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NORM-CLOSURE OF THE BARRIER CONE IN NORMED LINEAR SPACES

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Abstract. The aim of this note is to characterize the norm-closure of the barrier cone of a closed convex set in an arbitrary normed linear space by means of a new geometric object, the temperate cone.

1. Introduction and notation

Throughout the paper, we assume that X is a real normed linear space with continuous dual X^* and we denote by $\|\cdot\|$ and $\|\cdot\|_*$ the norms on X and X^*, respectively. Let C be a convex subset of X, and let

$$\mathcal{B}(C) = \left\{ f \in X^* : \sup_{x \in C} \langle f, x \rangle < \infty \right\}$$

denote the *barrier cone* to C. As usual (see Rockafellar [4] as a reference book for the finite-dimensional case), we use the notation C^∞ for the *recession cone* of the closed convex set C, that is,

$$C^\infty = \{ v \in X : \quad \forall \lambda > 0,\ x_0 \in C, x_0 + \lambda v \in C \}.$$

Denoting by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X, we recall that the negative polar cone C° of a closed convex set C is

$$C^\circ = \{ f \in X^* : \langle f, w \rangle \leq 0 \quad \forall w \in C \}.$$

The main objective of this note is to give a direct characterization of the norm-closure of the barrier cone for a closed convex set in a general normed linear space. It is well known (see for instance [6], Ex. 2.45) that the *weak***-closure* of the barrier cone of C is the polar of the recession cone of C,

$$\text{weak}^*\text{-cl}(\mathcal{B}(C)) = (C^\infty)^\circ.$$

When the underlying linear space X is reflexive, since the barrier cone of C is convex, its *weak***-closure* coincides with its *norm-closure,* and thus the norm-closure $\overline{\mathcal{B}(C)}$ of $\mathcal{B}(C)$ is characterized in X^* by the formula:

$$\overline{\mathcal{B}(C)} = (C^\infty)^\circ. \tag{1.1}$$

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When X is not a reflexive Banach space, the *weak***-closure* of $\mathcal{B}(C)$ may significantly be larger than its *norm-closure,* and thus relation (1.1) may fail. This situation occurs for instance when $X = L^1(0,1)$ and $C = \{f \in L^1(0,1) : 0 \leq f(x) \leq \frac{1}{x}\}$. Then, obviously, C is a closed convex set and $C^\infty = \{0\}$. The sets $K = \{f \in X^* = L^\infty(0,1) : 1 \leq f\}$ and $\mathcal{B}(C)$ are disjoint; since the interior of K is non-void, it follows that $\overline{\mathcal{B}(C)} \neq X^* = (C^\infty)^\circ$.

In order to characterize the norm-closure of the barrier cone in a general normed linear space, we introduce and study a new object that we call the *temperate cone* of a closed convex set. The main result, Theorem 2.1, states that the temperate cone of a closed convex set C is the norm-closure of the barrier cone of C for every closed and convex subset C of a normed linear space.

This result leads to the description (Proposition 2.5) of the class of all closed and convex sets satisfying relation (1.1) in a general normed linear space.

In the sequel, the symbol $\mathbb{B}_X$ is used to denote the closed unit ball in X, and a closed convex set is called *linearly bounded* if $C^\infty = \{0\}$.

2. The norm-closure of the barrier cone

Let C be a closed convex set in a normed linear space X. Along with the barrier cone $\mathcal{B}(C)$ of the set C, we define the *temperate cone* of C:

$$\mathcal{T}(C) = \left\{ f \in X^* : \lim_{r\to\infty} \left(\inf_{x\in C, \langle f,x\rangle \geq r} \frac{\|x\|}{r} \right) = \infty \right\}. \tag{2.1}$$

Since $\inf_{x\in\emptyset} \|x\| = \infty$, it follows that $\mathcal{B}(C) \subseteq \mathcal{T}(C)$.

The following theorem states the main result of this note.

Theorem 2.1. *For every closed convex set C of a normed linear space X the temperate cone is the norm-closure of the barrier cone.*

Let C be a given closed convex set. We recall that the *indicator function* of C is the functional ι_C defined by

$$\iota_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \in X \setminus C. \end{cases}$$

Given an extended real-valued function $\Psi : X \to \mathbb{R} \cup \{+\infty\}$, recall that the *Fenchel conjugate* of Ψ is the function $\Psi^* : X^* \to \mathbb{R} \cup \{+\infty\}$ given by

$$\Psi^*(f) := \sup_{x\in X} \{\langle f, x\rangle - \Psi(x)\}.$$

The barrier cone of a closed convex set C coincides with the effective domain of the Fenchel conjugate of its indicator function, i.e., the set of all the points where this Fenchel conjugate is finite. Since, obviously, the epigraph of ι_C denoted by $\operatorname{epi} \iota_C$ coincides with $C \times [0, +\infty)$, we have $\mathcal{B}(\operatorname{epi}\ \iota_C) = \mathcal{B}(C) \times (-\infty, 0]$ and therefore,

$$\mathcal{B}(\operatorname{epi}\ \iota_C) \cap \big(X^* \times \{-1\}\big) = \mathcal{B}(C) \times \{-1\}. \tag{2.2}$$

