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A PROOF OF W. T. GOWERS' c_0 THEOREM

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ABSTRACT. W. T. Gowers' c_0 theorem asserts that for every Lipschitz function $F : S_{c_0} \to \mathbb{R}$ and $\varepsilon > 0$, there exists an infinite-dimensional subspace Y of c_0 such that the oscillation of F on S_Y is at most ε. The proof of this theorem has been reduced by W. T. Gowers to the proof of a new Ramsey type theorem. Our aim is to present a proof of the last result.

1. Introduction and notation

In this paper we give a proof of W. T. Gowers' c_0 theorem (see [G1], [BL] and [G2]). As W. T. Gowers has shown, the theorem is essentially purely combinatorial and reduces to a Ramsey type result on the space of finitely supported functions $f : \mathbb{N} \to \{-k, ..., k\}$. The proof presented here is based on an abstract lemma, implicitly contained in Gowers' proof and which may be considered as an extension of the well-known lemma of the existence of an idempotent in a compact semigroup. Both the positive and the general case of the theorem are naturally derived from it. At the end of the paper we briefly describe some Hales-Jewett versions of the above combinatorial results.

Notation. Let us fix some notation. By $\mathbb{N} = \{0, 1, 2, \dots\}$ we denote the set of all non-negative integers. For $k \geq 1$ we define

$$X_k = \big\{ f : \mathbb{N} \to \{0, ..., k\} : \operatorname{supp} f \text{ is finite and } \exists n \in \mathbb{N} \text{ with } f(n) = k \big\} \text{ and}$$
$$X_{\pm k} = \big\{ f : \mathbb{N} \to \{-k, ..., k\} : \operatorname{supp} f \text{ is finite and } \exists n \in \mathbb{N} \text{ with } |f(n)| = k \big\}.$$

By θ we denote the constant zero map $\theta : \mathbb{N} \to \{0\}$ and we set

$$X_{[k]} = \bigcup_{i=0}^{k} X_i \quad \text{and } X_{[\pm k]} = \bigcup_{i=0}^{k} X_{\pm i}, \quad \text{where } X_0 = \{\theta\}.$$

For $f_1, f_2 \in X_{[\pm k]} \setminus X_0$, we write $f_1 < f_2$ if $\max \operatorname{supp} f_1 < \min \operatorname{supp} f_2$. A sequence $\vec{f} = (f_n)_n$ in $X_{[\pm k]}$ is called a block sequence if $f_n < f_{n+1}$ for every n.

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Let $T : X_{[\pm k]} \to X_{[\pm k]}$ be defined by

$$Tf(n) = \begin{cases} f(n)-1 & \text{if } f(n)>0, \\ f(n)+1 & \text{if } f(n)<0, \\ 0 & \text{if } f(n)=0. \end{cases}$$

For a block sequence $\vec{f} = (f_n)_n$ in X_k the *span* of $\vec{f}$ in X_k is the set

$$\langle \vec{f} \rangle_k = \big\{ T^{\epsilon_0} f_{n_0} + ... + T^{\epsilon_m} f_{n_m} \ : \ n_0 < ... < n_m, \ \epsilon_0, ..., \epsilon_m \in \{0, ..., k-1\} \text{ and } \exists i \in \{0, ..., m\} \text{ with } \epsilon_i = 0 \big\},$$

and for a block sequence $\vec{f} = (f_n)_n$ in $X_{\pm k}$ the *span* of $\vec{f}$ in $X_{\pm k}$ is the set

$$\langle \vec{f} \rangle_{\pm k} = \big\{ \pm T^{\epsilon_0} f_{n_0} \pm ... \pm T^{\epsilon_m} f_{n_m} \ : \ n_0 < ... < n_m, \ \epsilon_0, ..., \epsilon_m \in \{0, ..., k-1\} \text{ and } \exists i \in \{0, ..., m\} \text{ with } \epsilon_i = 0 \big\}.$$

A *coloring* of a set X is a partition of X into disjoint pieces, and a subset of X is called *monochromatic* if it is contained in one of these pieces. Under the above notation the positive case of W. T. Gowers' theorem is stated as follows.

Theorem 1. *For every finite coloring of X_k, $k \geq 1$, there exists an infinite block sequence $\vec{f}$ in X_k such that $\langle \vec{f} \rangle_k$ is monochromatic.*

For the general case we need one more definition. Two elements f_1, f_2 of $X_{[\pm k]}$ are called *neighbours* if $\|f_1 - f_2\|_\infty \leq 1$. For a set $C \subseteq X_{[\pm k]}$, by $\widehat{C}$ we denote the set of all neighbours of elements of C, i.e. $\widehat{C} = \{f \in X_{[\pm k]} : \exists g \in C \text{ such that } \|f-g\|_\infty \leq 1\}$.

Finally, if $X_{[\pm k]} = C_0 \cup \dots \cup C_p$ is a finite coloring of $X_{[\pm k]}$, then a set $A \subseteq X_{[\pm k]}$ is called *approximately monochromatic* if there exists an i such that $A \subseteq \widehat{C}_i$. The general and most interesting case of W. T. Gowers' theorem is the following.

Theorem 2. *For every finite coloring of $X_{\pm k}$, $k \geq 1$, there exists an infinite block sequence $\vec{f}$ in $X_{\pm k}$ such that $\langle \vec{f} \rangle_{\pm k}$ is approximately monochromatic.*

As has been shown by Gowers the above result yields the c_0 theorem. For the sake of completeness we include the proof here. Let F be a real-valued Lipschitz (or more generally a uniformly continuous) function on the unit sphere of c_0. Fix $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $|F(x) - F(y)| \leq \varepsilon/5$, for all $x, y \in S_{c_0}$ with $\|x-y\| \leq \delta$. Choose k with $(1+\delta)^{-(k-1)} < \delta$ and define Δ to be the set of all finitely supported vectors of S_{c_0} all of whose coordinates belong to $\{0\} \cup \{\pm(1+\delta)^{-(k-i)} : i = 1, \dots, k\}$. Then Δ is in one-to-one correspondence to $X_{\pm k}$ via the map $G : X_{\pm k} \to \Delta$, defined by $G(f) = f'$ where $f'(n) = (1+\delta)^{-(k-i)}$ if $f(n) = i > 0$, $f'(n) = -(1+\delta)^{-(k+i)}$ if $f(n) = i < 0$ and $f'(n) = 0$ if $f(n) = 0$. Since F is bounded, for the given $\varepsilon > 0$, there exist $I_0, \dots, I_p$ disjoint intervals of length $\varepsilon/5$ that cover the range of F. Define a coloring $\{C_r : r = 1, \dots, p\}$ of $X_{\pm k}$ by putting f in C_r iff $F(f') \in I_r$.

