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# TRIANGULAR $G_a$ ACTIONS ON $\mathbf{C}^4$

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Abstract. Every locally trivial action of the additive group of complex numbers on four-dimensional complex affine space that is given by a triangular derivation is conjugate to a translation. A criterion for a proper action on complex affine $n$-space to be locally trivial is given, along with an example showing that the hypotheses of the criterion are sharp.

## 1. Introduction

Let $G_a$ denote the additive group of complex numbers, and $X$ a complex affine variety. By an action of $G_a$ on $X$ we will mean an algebraic action. It is well known that every such action can be realized as the exponential of some locally nilpotent derivation $D$ of the coordinate ring $\mathbf{C}[X]$ and that every locally nilpotent derivation gives rise to an action. The ring $C_0$ of $G_a$ invariants in $\mathbf{C}[X]$ is equal to the ring of constants of the generating derivation.

Given an action $\sigma : G_a \times X \to X$, let $\bar{\sigma} : G_a \times X \to X \times X$ denote the graph morphism and $\hat{\sigma} : \mathbf{C}[X] \to \mathbf{C}[X, t]$ (resp. $\tilde{\sigma} : \mathbf{C}[X \times X] \to \mathbf{C}[X, t]$) denote the induced maps on coordinate rings.

The action is said to be proper if $\bar{\sigma}$ is a proper morphism (i.e., if $\mathbf{C}[X, t]$ is integral over the image of $\tilde{\sigma}$). The action is said to be equivariantly trivial if there is a variety $Y$ for which $X$ is $G_a$ equivariantly isomorphic to $G_a \times Y$, the action on $G_a \times Y$ being given by $g * (y, h) = (y, g + h)$. The action is locally trivial if there is a cover of $X$ by $G_a$ stable affine open subsets $X_i$ on which the action is equivariantly trivial. Equivariant triviality of an action on $X$ is equivalent with the existence of a regular function $s \in \mathbf{C}[X]$ for which $Ds = 1$. Such a function is called a slice and, if one exists, $\mathbf{C}[X] = C_0[s]$. If $X$ is factorial, i.e., its coordinate ring is a unique factorization domain, then local triviality is equivalent with the intersection of the kernel and image of $D$ generating the unit ideal in $\mathbf{C}[X]$.

The affine cancellation problem can be phrased in terms of $G_a$ actions on $X = \mathbf{C}^{n+1}$: If the action is equivariantly trivial, is then $Y \cong \mathbf{C}^n$? The answer is affirmative for $n = 2$, and for $n = 3$ provided the ring of invariants contains a coordinate function [16, Cor. 4.5.5]. It has recently been shown that the ring of $G_a$ invariants is finitely generated for actions on $\mathbf{C}^4$ whose generating derivation is triangulable (triangulable actions) [2]. These positive results suggest that a more complete understanding of actions on $\mathbf{C}^4$ is within reach. In section 1 we show that

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locally trivial triangulable actions on $\mathbf{C}^4$ are in fact equivariantly trivial, admitting a geometric quotient isomorphic to $\mathbf{C}^3$. Thus the example of Winkelmann [14] of a locally trivial, but not equivariantly trivial, triangular action on $\mathbf{C}^5$ is optimal.

Locally trivial actions are proper, and proper actions on $\mathbf{C}^n$ are locally trivial provided that $\mathbf{C}[X]$ is a flat ring extension of $C_0$ [4, Theorem 2.8]. This need not always be the case as shown in [5]. On the other hand, Holmann [12] showed that any proper holomorphic action on a complex manifold admits a quotient that is a manifold, while Popp [11, Lecture 3] showed that this quotient admits the structure of an algebraic space if the action is algebraic and the manifold is a smooth variety. Based on these results, we give in section 3 a ring-theoretic criterion for a proper action on $\mathbf{C}^n$ to be locally trivial and indicate where the hypotheses fail for the example in [5] of a nonlocally trivial proper action on $\mathbf{C}^5$.