Obviously, the domain of Ψ^* denoted by $\operatorname{dom} \Psi^*$ is connected to the barrier cone of the epigraph of Ψ through the following equivalence:

$$g \in \operatorname{dom} \Psi^* \Leftrightarrow (g, -1) \in \mathcal{B}(\operatorname{epi}\ \Psi).$$

This yields that dom $\Psi^* \times \{-1\}$ is the intersection of the barrier cone of epi Ψ with the hyperplane $X^* \times \{-1\}$ of $X^* \times \mathbb{R}$:

$$\text{dom } \Psi^* \times \{-1\} = \mathcal{B}(\text{epi } \Psi) \cap \big(X^* \times \{-1\}\big). \tag{2.3}$$

Finally, from relations (2.2) and (2.3) we deduce that

$$\mathcal{B}(C) = \text{dom } \iota_C^*.$$

Let us recall the following well-known characterization of the norm-closure of the domain of the Fenchel conjugate of an extended real-valued, proper, convex and lower semicontinuous functional.

Theorem 2.2. *Let $\Psi : X \mapsto \mathbb{R} \cup \{+\infty\}$ be an extended real-valued proper convex and lower semicontinuous functional. Then $0 \in \overline{\text{dom}\,(\Psi^*)}$ if and only if the map $x \mapsto (\Psi(x) + \varepsilon\|x\|)$ is bounded from below for every $\varepsilon > 0$.*

Remark 2.3. For the reflexive case, Theorem 2.2 goes back (in a slightly different form) to [5]. The non-reflexive setting has been considered for the first time in [3] (the reader is invited to consult Example 25 in [6], where a clear, detailed and self-contained analysis is given).

From Theorem 2.2 we deduce that a given element f of X^* belongs to the norm-closure of $\mathcal{B}(C)$ if and only if the map $x \mapsto \iota_C(x) - \langle f, x\rangle + \varepsilon\|x\|$ is bounded from below for every $\varepsilon > 0$.

Theorem 2.1 will thus be verified by proving that the above property characterizes the elements of the temperate cone of the closed convex set C.

For every $f \in X^*$, let us use the notation C_f for $\{x \in X : \langle f, x\rangle \geq \|x\|\}$. The proof of Theorem 2.1 relies on the following characterization of the temperate cone.

Lemma 2.4. *Let C be a closed convex set in a normed linear space X. A continuous linear functional $f \in X^*$ belongs to the temperate cone $\mathcal{T}(C)$ if and only if the set $C \cap C_{\lambda f}$ is bounded for every $\lambda > 0$.*

Proof of Lemma 2.4. Let us first prove that for every $f \in \mathcal{T}(C)$ and $\lambda > 0$, the set $C \cap C_{\lambda f}$ is bounded. From relation (2.1) it follows that there is $\overline{r} > 0$ such that

$$\inf_{x \in C, \langle f, x\rangle \geq r} \frac{\|x\|}{r} \geq (1 + \lambda) \quad \forall r \geq \overline{r}. \tag{2.4}$$

Let $x \in C$ be such that $\langle f, x\rangle \geq \overline{r}$. Applying relation (2.4) to $r = \langle f, x\rangle$ yields

$$\|x\| \geq (1 + \lambda)\,\langle f, x\rangle > \langle \lambda f, x\rangle \quad \forall x \in C, \langle f, x\rangle \geq \overline{r}. \tag{2.5}$$

Let $x \in C \cap C_{\lambda f}$, which means that $x \in C$ and $\|x\| \leq \langle \lambda f, x\rangle$. From relation (2.5) it follows that $\langle f, x\rangle \leq \overline{r}$. Accordingly, $\|x\| \leq \lambda\overline{r}$, and so $C \cap C_{\lambda f} \subseteq \lambda\overline{r}\mathbb{B}_X$.