By Theorem 2, there exists an infinite block sequence $\vec{f} = (f_n)_n$ in $X_{\pm k}$ and an r_0 such that $\langle \vec{f} \rangle_{\pm k} \subseteq \widehat{C}_{r_0}$. By the choice of k, for f, g neighbours of $X_{\pm k}$, we have that $\|f' - g'\| < \delta$, and this gives that the oscillation of F on $G(\langle \vec{f} \rangle_{\pm k})$ is at most $3\varepsilon/5$. We set $\vec{f'} = (f'_n)_n$, and let Y denote the linear span of $\vec{f'}$ in c_0. Then

$G\big(\langle \vec{f}\,\rangle_{\pm k}\big)$ is easily seen to be δ-dense in S_Y, and so the oscillation of F on S_Y is at most ε.

2. A lemma on compact semigroups

A *compact semigroup* is a semigroup $(S,+)$ with a topology with respect to which S is a compact Hausdorff space and the map $t \to t+s$, $t \in S$, is continuous for each $s \in S$. An element $\alpha \in S$ is called *idempotent* if $\alpha+\alpha=\alpha$. The following lemma due to R. Ellis is fundamental in the theory of compact semigroups.

Lemma 3. *If $(S,+)$ is a compact semigroup, then S contains an idempotent.*

Proof. Let $\mathcal{C}=\{S' \subseteq S : \ S' \text{ is a compact subsemigroup of } S\}$ ordered by inclusion. By Zorn's lemma there exists a minimal element M of $\mathcal{C}$. Pick an $\alpha \in M$ and observe that $M+\alpha$ is a compact subsemigroup of M and so $M+\alpha=M$. Hence the set $M'=\{\alpha' \in M : \ \alpha'+\alpha=\alpha\}$ is non-empty. Notice that M' is a compact subsemigroup of M, and so again by the minimality of M, we get that $M'=M$. Therefore $\alpha \in M'$ and $\alpha+\alpha=\alpha$. □

A *two-sided ideal* of a semigroup $(S,+)$, is a subset I of S such that $S+I \subseteq I$ and $I+S \subseteq I$. On the set of all idempotents of a semigroup the following partial order is defined:

$$\alpha \leq \beta \Longleftrightarrow \alpha+\beta=\beta+\alpha=\alpha.$$

Under the above notation a sequence $\alpha_0,\alpha_1,\alpha_2,\dots$ of idempotents will be called *increasing* if $\alpha_n \leq \alpha_{n+1}$ for all n. For $k=0,1,\dots$ and a map $T:S\to S$ we say that T is *k-stabilized* if $T^k=T^{k+1}$ or equivalently $T^k=T^{k+m}$, for all $m \geq 1$ (where by convention T^0 is the identity on S). Similarly a sequence $(\alpha_n)_n$ in S is called k-stabilized if $\alpha_k=\alpha_{k+m}$ for all $m\geq 1$. Notice that a map $T:S\to S$ is 0-stabilized if and only if T is the identity and that a sequence $(\alpha_n)_n$ is 0-stabilized if and only if it is constant.

Although the basic arguments below are essentially due to W. T. Gowers, the next lemma is inspired by a presentation of Gowers' theorem due to S. Todorcevic [T].

Lemma 4. *Let $(S,+)$ be a compact semigroup, I a closed two-sided ideal of S, $k=0,1,\dots$, and $T:S\to S$ a continuous k-stabilized endomorphism. Then there exists an $\alpha\in I$ such that $\alpha, T\alpha, T^2\alpha,\dots$ is an increasing (k-stabilized) sequence of idempotents of $(S,+)$.*

Proof. We proceed by induction on k. If $k=0$, then T is the identity on S and since I is a compact subsemigroup of S the result follows trivially by the previous lemma.

Suppose that the result is true for some $k\geqslant 0$ and let T be a continuous $(k+1)$-stabilized endomorphism of S. Let $S'=T(S)$, $I'=T(I)$ and notice that $(S',+)$ is a compact semigroup, I' is a two-sided closed ideal of S' and the restriction of T to $(S',+)$ is a continuous k-stabilized endomorphism of S'. By our inductive assumption there exists an idempotent $\alpha'\in I'$ such that the sequence $(T^n\alpha')_n$ is an increasing k-stabilized sequence of idempotents of $(S,+)$. We set

$$G=\{\beta\in I : \ T\beta=\alpha'\}+\alpha'$$

and we claim that $(G,+)$ is a semigroup. Indeed, let $\beta_1+\alpha'$, $\beta_2+\alpha'$ be in G. We have that $(\beta_1+\alpha')+(\beta_2+\alpha')=(\beta_1+\alpha'+\beta_2)+\alpha'$ and since $(T^n\alpha')_n$ is increasing,

$\alpha' \leq T\alpha'$. Hence, $T(\beta_1 + \alpha' + \beta_2) = T\beta_1 + T\alpha' + T\beta_2 = \alpha' + T\alpha' + \alpha' = \alpha'$, which gives that $(G, +)$ is a semigroup.