## 2. Locally trivial triangular actions on $\mathbf{C}^4$

From [4, Theorem 2.8] we know in general that the quotient of a locally trivial action on an affine factorial variety $X$ exists as a quasiaffine variety $Y \subset$ **Spec** $R$, where $R$ is the subring of $C_0$ constructed as follows: Let $\delta(a_1), ..., \delta(a_n) \in C_0$ generate the unit ideal in $\mathbf{C}[X]$, and set $R_i = \mathbf{C}[X, \frac{1}{\delta(a_i)}]^{G_a}$. Note that $\mathbf{C}[X, \frac{1}{\delta(a_i)}] = R_i[\frac{a_i}{\delta(a_i)}]$ so that $R_i$ is a finitely generated $\mathbf{C}$ algebra, say $R_i = \mathbf{C}[b_{i1}, ..., b_{im}, \frac{1}{\delta(a_i)}]$, with $b_{ij} \in C_0$. The ring $R = \mathbf{C}[b_{ij}, \delta(a_i) \,|\, 1 \leq i \leq n, 1 \leq j \leq m]$ is the required subring of $C_0$.

It is easy to see that $C_0$ is the factorial closure of $R$ (i.e., the intersection of all unique factorization domains containing $R$), and we ask whether $C_0$ is the integral closure of $R$. Of course a positive answer would solve Hilbert's $14^{th}$ problem for locally trivial $G_a$ actions. Since $Y$ is a geometric quotient, $C_0$ is the ring of global sections of its structure sheaf. With $I$ denoting the ideal defining the complement of $Y$ in **Spec** $R$, and $F$ the quotient field of $R$, the ring $C_0$ is isomorphic to $T_I R = \bigcup_{n \geq 0} \{\alpha \in F \,|\, \alpha I^n \subset R\}$, the ideal transform of $R$ with respect to $I$. A fuller discussion of these notions can be found in [6].

Consider a locally trivial $G_a$ action on $\mathbf{C}^4$ generated by the locally nilpotent derivation of $\mathbf{C}[x_1, x_2, x_3, x_4]$ defined by $\delta$

$$\begin{aligned} x_4 &\mapsto p(x_1, x_2, x_3), \\ x_3 &\mapsto q(x_1, x_2), \\ x_2 &\mapsto r(x_1), \\ x_1 &\mapsto 0. \end{aligned}$$

It was recently shown [2] that $C_0$ is finitely generated for any triangular action on $\mathbf{C}^4$. In the special case under consideration, we show that $Y \cong$ **Spec** $C_0$. Since the quotient $Y$ is then affine, the action is equivariantly trivial (locally trivial actions with quotient $Y$ correspond to elements of $H^1(Y, O(Y))$, which is 0 with $Y$ affine), and van Rossum's thesis [16] then shows that $Y \cong \mathbf{C}^3$.

**Theorem 2.1.** *Let $G_a$ act locally trivially on $X = \mathbf{C}^4$ via a triangular derivation as above. Then the action is equivariantly trivial with quotient isomorphic to $\mathbf{C}^3$.*

*Proof.* Set $Z =$**Spec** $C_0$, and denote by $\pi : X \to Z$ the $G_a$ equivariant morphism induced by the ring inclusion $C_0 \hookrightarrow \mathbf{C}[x_1, x_2, x_3, x_4]$. By hypothesis, $x_1 \in C_0$ and is prime, so that for each $\lambda \in \mathbf{C}$, $\pi_\lambda$, the restriction of $\pi$ to the hyperplane $X_\lambda$

defined by $x_1 - \lambda$, is a $G_a$ equivariant morphism to the surface $Z_\lambda \subset Z$ defined there by $x_1 - \lambda$. The assertion is proved by showing that $\pi_\lambda$ is surjective for all $\lambda$.

It suffices to consider only those $\lambda$ for which $x_1 - \lambda$ divides $r(x_1)$, since otherwise $\frac{x_2}{r(x_1)}$ defines a slice on $X_\lambda$. Assume that $r(0) = 0$, and we consider $\pi_0 : X_0 \to Z_0$. Note that $X_0 \cong \mathbf{C}^3$ so that the action on $X_0$ has a slice and $\pi_0$ is an orbit map, but $Z_0$ is not a priori the quotient (which is the open image of $\pi_0$). Indeed, $Z_0 \cong \mathbf{Spec}\ C_0/(x_1)$, and $C_0/(x_1) \subset [C[X]/(x_1)]^{G_a}$, the latter being the ring of functions defined on the image of $\pi_0$ in $Y_0$, and $X_0 \to \mathbf{Spec}\ [C[X]/(x_1)]^{G_a}$ is clearly surjective. Denote by $I$ the ideal in $C_0/(x_1)$ defining $Y_0 - \mathrm{im}(\pi_0)$. We have $T_I(C_0/(x_1)) \cong [\mathbf{C}[X]/(x_1)]^{G_a} \cong \mathbf{C}^{[2]}$, a polynomial ring in two variables.

