PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 10, Pages 2825–2831
S 0002-9939(04)07583-5
Article electronically published on June 2, 2004

REFLECTION QUOTIENTS IN RIEMANNIAN GEOMETRY. A GEOMETRIC CONVERSE TO CHEVALLEY'S THEOREM

R. MILSON

(Communicated by Wolfgang Ziller)

Abstract. Chevalley's theorem and its converse, the Sheppard-Todd theorem, assert that finite reflection groups are distinguished by the fact that the ring of invariant polynomials is freely generated. We show that, in the Euclidean case, a weaker condition suffices to characterize finite reflection groups, namely, that a freely-generated polynomial subring is closed with respect to the gradient product.

1. Introduction

From the standpoint of invariant theory, finite reflection groups are distinguished by the property that the corresponding algebra of invariants is freely generated. The existence of free generators of the invariant algebra is known as Chevalley's theorem [2]. The converse, i.e., the statement that a finite group with a freely generated invariant algebra is necessarily generated by reflections, is known as the Sheppard-Todd theorem [7]. The customary proofs of these results use algebraic methods.

The purpose of this note is to propose an alternate characterization of finite reflection groups over the reals, and to prove the result using the language and ideas of Riemannian geometry. The term *reflection* is being used here to refer to a linear automorphism of order 2 that fixes a codimension 1 hyperplane, while *reflection group* refers to a group of linear automorphisms that can be generated by reflections. Riemannian theory has some relevance to this topic, because every finite group acting linearly on a real vector space determines a Euclidean structure (obtained by averaging an arbitrary positive-definite quadratic form [3]), and therefore, without loss of generality, the elements of a finite reflection group can be assumed to be Euclidean automorphisms. Now the structure of Riemannian geometry is specified by a fundamental covariant: the gradient operation. Axiomatically then, the gradient operation is preserved by all Riemannian automorphisms.

Specializing to Euclidean space, if $P(x^1, \ldots, x^n)$, $Q(x^1, \ldots, x^n)$ are two polynomials that are invariant with respect to some Euclidean reflections, then the

Received by the editors December 3, 2001 and, in revised form, June 12, 2002.
2000 *Mathematics Subject Classification.* Primary 20H15, 14L24, 53B21.
Key words and phrases. Reflection groups, invariants, degenerate metrics.
The author was supported by NSERC grant 228057.

corresponding gradient product

$$\nabla P \cdot \nabla Q = \sum_i \frac{\partial P}{\partial x^i} \frac{\partial Q}{\partial x^i}$$

is also an invariant polynomial. It follows immediately that if $I^1, \ldots, I^n$ is a free basis of the invariant algebra, then the matrix of gradient cross-products also consists of polynomials in the basis elements, i.e.,

$$\nabla I^i \cdot \nabla I^j = g^{ij}(I^1, \ldots, I^n), \quad i, j = 1, \ldots, n \tag{1.1}$$

where the g^{ij} are n-variable polynomials.

We will show that this property characterizes the class of finite reflection groups in the following sense.

Theorem 1.1. *Let $I^1, \ldots, I^n$ be algebraically independent, homogeneous, real polynomials in n variables, and let G be the group of linear automorphisms that leaves them invariant. If all the corresponding Euclidean gradient cross-products are themselves polynomial in $I^1, \ldots, I^n$, then $I^1, \ldots, I^n$ freely generate the algebra of G-invariants.*

Accepting this theorem as true, the Shepard-Todd theorem then implies that G is generated by reflections.

The restriction of the field to $\mathbb{R}$ and the assumption of Euclidean signature are indispensable; the theorem is not true without them. Consider the following 2-dimensional counterexamples:

$$I^1 = x^1 + ix^2, \quad I^2 = (x^1)^2 + (x^2)^2.$$

The matrix of gradient cross-products is

$$\begin{pmatrix} 0 & 2I^1 \\ 2I^1 & 4I^2 \end{pmatrix}.$$

However, $x^1 - ix^2 = I^2/I^1$, and hence G is trivial. The conclusion of the theorem does not hold. Alternatively, one can change the above into a real counterexample based on non-Euclidean signature. Take

$$I^1 = x^1 + x^2, \quad I^2 = (x^1)^2 - (x^2)^2.$$

The key to the proof of Theorem 1.1 is to regard the matrix of gradient products, g^{ij}, as the contravariant form of a Riemannian metric tensor. The matrix of gradient cross-products has been studied in the context of singular projections [1], [6]. To the author's best knowledge, the Riemannian-geometric interpretation of g^{ij} is novel.