By the continuity properties of T and $+$, the semigroup $(G, +)$ is also compact. Hence by Lemma 3, the compact semigroup $(G, +)$ contains an idempotent $\gamma + \alpha'$ with $\gamma \in I$ and $T\gamma = \alpha'$. We put $\alpha = \alpha' + \gamma + \alpha'$, and it is easily checked that α is an idempotent in I, $\alpha \leq \alpha'$ and $T\alpha = \alpha'$. Therefore $T^{n+1}\alpha = T^n\alpha'$ for all n, and it is straightforward that $(T^n\alpha)_n$ is an increasing $(k+1)$-stabilized sequence of idempotents of $(S, +)$. □

3. The proof of W. T. Gowers' c_0 theorem

We will apply Lemma 4 to certain semigroups and two-sided ideals of ultrafilters. Before this let us start by recalling certain basic facts concerning ultrafilters. An *ultrafilter* α on a set X is a maximal filter on X, that is, a family of subsets of X that satisfies the following properties: (i) $\varnothing \notin \alpha$, (ii) if $A \in \alpha$ and $A \subseteq B \subseteq X$, then $B \in \alpha$, (iii) if $A, B \in \alpha$, then $A \cap B \in \alpha$, (iv) for every $A \subseteq X$, either A or $X \setminus A$ belongs to α. The set of all ultrafilters on X will be denoted by βX. An ultrafilter α is called *principal* if there exists $x \in X$ such that $\alpha = \{A \subseteq X : x \in A\}$. The principal ultrafilter determined by the element $x \in X$ is usually denoted by x^*. The set X is naturally identified with the set of principal ultrafilters via the map $X \ni x \to x^* \in \beta X$.

The set βX becomes a compact Hausdorff topological space, with basis the family of sets $A^* = \{\alpha \in \beta X : A \in \alpha\}$ where A ranges over all non-empty subsets of X. With this topology X is a dense open subset of βX. Of particular importance is the following property of βX. Whenever $\sigma : X \to X'$ is a map, then σ has a continuous extension $\beta\sigma : \beta X \to \beta X'$ defined by

$$\beta\sigma(\alpha) = \{A' \subseteq X' : \sigma^{-1}(A') \in \alpha\}.$$

It is easy to see that if $\sigma : X \to X'$ and $\tau : X' \to X''$, then $\beta(\tau \circ \sigma) = \beta\tau \circ \beta\sigma$. In particular for a map $\sigma : X \to X$ we have that $\beta\sigma^j = (\beta\sigma)^j$ for every j. In the sequel we shall denote the extension $\beta\sigma$ also by σ and we will write $\sigma\alpha$ instead of $\sigma(\alpha)$.

Besides the above set-theoretic description of βX there is another one which will be very useful for our purposes. To every ultrafilter α on X, we associate the quantifier (αx) reading as *"for α-almost $x \in X$"*, in the following sense. If $P(x)$ is a statement concerning the elements $x \in X$, we write

$$(\alpha x)P(x) \iff \{x \in X : P(x)\} \in \alpha,$$

that is, "for α-almost $x \in X$, $P(x)$ is true" if and only if the set of all x that satisfy $P(x)$ belongs to the ultrafilter α. Substituting $P(x)$ with $(x \in A)$, we get that

$$(\alpha x)(x \in A) \iff A \in \alpha.$$

Standard properties of ultrafilters yield that the quantifier (αx) commutes with conjunction $(\wedge)$ and negation $(\neg)$, namely

$$(\alpha x)P(x) \wedge (\alpha x)Q(x) \iff (\alpha x)(P(x) \wedge Q(x))$$

and

$$\neg(\alpha x)P(x) \iff (\alpha x)(\neg P(x)).$$

For example, if $\sigma : X \to X'$, then for every $\alpha \in \beta X$ the associated to $\sigma\alpha$ quantifier is defined by

$$(\sigma\alpha x')P(x') \iff (\alpha x)P(\sigma(x))$$

or equivalently,

$$A' \in \sigma\alpha \iff (\sigma\alpha x')(x' \in A') \iff (\alpha x)(\sigma(x) \in A').$$

Another important fact concerning βX is that it inherits the algebraic properties of X. Specifically if X is a semigroup, then it can be shown that βX is a compact semigroup. The classical example is the compact semigroup $(\beta\mathbb{N}, +)$. In our case the set $X_{[k]} = \bigcup_{i=0}^{k} X_i$, $k \geq 1$, with the usual operation $+$, although it is not a semigroup it is a partial semigroup, where $f + g$ is defined if either f or g are equal to $\theta \equiv 0$, or otherwise if $\max \operatorname{supp} f < \min \operatorname{supp} g$. However this is enough to guarantee that $\beta X_{[k]}$ contains a compact semigroup. To be more precise let us set

$$X^n_{[k]} = \{f \in X_{[k]} : f(0) = ... = f(n) = 0\}$$

for every $n = 0, 1, \ldots$. Then the family $\{X^n_{[k]}\}_n$ has the finite intersection property and therefore we can define

$$\delta X_{[k]} = \{\alpha \in \beta X_{[k]} :\ X^n_{[k]} \in \alpha,\ \text{for every}\ \ n\}.$$

Observe that $\delta X_{[k]} = \bigcap_n (X^n_{[k]})^*$, and so $\delta X_{[k]}$ is a non-empty compact subspace of $\beta X_{[k]}$. Also since $\theta \in X^n_{[k]}$ for all n, the principal ultrafilter $\theta^* = \{A \subseteq X_{[k]} :$ A contains the function $\theta \equiv 0\}$ belongs to $\delta X_{[k]}$ and moreover it is the only principal ultrafilter in $\delta X_{[k]}$.

On $\delta X_{[k]}$ we define the addition $+$ by

$$\alpha + \beta = \{A \subseteq X_{[k]}\ :\ (\alpha f)(\beta g)(f + g \text{ is defined and } f + g \in A)\}$$

or equivalently,

$$A \in \alpha + \beta \iff \{f \in X_{[k]} :\ A_f \in \beta\} \in \alpha$$

where $A_f = \{g \in X_{[k]} :\ \ f + g\ \text{ is defined and }\ f + g \in A\}$. The following three propositions are essentially from [BBH] where they are stated and proved in greater generality.

Proposition 5. *The set $\delta X_{[k]}$ with the operation $+$ is a compact semigroup and θ^* is its unit element.*

Proof. We leave to the reader the verification that $\alpha + \beta$ is indeed an ultrafilter, the operation $+$ is associative and the map $\alpha \to \alpha + \beta$, $\alpha \in \delta X_k$ is continuous for any $\beta \in \delta X_k$ that is obtained in a similar manner as in the case of $\beta\mathbb{N}$.

