocompactness is not preserved by perfect maps in the forward direction (see Remark 1.3 in [22]).

Question 3.2. Is base-cover para(meta)compactness preserved by perfect maps?

Question 3.3. If $X \times (\omega+1)$ is base-cover para(hypo)compact, then X has what property $\mathcal{P}$ (such that the Sorgenfrey line and the Michael line do not have $\mathcal{P}$)? Is X a paracompact p-space (i.e., the perfect preimage of a metric space)?

Question 3.4. If X is base-cover metacompact and Y is compact or metrizable, is $X \times Y$ base-cover metacompact? What can we say about X if $X \times (\omega+1)$ is base-cover metacompact? Is $S \times (\omega+1)$ base-cover metacompact, where S is the Sorgenfrey line?

Recall that $A(\kappa)$ denotes the *one-point compactification of a discrete space of cardinality* κ. We think of $A(\kappa)$ as $\kappa+1 = \{\alpha : \alpha \le \kappa\}$ as a set, with all ordinals $\alpha < \kappa$ isolated and neighborhoods of κ of the form $\kappa+1$ minus a finite subset of κ. As usual, κ^+ denotes the least cardinal greater than κ.

Theorem 3.5. *Suppose X is a regular paracompact space and $\chi(X) = \kappa \geq \omega$. The following conditions are equivalent.*

(a) X is locally compact.

(a') X is locally countably compact.

(b) $X \times A(\kappa^+)$ is base-cover paracompact.

(c) $X \times K$ is base-cover paracompact for every compact space K.

If X is first countable, then the following condition can be added to the list:

(d) X does not contain a closed copy of T, where $T = (\bigcup_{n<\omega} T_n) \cup \{\infty\}$, with each T_n a countably infinite set consisting of isolated points, and basic neighborhoods of ∞ of the form $\{\infty\} \cup (\bigcup_{n \geq i} T_n)$, $i < \omega$.

Proof. The equivalence $(a) \leftrightarrow (a')$ is well known for paracompact spaces. If X is paracompact, locally compact and K is compact, then $X \times K$ is paracompact, locally compact, hence base-cover paracompact by Theorem 2.5. Thus, $(a) \to (c)$.

$(c) \to (b)$ is trivial. We show $(b) \to (a')$. If X is not locally countably compact at a point p, then fix a local base $\{U_\beta : \beta < \kappa\}$ at p. Let $\mathcal{B}$ be any base for the product $Z = X \times A(\kappa^+)$, and for each $\alpha < \kappa^+$ pick a $B_\alpha \in \mathcal{B}$ and $\beta_\alpha < \kappa$ such that $\langle p, \alpha \rangle \in \overline{U_{\beta_\alpha}} \times \{\alpha\} \subset B_\alpha \subset X \times \{\alpha\}$. There is a set $A \subset \kappa^+$ of cardinality κ^+ such that the β_α are the same, say $\beta_\alpha = \gamma$, for all $\alpha \in A$, and there is a countably infinite set $\{\alpha_n : n < \omega\} \subset A$. There is a countably infinite closed discrete subspace $\{x_n : n < \omega\}$ of $\overline{U_\gamma}$. The set $F = \{\langle x_n, \alpha_n \rangle : n < \omega\}$ is closed in Z. If $\mathcal{C}' = \{B_{\alpha_n} : n < \omega\}$ and $\mathcal{C}''$ is a subfamily of $\mathcal{B}$ with $\bigcup \mathcal{C}'' = Z \setminus F$, then $\bigcup \mathcal{C}' \supset \bigcup \{(\overline{U_\gamma} \times \{\alpha_n\}) : n < \omega\} \supset F$, and hence $\mathcal{C}' \cup \mathcal{C}''$ is a subcover of $\mathcal{B}$. Any subcover of $\mathcal{C}' \cup \mathcal{C}''$ must contain $\mathcal{C}'$, which is not locally finite at $\langle x, \kappa^+ \rangle$ for any $x \in \overline{U_\gamma}$.

It was observed in [15] that $(a') \leftrightarrow (d)$ for first countable spaces. □

Remark 3.6. The Sorgenfrey line and the Michael line are base-cover paracompact spaces that are not perfect preimages of a metric space. If $\mathbb{Q}$ denotes the rationals with the usual topology, then $\mathbb{Q} \times A(\omega_1)$ is a paracompact p-space that is not base-cover paracompact (since $\mathbb{Q}$ is not locally compact).

4. Some related classes of spaces

Definition 4.1. A space X is *totally paracompact* [13] (*totally metacompact*) if *every* open base for X contains a locally finite (point finite) subcover. A space X is *base*-base paracompact [23] (*base*-base hypocompact, *base*-base metacompact) if X has an open *base* $\mathcal{B}$ such that every base $\mathcal{B}' \subset \mathcal{B}$ contains a locally finite (star finite, point finite) subcover. A space X is *base paracompact* [23], [24] if it has an open *base* $\mathcal{B}$ of cardinality equal to the weight of the space (i.e., $|\mathcal{B}| = w(X)$) such that every open cover $\mathcal{C}$ of X has a locally finite open refinement $\mathcal{C}' \subset \mathcal{B}$.

For totally paracompact metric spaces, large and small inductive dimension coincide [13]. The Sorgenfrey line S and the Michael line M are not totally paracompact, or totally metacompact (see [25] (for the Michael line), [18], [19], [2]). The irrationals $\mathbb{P}$ with the usual topology are not totally para(meta)compact [7], [1] (Konstantinov); see also [16]. Thus total paracompactness is very restrictive, which led John Porter to define and study base-base paracompactness and base paracompactness [23], [24]. His proof that metrizable spaces are base-base paracompact worked for what we called base-cover paracompact spaces [21], [22]. Total paracompactness and base-cover paracompactness each imply base-base paracompactness, which implies base-paracompactness [23], which implies paracompactness. S, M and $\mathbb{P}$ are base-cover hypocompact (Theorem 1.8 here, and [22]); the countable sequential fan is totally paracompact. Base-base paracompact spaces are D-spaces, and Lindelöf spaces are base paracompact, but it is unknown if paracompactness implies base paracompactness, or if the latter implies base-base paracompactness [23], [24]. This is related to the following question of Eric van Douwen, versions of which have been discussed in [4], [6], [9], [10], [12], [26]: Is every Lindelöf space a D-space? The next question is implicit in [23] and should be compared with Theorem 1.8 here.

Question 4.2. Is every subspace of the Sorgenfrey line base-base paracompact?

Consistent examples of base-base paracompact subspaces of the Sorgenfrey line that are not F_σ are any Lusin subspace and, under MA, any uncountable subspace of cardinality less than the continuum; these spaces are Hurewicz (see [14] (for the reals), [18], [2], [20]), and thus totally paracompact [8]. Other questions implicit in [23] include: Is base-base paracompactness hereditary with respect to closed sets? If X is base-base paracompact and Y is compact, is $X \times Y$ base-base paracompact? Is X base-base paracompact if it is paracompact and locally base-base paracompact?

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