Unfortunately, the idea cannot be made entirely rigorous because of an essential complication. Let δ denote the discriminant

$$\delta = \det g^{ij},$$

and note that the tensor g^{ij} is degenerate at points where $\delta = 0$. The presence of degeneracies means that the g^{ij} do not define a Riemannian structure in the usual sense of this term.

To put it another way, the g^{ij} permit us to regard the map

$$\Pi : \mathbb{R}^n \to \mathbb{R}^n,$$

where $\Pi = (I^1, \dots, I^n)$, as an isometry. However, there is a complication: Π is not a regular covering. Note that $\Pi^*\delta$ is the square of the Jacobian of Π,

$$J = \det \left[\frac{\partial I^j}{\partial x^i} \right],$$

and hence the equation $\delta = 0$ picks out the images of points where this map has less than maximal rank.

Thus, the resulting g^{ij} must be regarded as belonging to some generalization of Riemannian theory that permits degenerate metric tensors. At the present time, the geometric interpretation of degenerate metrics is not well understood. Some efforts in this direction can be found in [5] and [4]. There may also exist applications of these ideas to the theory of isoparametric manifolds [8], because an isoparametric map naturally gives rise to degenerate g^{ij}, and because reflection groups play an important role in both theories.

In the present note we restrict our attention to the proof of Theorem 1.1, and merely use the geometric ideas as a guiding principle. The fact of the matter is that the restriction of the g^{ij} to the domain $\{\delta \neq 0\}$ defines a perfectly standard, albeit incomplete, Riemannian structure. This observation underlies and motivates the technique of the proof.

2. Proof of the main theorem

We now turn to the proof of our theorem. Let us fix notation and hypotheses. Let $I^1, \dots, I^n$ be algebraically independent, real homogeneous polynomials in n indeterminates $x^1, \dots, x^n$. It will be convenient to carry out the proof over the field of complex numbers, and so we identify the polynomials in question with maps from $\mathbb{C}$ to $\mathbb{C}$ that commute with complex conjugation. In the same spirit, we regard $\Pi = (I^1, \dots, I^n)$ as a map from $\mathbb{C}^n$ to $\mathbb{C}^n$, and define G to be the group of complex linear automorphisms that preserve Π.

We shall write $\mathbb{C}[\mathbf{x}]$ and $\mathbb{C}[\mathbf{I}]$ for the polynomial algebras generated, respectively, by $x^1, \dots, x^n$, and by $I^1, \dots, I^n$. We shall write $\mathbb{C}(\mathbf{x})$ and $\mathbb{C}(\mathbf{I})$ for the corresponding fraction fields. It will also be convenient to introduce n additional indeterminates $\xi^1, \dots, \xi^n$ to serve as the coordinates of the codomain: $\Pi^*(\xi^j) = I^j$. Throughout we assume that the $\mathbb{C}[\mathbf{I}]$ is closed with respect to the gradient product, i.e., that there exist polynomials $g^{ij} \in \mathbb{R}[\boldsymbol{\xi}]$ such that (1.1) holds. We can now define a gradient operation on $\mathbb{C}[\boldsymbol{\xi}]$:

$$\nabla\alpha \cdot \nabla\beta = \sum_{ij} \frac{\partial\alpha}{\partial\xi^i} \frac{\partial\beta}{\partial\xi^j}\, g^{ij}, \quad \alpha, \beta \in \mathbb{C}[\boldsymbol{\xi}].$$

This operation is, by construction, compatible with the gradient operation on $\mathbb{C}[\mathbf{x}]$:

$$\Pi^*(\nabla\alpha \cdot \nabla\beta) = \nabla(\Pi^*\alpha) \cdot \nabla(\Pi^*\beta), \tag{2.1}$$

for all $\alpha, \beta \in \mathbb{C}[\boldsymbol{\xi}]$. We let $\delta \in \mathbb{R}[\boldsymbol{\xi}]$ and $J \in \mathbb{R}[\mathbf{x}]$ denote, respectively, the discriminant and the Jacobian as per above, and note that

$$\Pi^*\delta = J^2.$$

Proposition 2.1. *The group G of linear Π-automorphisms is a subgroup of $\mathrm{O}_n(\mathbb{C})$ and is isomorphic to the group of field automorphisms of $\mathbb{C}(\mathbf{x})$ over $\mathbb{C}(\mathbf{I})$. Furthermore, $\mathbb{C}(\mathbf{x})$ is a normal extension of $\mathbb{C}(\mathbf{I})$, and hence G is isomorphic to the Galois group of $\mathbb{C}(\mathbf{x})$ over $\mathbb{C}(\mathbf{I})$.*

Proof. By definition, each $g \in G$ defines an automorphism of $\mathbb{C}[\mathbf{x}]$ over $\mathbb{C}[\mathbf{I}]$ and hence of $\mathbb{C}(\mathbf{x})$ over $\mathbb{C}(\mathbf{I})$. We therefore have a natural inclusion

$$G \subset \mathrm{Aut}(\mathbb{C}(\mathbf{x})/\mathbb{C}(\mathbf{I})).$$

We now prove that this inclusion is, in fact, an isomorphism, as well as show that the elements of G are orthogonal.