We show the rest of the conclusions. We begin by showing that $\alpha + \beta \in \delta X_k$ for $\alpha, \beta \in \delta X_k$. By the definition of $\delta X_{[k]}$, it is enough to check that $X^n_{[k]} \in \alpha + \beta$ for all n. Fix n and set for simplicity $A = X^n_{[k]}$. Then for every $f \in A$, we have that $A_f \in \beta$. Indeed, if $f = \theta$, then $A_f = A$; otherwise $A_f = X^{n_0}_{[k]}$ where $n_0 = \max \operatorname{supp} f$. Therefore $A \subseteq \{f \in X_{[k]} :\ A_f \in \beta\}$, and since $A \in \alpha$, we obtain that $\{f \in X_{[k]} :\ A_f \in \beta\} \in \alpha$, which by definition means that $A \in \alpha + \beta$.

To show that $\alpha = \alpha + \theta^*$, observe that $A \in \alpha \implies \forall f \in A,\ \theta \in A_f \iff \forall f \in A,\ A_f \in \theta^* \iff A \subseteq \{f \in X_{[k]} :\ A_f \in \theta^*\} \implies \{f \in X_{[k]} :\ A_f \in \theta^*\} \in \alpha \iff A \in \alpha + \theta^*$. Hence $\alpha \subseteq \alpha + \theta^*$, and since both are ultrafilters $\alpha = \alpha + \theta^*$. The equality $\theta^* + \alpha = \alpha$ is similarly shown. □

We now describe the properties of the extension of $T : X_{[k]} \to X_{[k]}$ to $\beta X_{[k]}$. Observe that if $f+g$ is defined, then $Tf+Tg$ is also defined (the converse fails) and $T(f+g) = Tf+Tg$, that is, T is an endomorphism of the partial semigroup $(X_{[k]}, +)$.

Proposition 6. *The extension of $T : X_{[k]} \to X_{[k]}$ to $T : \beta X_{[k]} \to \beta X_{[k]}$ maps $\delta X_{[k]}$ into $\delta X_{[k]}$, and its restriction to $(\delta X_{[k]}, +)$ is a continuous k-stabilized endomorphism.*

Proof. First let us verify that $T(\alpha+\beta) = T\alpha + T\beta$ for $\alpha, \beta \in \delta X_{[k]}$. Indeed,

$$\begin{aligned}
A \in T(\alpha+\beta) &\iff T^{-1}(A) \in \alpha+\beta \\
&\iff (\alpha f)(\beta g)(f+g \text{ is defined and } f+g \in T^{-1}(A)) \\
&\iff (\alpha f)(\beta g)(f+g \text{ is defined and } T(f+g) \in A) \\
&\iff (\alpha f)(\beta g)(f+g \text{ is defined and } Tf+Tg \in A) \\
&\implies (\alpha f)(\beta g)(Tf+Tg \text{ is defined and } Tf+Tg \in A) \\
&\iff (T\alpha f')(T\beta g')(f'+g' \text{ is defined and } f'+g' \in A) \\
&\iff A \in T\alpha + T\beta.
\end{aligned}$$

Hence $T(\alpha+\beta) \subseteq T\alpha + T\beta$, and so $T(\alpha+\beta) = T\alpha+T\beta$. To see that T maps $\delta X_{[k]}$ into itself, observe that $X_{[k]}^n \subseteq T^{-1}(X_{[k]}^n)$ for all n and so $T\alpha \in \delta X_{[k]}$ for every $\alpha \in \delta X_{[k]}$. To show that T is k-stabilized notice that for every $f \in X_{[k]}$, $T^{k+1}f = T^k f = \theta$, which easily gives $T^{k+1}\alpha = T^k\alpha = \theta^*$ for all $\alpha \in \beta X_{[k]}$. Finally the continuity of T follows from the continuity of $T : \beta X_{[k]} \to \beta X_{[k]}$. □

We set

$$X_k^n = \{f \in X_k : f(0) = ... = f(n) = 0\}$$

for every $n = 0, 1, \ldots$ and we define

$$\delta X_k = \{\alpha \in \beta X_k : \ X_k^n \in \alpha \text{ for every } n\}.$$

Again by the finite intersection property of the family $\{X_k^n\}_n$, the set δX_k is a compact non-empty subset of βX_k. As in the case of $\delta X_{[k]}$, it can be shown that $(\delta X_k, +)$ is a compact semigroup with the operation $+$ having an analogous meaning. Notice that for $f, g \in X_{[k]}$, $f+g \in X_k$ whenever $f+g$ is defined and f or $g \in X_k$ i.e. X_k is a two-sided ideal of the partial semigroup $(X_{[k]}, +)$. This property is carried over to δX_k in the following sense.

Proposition 7. *The extension of the inclusion map $I : X_k \hookrightarrow X_{[k]}$ to $I : \beta X_k \to \beta X_{[k]}$ maps δX_k into $\delta X_{[k]}$, and its restriction to $(\delta X_k, +)$ is a continuous isomorphism onto a (closed) two-sided ideal of $(\delta X_{[k]}, +)$.*

Proof. As in Proposition 6, it can be checked that $I : \beta X_k \to \beta X_{[k]}$ maps $(\delta X_k, +)$ isomorphically into $(\delta X_{[k]}, +)$. Observe that for an ultrafilter $\beta \in \delta X_{[k]}$, we have that

$$\beta \in I(\delta X_k) \iff X_k \in \beta.$$

Since X_k is a two-sided ideal of $X_{[k]}$, it is easy to see that for $\alpha \in \delta X_{[k]}$ and $\beta \in I(\delta X_k)$, $X_k \in \alpha+\beta$, and so by the above remark $\alpha+\beta \in I(\delta X_k)$. Similarly it is shown that $\beta+\alpha \in I(\delta X_k)$, and so $I(\delta X_k)$ is a two-sided ideal of $\delta X_{[k]}$. □

In the sequel we will identify δX_k with the closed two-sided ideal $I(\delta X_k)$ of $(\delta X_{[k]}, +)$.

Finally, the semigroups $(\delta X_{[\pm k]}, +)$ and $(\delta X_{\pm k}, +)$ are defined analogously, and it is easy to see that they satisfy what corresponds to the above propositions. Since for every $f \in X_{[k]}$ and every $\beta \in \delta X_{[k]}$, we have that $(\beta g)(f + g$ is defined), in what follows we simply write $(\alpha f)(\beta g)(f + g \in A)$ to denote that $A \in \alpha + \beta$ for $\alpha, \beta \in \delta X_{[k]}$ (and similarly for $\delta X_{[\pm k]}$).