Lemma 2.5 of [2] can be interpreted in this context as saying that $C_0/(x_1) = R[u]$, where $u$ is transcendental over some subring $R$, i.e., that the algorithm [1] to construct the ring of invariants terminates with the adjunction of an element transcendental over a subring. Denote by $\hat{R}$ the integral closure of $R$ in its quotient field. Note that $\hat{R}[u] \subset T_I\ (R[u])$ since both are integrally closed. Moreover, $\hat{R}$ is Dedekind and rational since the quotient field, $qf(\hat{R}[u]) \cong \mathbf{C}^{(2)} \cong qf(R)(u)$. By the generalized Luroth theorem, $qf(R)$ is rational. Since $\hat{R}$ is a subring of a polynomial ring, $\hat{R} \cong \mathbf{C}^{[1]}$.

Note that $T_{I\hat{R}[u]}\ (\hat{R}[u]) \cong T_I\ (R[u]) \cong \mathbf{C}^{[2]}$. If $ht(I\hat{R}[u]) = 1$, then $I\hat{R}[u]$ is principal, say $I\hat{R}[u] = (f)$. But then $\frac{1}{f} \in \mathbf{C}^{[2]}$, a contradiction. Thus $ht(I\hat{R}[u]) = 2$, which implies that $T_{I\hat{R}[u]}\ (\hat{R}[u]) = \hat{R}[u]$. Finally, we obtain the chain of surjections:

$$X_0 \to \mathbf{Spec}\ \ [C[X]/(x_1)]^{G_a} = \mathbf{Spec}\ \ \hat{R}[u] \to \mathbf{Spec}\ \ R[u] = Z_0.$$

That the quotient is isomorphic to $\mathbf{C}^3$ follows from a special case of [16, Cor. 4.5.5]. □

*Remark* 2.2. It appears to be true that every fixed point free action on $\mathbf{C}^3$ has a slice (S. Kaliman, preprint), while Winkelmann produced an example of a locally trivial triangular derivation on $\mathbf{C}^5$ that has no slice, and there is an example of a triangular proper action on $\mathbf{C}^5$ that is not locally trivial. The situation for $\mathbf{C}^4$ is not so clear, but the next section proposes an avenue of attack on the proper triangular case.

## 3. Proper actions

Consider a proper $G_a$ action on $X = \mathbf{C}^n$ generated by the locally nilpotent derivation $D$ . Assume that the ring of invariants $C_0$ is finitely generated defining the affine variety $Y = \mathbf{Spec}\ C_0$. Let $\pi : \mathbf{C}^n \to Y$ as above be the morphism induced by the ring inclusion $C_0 \subset \mathbf{C}[x_1, ..., x_n]$, and let $I$ denote the ideal $C_0 \cap \mathrm{im} D$ ($I = C_0$ if and only of the action is equivariantly trivial). Assuming that the action is not equivariantly trivial, in particular $n \geq 4$, denote by $Z$ the closed subset of $Y$ defined by $I$. From [7] we know that every irreducible component of $Z$ has codimension exactly two and that $\pi\ |_{X - \pi^{-1}Z}$: $X - \pi^{-1}Z \to Y - Z$ is a principal $G_a$ bundle. The action is locally trivial if and only if $\pi^{-1}Z = \emptyset$.

From Holmann [12]: we know that the space of orbits carries the structure of an analytic space $X/G_a$ (in fact, $X/G_a$ is a manifold) and from Popp [11] that $X/G_a$ is an algebraic space. The simplicity of our context enables us to make this even more explicit. The orbit $G_a x$ of any point $x \in X$ is isomorphic to a line. As such it is a coordinate line in some coordinate system, $(x_1, ..., x_n)$ for $X$, say