Let d be the degree of the extension. Using a primitive element, one can straightforwardly show that there exists a dense open subset $U \subset \mathbb{C}^n$ such that for every $\boldsymbol{\xi} \in U$ the set $\Pi^{-1}(\boldsymbol{\xi})$ consists of d distinct preimages. We avoid the locus $\{\delta = 0\}$ and choose an open $U_0 \subset U$ sufficiently small so that $\Pi^{-1}(U)$ is the union of d disjoint open sets $V_1, \ldots, V_d$ and such that the restriction of Π to each of these is nonsingular. We therefore have d complex-analytic maps

$$\sigma_i : U_0 \to V_i, \quad i = 1, \ldots, d$$

that are local inverses of Π. For each $i = 1, \ldots, d$ we define $g_i : V_1 \to V_i$ by

$$g_i = \sigma_i \circ \Pi\Big|_{V_1}.$$

We then note that each g_i preserves the gradient product, and hence defines a local automorphism of the dot product structure on the tangent bundle of $\mathbb{C}^n$. Such a local automorphism preserves straight lines, and hence must be the restriction of a global, semi-linear transformation:

$$g_i \in \mathrm{O}_n(\mathbb{C}) \ltimes \mathbb{C}^n.$$

However, the components of Π are homogeneous polynomials and

$$\Pi \circ g_i = \Pi,$$

by definition. Hence, each g_i maps the origin to itself, hence is linear, and hence is an element of G. Thus, we have shown that G contains d elements of $\mathrm{O}_n(\mathbb{C})$. However, the order of the automorphism group cannot exceed d, and the desired conclusions follow.

We now prove that $\mathbb{C}(\mathbf{x})$ is Galois over $\mathbb{C}(\mathbf{I})$ by showing that $\mathbb{C}(\mathbf{I})$ is the fixed field of G. Let $p \in \mathbb{C}(\mathbf{x})$ be such that $p \notin \mathbb{C}(\mathbf{I})$. The minimal polynomial of p over $\mathbb{C}(\mathbf{I})$ has degree 2 or more, and hence there will exist points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^n$ such that $\Pi(\mathbf{x}_1) = \Pi(\mathbf{x}_2)$ but such that $p(\mathbf{x}_1) \neq p(\mathbf{x}_2)$. However, using the technique in the preceding paragraph one can find a $g \in G$ such that $g(\mathbf{x}_1) = \mathbf{x}_2$, and hence p is not G-invariant. Therefore, the fixed field of G coincides with $\mathbb{C}(\mathbf{I})$. □

Lemma 2.2. *The open $\{\delta \neq 0\} \subset \mathbb{C}^n$ is in the image of Π.*

Proof. Note that the open subset in question is path connected. Consequently, if $\Gamma \subset \{\delta \neq 0\}$ is a continuous path between points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$, and if $\boldsymbol{\xi}_1$ is in the image of Π, then so is $\boldsymbol{\xi}_2$. This is because Π has a local inverse wherever $J \neq 0$, and hence Γ can be lifted to a continuous path. However, at least one such $\boldsymbol{\xi}_1$ is bound to exist, and the desired conclusion follows. □

Proposition 2.3. *If $\lambda \in \mathbb{C}[\boldsymbol{\xi}]$ is such that $\delta = 0$ wherever $\lambda = 0$, then $\nabla \log \lambda$ is a well-defined derivation of $\mathbb{C}[\boldsymbol{\xi}]$. In other words, for every $\rho \in \mathbb{C}[\boldsymbol{\xi}]$ there exists a $\sigma \in \mathbb{C}[\boldsymbol{\xi}]$ such that*

$$\nabla\lambda(\rho) = \sigma\lambda.$$

Proof. It suffices to give a proof for irreducible λ. Suppose the proposition is false. Then, there exists a $\rho \in \mathbb{C}[\boldsymbol{\xi}]$ such that $\nabla\rho$ is transverse to the subvariety $\{\lambda = 0\}$. In other words, $\nabla\rho(\lambda)$ does not vanish identically on $\{\lambda = 0\}$. The covariant formula for the Laplacian gives

$$\Delta\rho = \sum_i \frac{\partial}{\partial\xi^i}(\nabla\xi^i(\rho)) - \frac{1}{2}(\nabla\log\delta)(\rho). \tag{2.2}$$

It follows that

$$\Pi^*\left((\nabla\log\delta)(\rho)\right) \in \mathbb{C}[\mathbf{x}],$$

and since $\Pi^*\lambda$ divides $\Pi^*\delta$,

$$\Pi^*\left((\nabla\log\lambda)(\rho)\right) \in \mathbb{C}[\mathbf{x}]$$

as well.