We are now ready to prove W. T. Gowers' theorem.

Proof of Theorem 1. We set

$$\mathcal{I}(\delta X_k, T) = \{\alpha \in \delta X_k : \alpha + T^j\alpha = T^j\alpha + \alpha = \alpha \quad \text{for every } \ j = 0, ..., k\}.$$

By the above propositions and Lemma 4, for $S = \delta X_{[k]}$ and $I = \delta X_k$, we obtain that

$$\mathcal{I}(\delta X_k, T) \neq \emptyset,$$

and so we can choose an $\alpha \in \mathcal{I}(\delta X_k, T)$.

Lemma 8. *For every $A \in \alpha$ there exists a block sequence $\vec{f}$ in X_k such that $\langle \vec{f} \rangle_k \subseteq A$.*

Proof. Since $\alpha \in \mathcal{I}(\delta X_k, T)$ we have that for every $A \in \alpha$,

$$(\alpha f)(\alpha g)(\{f, g, f + T^j g, T^j f + g : j = 0, 1, ..., k-1\} \subseteq A)$$

or equivalently,

$$(1) \qquad (\alpha f)(\alpha g)\big(\langle f, g\rangle_k \subseteq A\big).$$

By induction pick $A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \dots$ and $f_0 < f_1 < \cdots < f_n < \dots$ such that $\forall n$, $A_n \in \alpha$, $f_n \in A_n$ and $A_{n+1} = \{g \in X_k : \langle f_n, g\rangle_k \subseteq A_n\}$.

To start set $A_0 = A$ and by (1) choose an $f_0 \in X_k$ so that $(\alpha g)(\langle f_0, g\rangle_k \subseteq A_0)$ and put $A_1 = \{g \in X_k : \langle f_0, g\rangle_k \subseteq A_0\}$. Then $f_0 \in A_0$, $A_1 \in \alpha$ and $A_0 \supseteq A_1$. Suppose that $A_0, \dots, A_n$ and $f_0, \dots, f_{n-1}$ have been constructed so as to satisfy the required properties. Since A_n belongs to α, (1) holds for A_n in place of A, and so we can pick an $f_n \in X_k$ so that $f_{n-1} < f_n$ and $(\alpha g)(\langle f_n, g\rangle_k \subseteq A_n)$. We set $A_{n+1} = \{g \in X_k : \langle f_n, g\rangle_k \subseteq A_n\}$ and clearly we have that $A_{n+1} \in \alpha$, $f_n \in A_n$, $A_{n+1} \subseteq A_n$, and so the induction can be carried out. By downwards induction it is easily checked that $\langle f_i, \dots, f_n\rangle_k \subseteq A_i$, for all $i \leq n$ and so $\langle f_0, f_1, \dots, f_n\rangle_k \subseteq A_0 = A$, for all n. □

Since one of the pieces of every finite coloring of X_k belongs to α the proof of Theorem 1 is complete. □

Proof of Theorem 2. The main idea here is to apply Lemma 4 for $S = \delta X_{[\pm k]}$, $I = \delta X_{\pm k}$ and $-T$ instead of T defined by $-Tf = -(Tf) = T(-f)$ for all $f \in X_{[\pm k]}$.

We set

$$\mathcal{I}(\delta X_{\pm k}, -T) = \{\alpha \in \delta X_{\pm k} : \alpha + (-T)^j\alpha = (-T)^j\alpha + \alpha = \alpha \text{ for every } \ j = 0, ..., k\}.$$

It is easily shown that $-T$ satisfies the obvious analogue of Proposition 6, and so Lemma 4 gives that for every $k \geq 1$,

$$\mathcal{I}(\delta X_{\pm k}, -T) \neq \emptyset.$$

For a block sequence $\vec{f} = (f_n)_n$ in $X_{\pm k}$ we define

$$\langle \vec{f} \rangle_{(-T)} = \{(-T)^{\epsilon_0} f_{n_0} + ... + (-T)^{\epsilon_m} f_{n_m} :\ 0 \le \epsilon_i \le k-1,\ \ n_0 < ... < n_m \text{ and } \exists i \text{ with } \epsilon_i = 0\}.$$

Lemma 9. *For a symmetric set $A \subseteq X_{\pm k}$ (i.e. $A = -A$) and a block sequence $\vec{f}$ in $X_{\pm k}$, if $\langle \vec{f} \rangle_{(-T)} \subseteq A$, then $\langle \vec{f} \rangle_{\pm k} \subseteq \widehat{A}$.*

Proof. Let $f \in \langle \vec{f} \rangle_{\pm k}$. Then there exist $m \in \mathbb{N}$, $n_0 < \cdots < n_m$, $\epsilon_0, \ldots, \epsilon_m \in \{0, \ldots, k-1\}$, with $\epsilon_i = 0$ for some i and $\delta_0, \ldots, \delta_m \in \{-1, +1\}$, such that

$$f = \delta_0 T^{\epsilon_0} f_{n_0} + ... + \delta_m T^{\epsilon_m} f_{n_m}.$$

We distinguish two cases:

Case 1. For some $i \in \{0, \ldots, m\}$, $\epsilon_i = 0$ and $\delta_i = +1$. Then for every $i \in \{0, \ldots, m\}$ we set $g_i = \delta_i T^{\epsilon_i} f_{n_i}$ and

(a) If $\delta_i = -1$ and ϵ_i is odd, or $\delta_i = +1$ and ϵ_i is even, we set $g_i' = g_i$. Observe that in these subcases $g_i' = g_i = (-T)^{\epsilon_i} f_{n_i}$.

(b) If $\delta_i = -1$ and ϵ_i is even, or $\delta_i = +1$ and ϵ_i is odd, we set $g_i' = Tg_i$. Then it is easy to see that $g_i' = (-T)^{\epsilon_i + 1} f_{n_i}$.

In both subcases, by the choice of g_i', we have that g_i', g_i are neighbours for every $i = 0, ..., m$. We set $f' = g_0' + \cdots + g_m'$. Then $f' \in \langle \vec{f} \rangle_{(-T)} \subseteq A$ and $\|f - f'\|_\infty \le 1$. Hence $f \in \widehat{A}$.