$G_a x$ is the $x_1$-axis, and we can take $x$ to be the origin. If $H_x$ is the hyperplane $x_1 = 0$, then it is clear that the morphism $G_a \times H_x \cong \mathbf{C}^n \to X = \mathbf{C}^n$ given by $\rho : (\lambda, y) \mapsto \sigma(\lambda, y)$ is étale in an affine neighborhood $U$ of $(\lambda, x)$ (the principal open subset defined by the Jacobian determinant $d$ of the regular mapping). Indeed, $\rho$ is $G_a$ equivariant with respect to the action on $G_a \times H_1$ given by $(\mu, (\lambda, y)) \mapsto (\mu+\lambda, y)$. Thus $d \in \mathbf{C}[G_a \times H_x]^{G_a} = \mathbf{C}[H_x]$, and $U = G_a \times U_x$ with $U_x$ the principal open subset of $H_x$ defined by $d$. Identifying $U_x$ with the 0 section of the trivial $G_a$ bundle, and therefore the quotient of $U$ with respect to the $G_a$ action, the restriction $\rho|_{U_x} : U_x \to X/G_a$ gives an étale morphism. The images of finitely many such morphisms $\rho|_{U_{x_i}} : U_{x_i} \to X/G_a$ cover $X/G_a$. That $\amalg_i U_{x_i} \overset{\rho|_{U_{x_i}}}{\to} X/G_a$ provides an affine étale covering making $X/G_a$ an algebraic space is explained in [11, p. 39].

This description of the quotient as an algebraic space uses the complex structure. An alternative realization of the quotient of a variety $X$ by a proper action of an algebraic group $G$ as an algebraic space, valid in any characteristic, is given by Seshadri. Indeed, the construction is similar, differing in that Seshadri builds a variety $Z$ finite over $X$ from affine varieties analogous to $U$ above. The action of $G$ extends to $Z$ and is locally trivial. The quotient $W$ is separated but need not be quasiprojective. However, $\mathbf{C}(Z)/\mathbf{C}(X)$ is Galois with group $\Gamma$, $\Gamma$ acts on $W$, and the quotient of $X$ by $G$ is the algebraic space $W/\Gamma$. For the purposes of this paper the first description of the quotient is more convenient. For example, we can give a nice description of the stalks of the structure sheaf of the algebraic space $X/G_a$.

For a local ring $R$ with maximal ideal $\mathfrak{m}$, denote by $R^h$ the henselization of $R$ and by $\widehat{R}$ its completion at $\mathfrak{m}$. Recall that for $R$ equal to the localization of an affine domain, $R^h$ is the algebraic closure of $R$ in $\widehat{R}$.

**Proposition 3.1.** *Let $z \in X/G_a$. Then $O_{z,X/G_a} \equiv \varinjlim(O(U \times_{X/G_a} X)^{G_a}) \cong [\varinjlim(O(U \times_{X/G_a} X))]^{G_a} \cong (\mathbf{C}[x_2, ..., x_n]_{(x_2,...,x_n)})^h$, where the limit is taken over all étale open subsets $U \to X/G_a$ of $X/G_a$.*

*Proof.* The isomorphism between $O_{z,X/G_a}$ and $(\mathbf{C}[x_2, ..., x_n]_{(x_2,...,x_n)})^h$ is clear from the above construction of $X/G_a$. It is also clear that $\varinjlim(O(U \times_{X/G_a} X)^{G_a})$ maps injectively into $[\varinjlim(O(U \times_{X/G_a} X))]^{G_a}$. Take $\overline{h} \in (\varinjlim O(U \times_{X/G_a} X))^{G_a}$. Because the action is proper, we can find an étale open $V \to X/G_a$ of $X/G_a$ with $V$ affine and $V \times_{X/G_a} X$ equivariantly isomorphic to $V \times \mathbf{C}$ with $\overline{h}$ represented by some element $h \in O(V \times \mathbf{C})$. Because $\overline{h}$ is $G_a$ invariant, for each $\lambda \in G_a$ there is an open subset $V_\lambda \times \mathbf{C} \subset V \times \mathbf{C}$ with $\lambda(h)|_{V_\lambda \times \mathbf{C}} = h|_{V_\lambda \times \mathbf{C}}$. But then the cyclic subgroup of $G_a$ generated by $\lambda$ stabilizes $h$ on $V_\lambda \times \mathbf{C}$. The stabilizer of $h$ is an algebraic subset of $G_a$ and is therefore all of $G_a$ for any $\lambda \neq 0$. Thus $h \in O(V \times \mathbf{C})^{G_a}$ and the image of $h$ in $\varinjlim(O(U \times_{X/G_a} X)^{G_a})$ is the desired preimage of $\overline{h}$. □

**Example 1.** *The action on $X = \mathbf{C}^5$ determined by the locally nilpotent derivation of $\mathbf{C}[x_1, x_2, y_1, y_2, z]$, namely*

$$\delta : x_2 \mapsto x_1 \mapsto 0, \quad y_2 \mapsto y_1 \mapsto 0, \quad z \mapsto (1 + x_1 y_2^2),$$