Next, let Φ_t be the one-parameter flow generated by $\nabla\rho$, and let ϕ_t be the one-parameter flow generated by $\nabla(\Pi^*\rho)$. By (2.1), the two flows intertwine with Π:

$$\Phi_t \circ \Pi = \Pi \circ \phi_t. \tag{2.3}$$

Choose a $\xi_0 \in \{\lambda = 0\}$ such that

$$\nabla\rho(\lambda)_{\boldsymbol{\xi}_0} \neq 0.$$

It follows immediately that

$$\nabla\rho(\delta)_{\boldsymbol{\xi}_0} \neq 0,$$

and hence, for all sufficiently small t we have

$$\delta(\Phi_t(\xi_0)) \neq 0.$$

By the lemma, $\Phi_t(\xi_0)$ is in the image of Π , and hence by (2.3) so is ξ_0. However,

$$\Pi^*(\nabla\rho(\lambda)) = \Pi^*(\lambda)\Pi^*\left((\nabla\log\lambda)(\rho)\right).$$

The right-hand side is zero at the pre-image of ξ_0, and hence so is the left-hand side, a contradiction. □

Proposition 2.4. *If $\lambda_1, \lambda_2 \in \mathbb{C}[\boldsymbol{\xi}]$ are distinct irreducible factors of the discriminant δ, then*

$$\nabla\lambda_1 \cdot \nabla\lambda_2 = 0.$$

Proof. By the preceding proposition, $\nabla\lambda_1 \cdot \nabla\lambda_2$ is divisible by $\lambda_1\lambda_2$. However, the degree of Π^* of the former is smaller than the degree of Π^* of the latter, and the desired conclusion follows. □

Proposition 2.5. *The Jacobian J is a harmonic polynomial.*

Proof. The condition $\Delta J = 0$ is equivalent to

$$\Delta\log\delta = -\frac{(\nabla\delta)(\log\delta)}{2\delta}. \tag{2.4}$$

We show that the latter equation holds. Write

$$\nabla\log\delta = \sum_i \alpha_i \frac{\partial}{\partial\xi^i},$$

where

$$\alpha_i = \nabla\xi^i(\log\delta) = (\nabla\log\delta)(\xi^i).$$

By the preceding proposition the $\alpha_i \in \mathbb{C}[\boldsymbol{\xi}]$, and since the degree of $\Pi^*\alpha_i$ must be lower than the degree of $\Pi^*\xi^i$ we must have

$$\frac{\partial \alpha_i}{\partial \xi^i} = 0$$

for all i. Equation (2.4) now follows from the formula for the Laplacian shown in (2.2). □

Proposition 2.6. $\mathbb{C}[\mathbf{x}]$ *is an integral extension of* $\mathbb{C}[\mathbf{I}]$.

Proof. Suppose not. Consider a $p \in \mathbb{C}[\mathbf{x}]$ that is not integral over $\mathbb{C}[\mathbf{I}]$. The coefficients of the minimal polynomial over $\mathbb{C}(\mathbf{I})$ are symmetric functions of the G-conjugates of p, and hence, by Proposition 2.1, are polynomials in $\mathbb{C}[\mathbf{x}]$. We have now demonstrated that if the extension is non-integral, then there must exist non-integral elements of degree 1, i.e., $r \in \mathbb{C}[\mathbf{x}]$ such that

$$r = \Pi^*(\alpha/\beta)$$

for some relatively prime $\alpha, \beta \in \mathbb{C}[\boldsymbol{\xi}]$. Choose one such r. The repeated application of the Laplacian will eventually reduce every polynomial to a constant, and hence without loss of generality, we may assume that Δr is a constant. We then have the following identity in $\mathbb{C}(\boldsymbol{\xi})$:

$$\beta\,(\Delta\alpha - 2(\nabla \log\beta)(\alpha) - \beta\Delta r) = \alpha(\Delta\beta - 2(\nabla\log\beta)(\beta)). \tag{2.5}$$