Case 2. For every $i \in \{0, \ldots, m\}$ if $\epsilon_i = 0$, then $\delta_i = -1$. Let $g = -f$. Then the representation of g satisfies the condition of Case 1, and so there exists $g' \in \langle \vec{f} \rangle_{(-T)}$ such that $\|g - g'\|_\infty \le 1$. Since $\langle \vec{f} \rangle_{(-T)} \subseteq A$ and A is symmetric we have that $h = -g' \in A$ and $\|f - h\|_\infty = \|g - g'\|_\infty \le 1$.

Therefore for every $f \in \langle \vec{f} \rangle_{\pm k}$, we have that $f \in \widehat{A}$, that is, $\langle \vec{f} \rangle_{\pm k} \subseteq \widehat{A}$. □

In the following two lemmas, we assume that $\alpha \in \mathcal{I}(\delta X_{\pm k}, -T)$.

Lemma 10. *For every $A \in \alpha$ there exists a block sequence $\vec{f}$ in $X_{\pm k}$ such that $\langle \vec{f} \rangle_{(-T)} \subseteq A$.*

Proof. This is identical to that of Lemma 8. □

Lemma 11. *For every $A \in \alpha$, $-\widehat{A} \in \alpha$.*

Proof. Since $\alpha \le -T\alpha$, we have that

$$\begin{aligned} A \in \alpha = -T\alpha + \alpha \quad &\Longleftrightarrow \quad (\alpha f)(\alpha g)(-Tf + g \in A) \\ &\Longrightarrow \quad (\alpha f)(\alpha g)(-f + Tg \in \widehat{A}) \text{ (since } f < g) \\ &\Longleftrightarrow \quad (\alpha f)(\alpha g)(f - Tg \in -\widehat{A}) \\ &\Longleftrightarrow \quad -\widehat{A} \in \alpha - T\alpha = \alpha. \end{aligned}$$

□

For $A \subseteq X_{\pm k}$, by $\widehat{\widehat{A}}$ we denote the set of all neighbours of the elements of $\widehat{A}$. It is straightforward that $\widehat{\widehat{A}} = \{f \in X_{[\pm k]} :\ \exists g \in A \text{ such that } \|f - g\|_\infty \le 2\}$. The folllowing proposition is a weak version of Theorem 2.

Proposition 12. *For every $\alpha \in \mathcal{I}(\delta X_{\pm k}, -T)$ and every $A \in \alpha$, there exists a block sequence $\vec{f}$ in $X_{\pm k}$ such that $\langle \vec{f} \rangle_{\pm k} \subseteq \widehat{\widehat{A}}$.*

Proof. Applying Lemma 11 we get that the set $A' = \widehat{A} \cap -\widehat{A}$ belongs to α. By Lemma 10 there exists a block sequence $\vec{f}$ in $X_{\pm k}$ such that $\langle \vec{f} \rangle_{(-T)} \subseteq A'$, and since A' is symmetric Lemma 9 gives that $\langle \vec{f} \rangle_{\pm k} \subseteq \widehat{A'}$. Since $A' \subseteq \widehat{A}$ we have that $\langle \vec{f} \rangle_{\pm k} \subseteq \widehat{\widehat{A}}$. $\square$

Remark 1. Notice that the above weak version of Theorem 2 suffices to prove that every real-valued Lipschitz function on S_{c_0} is oscillation stable.

To complete the proof we define the map $\Phi : X_{\pm 2k} \to X_{\pm k}$, where for every $f \in X_{\pm 2k}$ and $n \in \mathbb{N}$, $\Phi f(n)$ is defined as follows:

$$\Phi f(n) = \begin{cases} \frac{f(n)}{2} & \text{if } f(n) \text{ is even,} \\ \frac{f(n)-1}{2} & \text{if } f(n) > 0 \text{ and } f(n) \text{ is odd,} \\ \frac{f(n)+1}{2} & \text{if } f(n) < 0 \text{ and } f(n) \text{ is odd.} \end{cases}$$

The following properties of Φ are straightforward:

(i) It is onto $X_{\pm k}$.
(ii) For every $f_1 < f_2$ in $X_{\pm 2k}$, $\Phi(f_1 + f_2) = \Phi f_1 + \Phi f_2$.
(iii) For every $f \in X_{\pm 2k}$, $\Phi(-f) = -(\Phi f)$.
(iv) For every $f_1 < f_2$ in $X_{\pm 2k}$ and every $\epsilon_1, \epsilon_2 \in \{0, \dots, k\}$ with $\epsilon_i = 0$ for at least one $i \in \{1, 2\}$, $T^{\epsilon_1}(\Phi f_1) + T^{\epsilon_2}(\Phi f_2) = \Phi(T^{2\epsilon_1} f_1 + T^{2\epsilon_2} f_2)$.
(v) For every $f_1, f_2 \in X_{\pm 2k}$, if $\|f_1 - f_2\|_\infty \leq 2$, then $\|\Phi f_1 - \Phi f_2\|_\infty \leq 1$.

We recall that Φ is extended to $\Phi : \beta X_{\pm 2k} \to \beta X_{\pm k}$, where for every $\alpha \in \beta X_{\pm 2k}$, $\Phi\alpha = \{A' \subseteq X_{\pm k} : \ \Phi^{-1}(A') \in \alpha\}$.

Lemma 13. *For every $\alpha \in \mathcal{I}(\delta X_{\pm 2k}, -T)$ and every $A' \in \Phi\alpha$ there exists a block sequence $\vec{f'}$ in $X_{\pm k}$ such that $\langle \vec{f'} \rangle_{\pm k} \subseteq \widehat{A'}$.*

Proof. By the definition of $\Phi\alpha$ we have that $A = \Phi^{-1}(A') \in \alpha$. By Proposition 12 there exists a block sequence $\vec{f}$ in $X_{\pm 2k}$ such that $\langle \vec{f} \rangle_{\pm 2k} \subseteq \widehat{\widehat{A}}$. We set $f'_n = \Phi f_n$ and $\vec{f'} = (f'_n)_n$. It is clear that $\vec{f'}$ is a block sequence in $X_{\pm k}$. We will show that $\langle \vec{f'} \rangle_{\pm k} \subseteq \widehat{A'}$.