In order to complete Lemma 2.4 it suffices to prove that $f \in \mathcal{T}(C)$ provided that the set $C \cap C_{\lambda f}$ is bounded for every $\lambda > 0$. Pick $M > 0$. Since the set $C \cap C_{Mf}$ is bounded, there is $\overline{r} > 0$ such that $(C \cap C_{Mf}) \subseteq \overline{r}\mathbb{B}_X$. Accordingly, for every $x \in C \cap C_{Mf}$ it follows that $\|x\| \leq \overline{r}$; since $\langle f, x\rangle \leq \|f\|_*\|x\|$ for every x from X, we deduce that $\langle f, x\rangle \leq \|f\|_*\overline{r}$ for every $x \in C \cap C_{Mf}$. Thus, for every $x \in C$ such that $\langle f, x\rangle > \|f\|_*\overline{r}$ we have $x \notin C_{Mf}$; that is, $\|x\| > M\,\langle f, x\rangle$. Consequently, for every $r > \|f\|_*\overline{r}$ and $x \in C$ such that $\langle f, x\rangle \geq r$ we have

$$\frac{\|x\|}{r} > \frac{M\,\langle f, x\rangle}{r} \geq M.$$

Hence,

$$\inf_{x\in C,\langle f,x\rangle\geq r}\frac{\|x\|}{r}\geq M\quad\forall r>\|f\|_*\overline{r},$$

and the conclusion of Lemma 2.4 follows. □

Proof of Theorem 2.1. Remark that the map $x\mapsto\iota_C(x)-\langle f,x\rangle+\varepsilon\|x\|$ is bounded from below if and only if

$$\exists\mu_\epsilon\in\mathbb{R}\text{ such that }\langle f,x\rangle\leq\varepsilon\|x\|+\mu_\varepsilon,\ \ \forall x\in C. \tag{2.6}$$

In view of Lemma 2.4, we have only to prove that relation (2.6) holds for every $\varepsilon>0$ if and only if $C\cap C_{\lambda f}$ is bounded for every $\lambda>0$.

Assume that for some $f\in X^*$ relation (2.6) holds for every $\varepsilon>0$, and fix $\lambda>0$. Taking $\varepsilon=\frac{1}{2\lambda}$ in relation (2.6) yields

$$\exists\overline{\mu}\in\mathbb{R}\text{ such that }\langle f,x\rangle\leq\frac{1}{2\lambda}\|x\|+\overline{\mu},\ \ \forall x\in C.$$

According to the definition of $C_{\lambda f}$, for every $x\in C\cap C_{\lambda f}$ we have

$$\|x\|\leq\langle\lambda f,x\rangle.$$

From the two previous relations we deduce

$$\|x\|\leq\frac{1}{2}\|x\|+\lambda\overline{\mu}\ \ \forall x\in C\cap C_{\lambda f},$$

and consequently we have $C\cap C_{\lambda f}\subseteq 2\lambda\overline{\mu}\mathbb{B}_X$.

Thus, if the map $x\mapsto\iota_C(x)-\langle f,x\rangle+\varepsilon\|x\|$ is bounded from below for every $\varepsilon>0$, then $C\cap C_{\lambda f}$ is bounded for every $\lambda>0$.

Conversely, assume that $C\cap C_{\lambda f}$ is bounded for every $\lambda>0$, and fix $\varepsilon>0$. The set $C\cap C_{f/\varepsilon}$ being bounded, there exists $\gamma>0$ such that $\|x\|\leq\gamma$ for every $x\in C\cap C_{f/\varepsilon}$.

Let $x\in C\setminus C_{f/\varepsilon}$; accordingly,

$$\left\langle\frac{f}{\varepsilon},x\right\rangle\leq\|x\|. \tag{2.7}$$

On the other hand, for every $x\in C\cap C_{f/\varepsilon}$ it follows that

$$\left\langle\frac{f}{\varepsilon},x\right\rangle\leq\frac{\|f\|_*}{\varepsilon}\|x\|\leq\frac{\|f\|_*}{\varepsilon}\gamma. \tag{2.8}$$

From relations (2.7) and (2.8) it follows that

$$\langle f,x\rangle\leq\varepsilon\|x\|+\tilde{\mu}\quad\forall x\in C,$$

where $\tilde{\mu}=\gamma\|f\|_*$. Thus, the map $x\mapsto\iota_C(x)-\langle f,x\rangle+\varepsilon\|x\|$ is bounded from below for every $\varepsilon>0$, and the proof of Theorem 2.1 is completed. □

From Theorem 2.1 and Lemma 2.4 we obtain the following characterization of the family of all closed convex sets fulfilling condition (1.1).

Proposition 2.5. *For every closed convex set C of a normed linear space X, the following two facts are equivalent:*

i) $\overline{\mathcal{B}(C)}=(C^\infty)^\circ$*;*

ii) $C\cap C_f$ *is bounded for every* $f\in(C^\infty)^\circ$*.*

Remark 2.6. When C is linearly bounded, Proposition 2.5 states that $\overline{\mathcal{B}(C)} = X^*$ if and only if $C \cap C_f$ is bounded for every $f \in X^*$. Condition *ii*) from Proposition 2.5 defines the family of *conically bounded* sets, which is a subclass of the class of linearly bounded sets for which holds the property of density of the barrier cone in X^*.

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