*is proper. Its quotient is an algebraic space that is not a scheme* [5]. *In particular, $W$, as in Seshadri's construction above, is not quasiprojective.*

The ring of invariants $C_0$ is generated by the five polynomials

$$\begin{aligned} c_1 &= x_1, \\ c_2 &= x_2, \\ c_3 &= x_1y_2 - x_2y_1, \\ c_4 &= 3y_1z - x_1y_2^3 - 3y_2, \\ c_5 &= \frac{x_1^2c_4 + c_3^3 + 3x_1c_3}{y1}. \end{aligned}$$

Set $Y = \mathbf{Spec}\ \mathbf{C}[c_1, c_2, c_3, c_4, c_5]$, and let $\pi : X \to Y$ be the morphism defined by the rings inclusion. One checks that the $C_0$ ideal $\sqrt{C_0 \cap \mathrm{im}(\delta)} = (c_1, c_2, c_3)$ has height 2, and that $[C_0 \cap \mathrm{im}(\delta)]\mathbf{C}[X] = (x_1, y_1)$. The singular locus $S$ of $Y$ is one dimensional, properly contained in the zeros locus $Z$ of $(c_1, c_2, c_3)$, and $\pi(\pi^{-1}(Z)) \subset S$. $\pi|_{X-\pi^{-1}(Z)}$ is a quotient morphism, but fibers over points in $S$ are all two dimensional.

In general, for a proper action with finitely generated $C_0$, the universal property for geometric quotients yields a morphism of algebraic spaces $\overline{\pi} : X/G_a \to Y$ that is an isomorphism outside of a closed subset of codimension 2 in $X/G_a$ and $Y$ (the zero loci of $(c1, c2)$ in the respective spaces). Note that $\mathbf{C}[Y]$ is a unique factorization domain (UFD), so that if $X/G_a$ had the structure of a variety, $\overline{\pi}$ would be an isomorphism into its image [18, Prop. 1, p. 289]. In our example, however, the completions (and henselizations) of the local rings over points in $S$ do not retain the unique factorization domain property.

To see this, we rely on the paper [15] where it is shown that the localization of the UFD $A = \mathbf{C}[X, Y, Z, T]/(XY - ZT + X^3 + Y^3)$ at the maximal ideal generated by the classes of $X, Y, Z, T$ does not remain a UFD upon completion. In fact, the completion $\hat{A}$ is shown to be isomorphic to $\mathbf{C}[[X, Y, Z, T]]/(XY - ZT)$, i.e., $X^3 + Y^3 \in (XY - ZT)\ \mathbf{C}[[X, Y, Z, T]]$. In the example above, $\mathbf{C}[Y] \cong \mathbf{C}[C_1, C_2, C_3, C_4, C_5]/(C_2C_5 - C_1^2C_4 - C_3^3 - 3C_1C_3)$. With a simple change of variables (replacing $C_3$ by $3C_3$) and the observation that $C_1^3 + C_3^3 \in (C_2C_5 - C_1C_3)\mathbf{C}[[C_1, C_2, C_3, C_5]]$, we can realize the completion of $\mathbf{C}[Y]$ at the maximal ideal $(C_1, C_2, C_3, C_4, C_5)$ as isomorphic to $\mathbf{C}[[C_1, C_2, C_3, C_4, C_5]]\ /(C_2C_5 - C_1K)$ for some $K$.

**Lemma 1.** *Let $A$ be the localization of a finitely generated domain over $\mathbf{C}$ at the maximal ideal $\mathfrak{m}$. Then the henselization $A^h$ of $A$ is a unique factorization domain if and only if the completion $\hat{A}$ of $A$ is.*

*Proof.* In this context, the henselization and strict henselization of $A$ are equal and $\widehat{A^h} \cong \hat{A}$ [19, p. 38]. From [3] (proof of Theorem 1), we have $C(A^h) = C(\hat{A})$, where $C(-)$ denotes the divisor class group. □