So let $f' \in \langle \vec{f'} \rangle_{\pm k}$. Then there exist $m \in \mathbb{N}$, $n_0 < \cdots < n_m$, $\epsilon_0, \dots, \epsilon_m \in \{0, \dots, k-1\}$, with $\epsilon_i = 0$ for some i and $\delta_o, \dots, \delta_m \in \{-1, +1\}$, such that

$$\begin{aligned} f' &= \delta_0 T^{\epsilon_0} f'_{n_0} + ... + \delta_m T^{\epsilon_m} f'_{n_m} \\ &= \delta_0 T^{\epsilon_0} \Phi f_{n_0} + ... + \delta_m T^{\epsilon_m} \Phi f_{n_m} = \Phi(\delta_0 T^{2\epsilon_0} f_{n_0} + ... + \delta_m T^{2\epsilon_m} f_{n_m}) \end{aligned}$$

by the properties (ii)-(iv) of Φ. Setting $f = \delta_0 T^{2\epsilon_0} f_{n_0} + ... + \delta_m T^{2\epsilon_m} f_{n_m}$ we have that $f' = \Phi f$ and $f \in \langle \vec{f} \rangle_{\pm 2k}$. Since $\langle \vec{f} \rangle_{\pm 2k} \subseteq \widehat{\widehat{A}}$ where $A = \Phi^{-1}(A')$, there exists $g \in \Phi^{-1}(A')$, such that $\|f - g\|_\infty \leq 2$. We set $g' = \Phi g$ and we have that $g' \in A'$ and by the property (v) of Φ, $\|f' - g'\|_\infty = \|\Phi f - \Phi g\|_\infty \leq 1$. Hence for every $f' \in \langle \vec{f'} \rangle_{\pm k}$, we have that $f' \in \widehat{A'}$, that is, $\langle \vec{f'} \rangle_{\pm k} \subseteq \widehat{A'}$. $\square$

Since for every $\alpha \in \mathcal{I}(\delta X_{\pm 2k}, -T)$ and every finite coloring of $X_{\pm k}$ one of the pieces of the partition belongs to $\Phi\alpha$, the proof of Theorem 2 is complete. $\square$

4. Notes

1. Theorems 1 and 2 have interesting corollaries that are actually equivalent to them and they illustrate the property $\alpha \leq T\alpha \leq \cdots \leq T^k\alpha$ of the sequence $(T^n\alpha)_n$ obtained in Lemma 4. To state them we need to extend slightly the definition of the span of a block sequence as follows. For an infinite block sequence $\vec{f} = (f_n)_n$ in $X_{[k]}$ define its span in $X_{[k]}$ to be the set

$$\langle \vec{f} \rangle_{[k]} = \big\{T^{\epsilon_0} f_{n_0} + T^{\epsilon_1} f_{n_1} + ... + T^{\epsilon_m} f_{n_m} : m \in \mathbb{N}, \ \ \epsilon_i = 0, ..., k-1 \text{ and } n_0 < n_1 < ... < n_m\big\},$$

and for all $i = 1, ..., k$ define the *relative* span in X_i by $\langle \vec{f} \rangle_i = \langle \vec{f} \rangle_{[k]} \cap X_i$. We will also use the notation $T^j\vec{f} = (T^j f_n)_n$ for a block sequence $\vec{f} = (f_n)_n$ of X_k and $j = 1, \ldots, k-1$. Then $\langle \vec{f} \rangle_i = \langle T^{k-i}\vec{f} \rangle_i = T^{k-i}\big(\langle \vec{f} \rangle_k\big)$ for all $i = 1, \ldots, k$.

Given two block sequences $\vec{f}, \vec{g}$ in X_k, we say that $\vec{f} = (f_n)$ is a block subsequence of $\vec{g}$, denoted by $\vec{f} \leqslant \vec{g}$, if $f_n \in \langle \vec{g} \rangle_k$ for all n. Under the above notation we have the following corollary of Theorem 1.

Corollary 14. *Let $k \geq 1$ and suppose that for each $i = 1, \ldots, k$, the set X_i is finitely colored. Then for every block sequence $\vec{g} = (g_n)_n$ in X_k there exists an infinite block subsequence $\vec{f} \leqslant \vec{g}$ such that for each $i = 1, \ldots, k$, the span $\langle \vec{f} \rangle_i$ is monochromatic.*

Proof. Let us denote by $\vec{e} = (e_n)_n$ the block sequence in X_k consisting of the functions e_n where $e_n(m) = k$ if $n = m$ and $e_n(m) = 0$ otherwise. Notice that $X_k = \langle \vec{e} \rangle_k$ and every $f \in X_k$ has a unique representation $f = T^{\epsilon_0} e_{n_0} + T^{\epsilon_1} e_{n_1} + ... + T^{\epsilon_m} e_{n_m}$, where $\epsilon_0, \ldots, \epsilon_m \in \{0, \ldots, k-1\}$. We define the map $P : X_k \to \langle \vec{g} \rangle_k$ by $P(T^{\epsilon_0} e_{n_0} + T^{\epsilon_1} e_{n_1} + ... + T^{\epsilon_m} e_{n_m}) = T^{\epsilon_0} g_{n_0} + T^{\epsilon_1} g_{n_1} + ... + T^{\epsilon_m} g_{n_m}$.

It is easily verified that P is bijective and carries block sequences in X_k to block subsequences of $\vec{g}$. Since the coloring of $\langle \vec{g} \rangle_k$ induces via P^{-1} a corresponding coloring of X_k, Theorem 1 and the above property of P yield that there exists a block sequence $\vec{f_1}$ of X_k with $\vec{f_1} \leqslant \vec{g}$ such that $\langle \vec{f_1} \rangle_k$ is monochromatic. Using the same arguments for the case $k-1$ instead of k and the block sequence $T\vec{f_1}$ in place of $\vec{g}$, we obtain a block sequence $\vec{f_2'} \leqslant T\vec{f_1}$ such that $\langle \vec{f_2'} \rangle_{k-1}$ is monochromatic. It is now easily shown that there exists a block sequence $\vec{f_2}$ of X_k such that $\vec{f_2} \leqslant \vec{f_1}$ and $T\vec{f_2} = \vec{f_2'}$. Then $\langle \vec{f_2} \rangle_{k-1} = \langle \vec{f_2'} \rangle_{k-1}$, $\langle \vec{f_2} \rangle_k \subseteq \langle \vec{f_1} \rangle_k$ and so the relative spans of $\vec{f_2}$ in X_k and X_{k-1} are monochromatic. Repeating the same procedure, after k steps we obtain an $\vec{f} = \vec{f_k} \leqslant \vec{g}$ with monochromatic relative span in each X_i, $i = 1, \ldots, k$. □

Stating the analogous definitions concerning the general case $X_{[\pm k]}$ we get the next corollary of Theorem 2.