We can also assert that $\delta = 0$ wherever $\beta = 0$. If not, there would exist a $\boldsymbol{\xi}_0$ such that

$$\alpha(\boldsymbol{\xi}_0) \neq 0, \quad \beta(\boldsymbol{\xi}_0) = 0, \quad \delta(\boldsymbol{\xi}_0) \neq 0.$$

However, then by Lemma 2.2, $\boldsymbol{\xi}_0 = \Pi(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in \mathbb{C}^n$, and hence

$$\alpha(\boldsymbol{\xi}_0) = \beta(\boldsymbol{\xi}_0) r(\mathbf{x}_0) = 0,$$

a contradiction.

Hence, by Proposition 2.3, $\nabla\log\beta$ is a well-defined derivation of $\mathbb{C}[\boldsymbol{\xi}]$. It follows that (2.5) is, in fact, a relation between polynomials. Since α and β are relatively prime, equation (2.5) implies that β divides

$$\Delta\beta - 2(\nabla\log\beta)(\beta).$$

However, the degree of Π^* of this expression is smaller than the degree of $\Pi^*\beta$, and hence

$$\Delta\beta = 2(\nabla\log\beta)(\beta). \tag{2.6}$$

Next, let λ be an irreducible factor of β; let us say with multiplicity j. We have shown that λ is also an irreducible factor of δ; let us say with multiplicity k. By Proposition 2.4 all irreducible factors of δ are mutually orthogonal, and hence equation (2.6) implies that

$$\Delta\lambda = (j+1)(\nabla\log\lambda)(\lambda). \tag{2.7}$$

However, in Proposition 2.5 we showed that

$$\Delta\delta = \frac{1}{2}(\nabla\log\delta)(\delta),$$

and hence

$$\Delta\lambda = \left(1 - \frac{k}{2}\right)(\nabla\log\lambda)(\lambda). \tag{2.8}$$

Combining (2.7) and (2.8) we conclude that

$$\Delta\lambda = \Delta\lambda^2 = 0.$$

There does not exist a real polynomial that obeys such relations. However, δ is a polynomial with real coefficients, and hence $\bar{\lambda}$ is also an irreducible factor of λ that obeys the same relations. By Proposition 2.4, λ and $\bar{\lambda}$ are mutually orthogonal, and hence

$$\Delta\left(\lambda\bar{\lambda}\right) = \Delta\left(\lambda\bar{\lambda}\right)^2 = 0.$$

This is impossible, and the desired conclusion follows. □

We are now ready to give the proof of Theorem 1.1. Suppose $r \in \mathbb{C}[\mathbf{x}]$ is a G-invariant. Since G is isomorphic to the Galois group of $\mathbb{C}(\mathbf{x})$ over $\mathbb{C}(\mathbf{I})$, there exists a $\rho \in \mathbb{C}(\boldsymbol{\xi})$ such that $r = \Pi^*\rho$. However, we have just shown that the extension is integral, and so in fact $\rho \in \mathbb{C}[\boldsymbol{\xi}]$. This proves the theorem.

References

1. Arnol'd, V.I., Indexes of singular points of 1-forms on manifolds with boundary, convolutions of invariants of groups generated by reflections, and singular projections of smooth surfaces. *Uspekhi Mat. Nauk* **34** (1979), pp. 3–38. MR 81e:58041
2. Chevalley, C., Invariants of finite groups generated by reflections. *Amer. J. Math.* **77** (1955), pp. 778-782. MR 17:345d
3. Flato, L., Invariants of finite reflection groups. *Enseign. Math.* **24** (1978), pp. 237–292. MR 80j:14010
4. Hermann, R. , Geodesics of singular Riemannian metrics. *Bull. Amer. Math. Soc.* **79** (1973), pp. 780–782. MR 50:3271
5. Milson, R., Imprimitively generated Lie-algebraic Hamiltonians and separation of variables. *Canad. J. Math.* **50** (1998), pp. 1298–1322. MR 2000a:81246
6. Saito, K., T. Yano and J. Sekiguchi., On a certain generator system of the ring of invariants of a finite reflection group. *Comm. Algebra* **8** (1980), pp. 373–408. MR 81g:20095
7. Shephard, G.C. and J.A. Todd., Finite unitary reflection groups. *Canad. J. Math.* **6** (1954), pp. 274-304. MR 15:600b
8. Terng, C.L. , Isoparametric submanifolds and their Coxeter groups. *J. Differential Geom.* **21** (1985), pp. 79–107. MR 87e:53095

Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

E-mail address: milson@mscs.dal.ca