**Lemma 2.** *Let $G_a$ act properly on $X = \mathbf{C}^n$ with geometric quotient the algebraic space $X/G_a$. Assume that $C_0 = \mathbf{C}[X]^{G_a}$ is finitely generated defining the affine variety $Y$. Denote by $q$ the morphism $X \to Y$ induced by the ring inclusion, by $\pi$ the quotient morphism $X \to X/G_a$, and by $\overline{\pi}$ the canonical morphism $X/G_a \to Y$. For $z \in X/G_a$, let $K_z$ be the quotient field of the stalk at $z$ of the structure sheaf of $X/G_a$, and let $F_{\overline{\pi}(z)}$ be the quotient field of the henselization of $C_{0,\mathfrak{m}_{\overline{\pi}(z)}}$. Then $\overline{\pi}$ induces an isomorphism of $F_{\overline{\pi}(z)}$ and $K_z$.*

*Proof.* If the action is locally trivial in the Zariski topology, then there is nothing to show (note that since $Y$ is a variety, $O_{\overline{\pi}(z),Y}$ is the henselization of $C_{0,\mathfrak{m}_{\overline{\pi}(z)}}$). If the action is not Zariski locally trivial, then it is nevertheless geometrically irreducible in codimension one (GICO) [7], i.e., the intersection of the kernel and image of the generating derivation $\delta$ lies in no height one prime ideal of $\mathbf{C}[X]$ (or of $C_0$). As a consequence, there is a closed subset $Z$ of codimension precisely 2 in $Y$ so that $\pi^{-1}(Z)$ has codimension 2 in $X$, and $q : X - \pi^{-1}(Z) \to Y - Z$ is a principal $G_a$ bundle, locally trivial in the Zariski topology with quotient $Y - Z$ [6, (and Theorem 4.2 below) below]. By the uniqueness of geometric quotients, $Y - Z \cong X/G_a - \overline{\pi}^{-1}Z$. The result follows by appealing to an affine étale covering of $X/G_a$ enabling the reduction to an affine neighborhood $U$ of $z$. A rational function on $U$ representing an element of $K_z$ is clearly in the function field of $Y$ at $\pi(z)$. □

**Theorem 3.2.** *Consider a proper action of $G_a$ on $X = \mathbf{C}^n$ with finitely generated ring of invariants $C_0$ defining the affine variety $Y$ and morphism $q : X \to Y$. The action is locally trivial with quasiaffine quotient if and only if for each $x \in X$ the completion of the local ring of $q(x)$ on $Y$ is a unique factorization domain.*

*Proof.* If the action is locally trivial, then $q$ is a flat morphism (in the Zariski topology) whose image is in the smooth locus of $Y$.

To prove the converse, the argument is essentially that of [18, p. 289, Proposition 1]. Let $y = q(x)$, $\mathfrak{m}$ the corresponding maximal ideal of $C_0$ and $A = C_{0\mathfrak{m}}$. If $y$ is a smooth point of $Y$, then the action is locally trivial in an affine neighborhood of $q^{-1}(y)$ [7]; so assume that $Y$ is singular at $y$. From Lemma 3.2, we know that $A^h$ is a unique factorization domain. Since the action is proper, $X/G_a$ exists as an algebraic space. Let $\overline{\pi}(z) = y$ for $z \in X/G_a$, and denote by $B = O_{z,X/G_a}$. Since $X/G_a$ is smooth but $Y$ is singular at $y$, we can view $A^h$ as a proper subring of $B$. If $\varphi \in B - A$, from Lemma 3.3, $\varphi = \frac{r}{s}$, $r, s \in A^h$. By choosing a suitable affine neighborhood of $y$, we can assume that there is a morphism of affine varieties $\overline{\pi} : V \to W$, subvarieties $W_1, W_2$ of $W$ (the zero loci of $r$ and $s$), whose intersection has pure codimension 2 in $W$, and a subvariety $V_1$ of $V$ (the zero locus of $s$) of codimension 1. Since $r = \varphi s$, every component of $V_1$ maps to $W_1 \cap W_2$. In particular, the set of points at which $\overline{\pi}$ is not an isomorphism has codimension 1, a contradiction. □

**Problem 1.** *For which actions does $C_{0,\mathfrak{m}}$ remain a unique factorization domain upon completion? In particular, does this hold for proper actions on $C^4$?*

## 4. Remarks

Related to Lemma 3.2 we have the following.