Corollary 15. *Let $k \geq 1$ and suppose that for each $i = 1, \ldots, k$, the set $X_{\pm i}$ is finitely colored. Then for every block sequence $\vec{g} = (g_n)_n$ in $X_{\pm k}$ there exists an infinite block subsequence $\vec{f} \leqslant \vec{g}$ such that for each $i = 1, \ldots, k$, the span $\langle \vec{f} \rangle_{\pm i}$ is approximately monochromatic.*

2. The sequence of idempotents obtained by Lemma 4 can be chosen so that all its terms $\alpha, T\alpha, \ldots$ will, in addition, be minimal. This can be done by using in the inductive step of the proof the fact that for every idempotent α and every two-sided ideal I there exists a minimal idempotent α' with $\alpha' \leq \alpha$ and $\alpha' \in I$ (see for example [FK] for a proof of this statement). However we do not see any further property resulting from the minimality of the idempotent element.

3. It is also possible to derive some Hales-Jewett versions of W. T. Gowers' combinatorial theorems in the context of variable words. Given a non-empty set Γ, a *word* over Γ is a map from a proper initial segment of $\mathbb{N}$ into Γ (i.e. a finite sequence in Γ). The set of all words over Γ, denoted by $W(\Gamma)$, with the operation of concatenation $*$ is a semigroup.

To state the first abstract version take for an alphabet a singleton set $A = \{u_0\}$ and for $k \geq 1$ let $u_1, ..., u_k$ be distinct variables. For $i = 0, \ldots, k$ we set $W_{[i]} = W(\{u_0, ..., u_i\})$ and $V_i = W_{[i]} \setminus W_{[i-1]}$, for all $i = 0, ..., k$ (where by convention $W_{[-1]} = \emptyset$). Let T be the homomorphic extension to $W_{[k]}$ of the map $u_i \to u_{i-1}$ for $i = 1, ..., k$ and $u_0 \to u_0$. For an infinite sequence $\vec{w} = (w_n)_n$ of words of V_k define its span in V_k to be the set

$$\langle \vec{w} \rangle_k = \{T^{\epsilon_0} w_{n_0} * T^{\epsilon_1} w_{n_1} * ... * T^{\epsilon_m} w_{n_m} : \epsilon_i = 0, ..., k, \;\; n_0 < n_1 < ... < n_m, \quad \text{and } \exists i \;\text{ with } \epsilon_i = 0\}.$$

Under the above definitions, we have the following analogue of Theorem 1, which is related to Carlson's results [C].

Theorem 16. *For $k \geq 0$ and every finite coloring of V_k there exists an infinite sequence $\vec{w}$ in V_k such that the span $\langle \vec{w} \rangle_k$ is monochromatic.*

Identifying $n \in \{1, 2, ...\}$ with the word $w = u_0 * ... * u_0$ (n times), the case $k = 0$ of the above theorem is Hindman's theorem for monochromatic finite sums of a sequence of positive integers. The proof of Theorem 16 is similar to that of Theorem 1 and actually it is simpler, since Lemma 4 is applied directly to $S = \beta W_k$ and $I = \beta V_k$.

We have that Theorem 1 can be derived from the above theorem. To show this we define the map $\Psi : V_k \to X_k$ for $k \geq 1$ where for every $w \in V_k$, $\Psi w(n) = i$ if $w(n) = u_i$ and $\Psi w(n) = 0$ if $w(n)$ is undefined. The coloring of X_k induces a coloring of V_k by lifting via Ψ. Let $\vec{w} = (w_n)_n$ be the resulting (by Theorem 16) sequence in V_k, and define $f_0 = \Psi w_0$ and $f_n = \Psi\big(T^k(w_0 * \cdots * w_{n-1}) * w_n\big)$, for all $n \geq 1$. It is easy to show that $\langle \vec{f} \rangle_k \subseteq \Psi\big(\langle \vec{w} \rangle_k\big)$ and so $\langle \vec{f} \rangle_k$ is monochromatic.

Other abstract versions can be derived using the concept of located words, which was introduced by V. Bergelson, A. Blass and N. Hindman in [BBH], in order to extend the results of T. Carlson in [C]. For a set Γ, a *located* word over Γ is a map from a nonempty finite subset of $\mathbb{N}$ into Γ. Note that this notion is actually a generalization of the common notion of a word over Γ since its domain is not necessarily an initial segment of $\mathbb{N}$. The set of located words over Γ, denoted by $L(\Gamma)$ becomes a partial semigroup under the operation $+$, defined by $w + w' = w \cup w'$ whenever $\mathrm{dom}(w) < \mathrm{dom}(w')$. As above let $A = \{u_0\}$ and for $k \geq 1$ let $u_1, ..., u_k$ be distinct variables. Set $L_{[i]} = L(\{u_0, ..., u_i\})$ and $L_i = L_{[i]} \setminus L_{[i-1]}$, for all $i = 0, ..., k$ (where $L_{[-1]} = \emptyset$). An infinite sequence of located words $\vec{w} = (w_n)_n$ is said to be a block sequence if $\mathrm{dom}(w_n) < \mathrm{dom}(w_{n+1})$ for every n. The span of a block sequence $\langle \vec{w} \rangle_k$ is defined as in the case of ordinary words.

Theorem 17. *For $k \geq 0$ and every finite coloring of L_k there exists an infinite block sequence $\vec{w}$ in L_k such that the span $\langle \vec{w} \rangle_k$ is monochromatic.*

One also could consider abstract versions of Theorem 2 by introducing “negative” variables $u_{-1}, ..., u_{-k}$ and defining the span of a sequence of words (or of a block sequence of located words) analogously.

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