**Proposition 4.1.** *Let $k$ be a field of characteristic 0 and $A$ an affine $k$ algebra satisfying the following conditions:*

1. *$A$ is a unique factorization domain;*
2. *with $T$ denoting an indeterminate, $A[[T]]$ is a unique factorization domain (i.e., $A$ has discrete divisor class group, e.g. $A$ is a regular UFD);*
3. *$G_a = G_a(k)$ acts on $A$ via the locally nilpotent derivation $d$ with kernel $A^d$.*

*Then $A^d$ satisfies Serre's $S_3$ condition.*

*Proof.* Consider the extension $D$ of the derivation to $A[[T]]$, defined by $D\sum_{i=0}^{\infty} a_iT^i = \sum_{i=0}^{\infty} d(a_i)T^i$, and the extension of the $G_a$ action by $\sigma_t \sum_{i=0}^{\infty} a_iT^i = \sum_{i=0}^{\infty} \exp(td)(a_i)T^i$ for each element $t \in k$. It is straightforward to check that

$A[[T]]^{G_a} = A[[T]]^D = A^d[[T]] = A^{G_a}[[T]]$. Moreover, $A[[T]]^{G_a}$ is factorially closed in $A[[T]]$. Indeed, suppose that $a \in A[[T]]^{G_a}$ has the factorization $a = a_1 a_2 ... a_k$ in $A[[T]]$. Then $G_a$ permutes the ideals $(a_i)$ inducing a homomorphism from $G_a$ to the symmetric group on $k$ letters. However, only the trivial homomorphism exists, so that $\sigma_t(a_i) = \lambda(t) a_i$ where $\lambda : G_a \to A[[T]]^*$. Comparing coefficients of $T^j$, we find that $(\lambda(t) - 1)a_i = \sum_{j=1}^{N} \frac{t^j}{j!} d^j(a_i)$, where $N+1$ is the least power of $d$ annihilating $a_i$. Note that $d^N$ annihilates $\sum_{j=1}^{N} \frac{t^j}{j!} d^j(a_i)$. Unless $\lambda(t) = 1$, we obtain a contradiction. Thus each $a_i \in A[[T]]^{G_a}$, and therefore this ring is a unique factorization domain. Thus $A^d$ has discrete divisor class group and consequently satisfies Serre's condition $S_3$ [13] . □

**Theorem 4.2.** *Let $X$ be a smooth factorial quasiaffine variety. Suppose that $G_a$ acts algebraically on $X$ and that $O(X)^{G_a}$ is finitely generated over $\mathbf{C}$. If $\dim X \leq 5$, then $O(X)^{G_a}$ is Gorenstein.*

*Proof.* Since $O(X)^{G_a}$ has dimension at most 4 and satisfies $S_3$, [17, Corollary 1.8] shows that all of its localizations are Gorenstein. □

The proposition also enables a strengthening of [6, Theorem 3.1] by removing the Cohen-Macaulay hypothesis on the ring of invariants.

**Theorem 4.3.** *Let $X$ be a smooth factorial complex affine variety of dimension $n \geq 4$ with a GICO $G_a$ action generated by the locally nilpotent derivation $\delta$ of $O(X)$. If $O(X)^{G_a}$ is finitely generated and the height of the ideal image$(\delta) \cap O(X)^{G_a}$ is at least 3, then the action is equivariantly trivial.*

*Proof.* Let $P$ be a prime ideal of $O(X)^{G_a}$ minimal over image$(\delta) \cap O(X)^{G_a}$. Set $R = O(X)_P^{G_a}$, denote the closed point of **Spec** $R$ by $M$, and let $U = \mathbf{Spec}\ R - \{M\}$. The Cohen-Macaulay hypothesis was used to show that $\mathrm{Ext}^1(O_U, O_U) \cong H_M^2(W, O_W) = 0$. But this follows from the $S_3$ condition. □

Greuel and Pfister have conjectured [8] that any proper action of a unipotent group on an affine scheme $X$ lifts to locally trivial action on some étale covering of $X$. If by étale covering one means a finite étale morphism, then the conjecture fails for $X = \mathbf{C}^n$ and the connected unipotent group by the simple conectivity of $\mathbf{C}^n$ [10]. Indeed, suppose $X = \bigcup_{i=1}^{m} X_i$, with $q_i : X_i \cong X$ for each $i$. Connectivity implies that each orbit will lie in exactly one $X_i$ so that the action is locally trivial on $X_i$ and $q_i$ is $G_a$ equivariant (i.e., the action was already locally trivial on $X$). On the other hand, if one drops the finiteness requirement, then section 3 indicates why the conjecture does hold for $X = \mathbf{C}^n$ and proper $G_a$ actions.

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