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CELLULAR GENERATORS

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Abstract. The aim of this paper is twofold. On the one hand, we show that the kernel $\overline{C(A)}$ of the Bousfield periodization functor P_A is cellularly generated by a space B, i.e., we construct a space B such that the smallest closed class $C(B)$ containing B is exactly $\overline{C(A)}$. On the other hand, we show that the partial order $(Spaces, \gg)$ is a complete lattice, where $B \gg A$ if $B \in C(A)$. Finally, as a corollary we obtain Bousfield's theorem, which states that $(Spaces, >)$ is a complete lattice, where $B > A$ if $B \in \overline{C(A)}$.

1. Introduction

The main objects of our investigation are classes of *unpointed* spaces $\mathcal{C}$ that are closed under certain fundamental operations. We say that:

a. $\mathcal{C}$ is closed under weak equivalences if $X \in \mathcal{C}$ weakly equivalent to Y implies that $Y \in \mathcal{C}$.
b. $\mathcal{C}$ is closed under taking homotopy colimits indexed by contractible categories if, for any diagram $F : I \to Spaces$ where $F(i) \in \mathcal{C}$, $i \in I$, and I is contractible, $hocolim_I F \in \mathcal{C}$.
c. $\mathcal{C}$ is closed under taking *pointed* homotopy colimits if, for any pointed diagram $F : I \to Spaces_*$ where $F(i) \in \mathcal{C}$, $i \in I$, the pointed homotopy colimit $hocolim^*_I F \in \mathcal{C}$.
d. $\mathcal{C}$ is closed under taking basic operations if the following conditions are satisfied: for any set I, if $X_i \in \mathcal{C}$, $i \in I$, then, for any choice of basepoints in X_i, $\bigvee_I X_i \in \mathcal{C}$; if $X_i \in \mathcal{C}$, for $i = 0, 1, 2$, then $hocolim(X_0 \leftarrow X_1 \to X_2) \in \mathcal{C}$.
e. We say that $\mathcal{C}$ is closed under extensions by fibration if, for any map $X \to B$ such that $B \in \mathcal{C}$ and over any basepoint in B the homotopy fiber $Fib(X \to B) \in \mathcal{C}$, the space $X \in \mathcal{C}$.

Our primary interest is in *closed* classes which were introduced by E. Dror Farjoun [8, 9].

1.1. **Definition.** A class of unpointed spaces is *closed* if it consists of nonempty spaces, it is closed under weak equivalences, and it is closed under taking homotopy colimits indexed by contractible categories.

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1.2. **Remark.** Although we deal with unpointed spaces, closed classes can be characterized using *pointed* conditions. These conditions are usually easier to check. It is a nontrivial result of E. Dror Farjoun that $\mathcal{C}$ is closed if and only if it is closed under weak equivalences and taking pointed homotopy colimits (see [4]). It is then not difficult to show that $\mathcal{C}$ is closed if and only if it is closed under weak equivalences and basic operations (see [4]). Throughout the paper we will use these characterizations of closed classes interchangeably.

The motivation for studying closed classes comes from the theory of unstable localization and colocalization functors (see [2, 9]), which are related to our key examples:

1.3. **Example** (Dror Farjoun, [8])**.** Let A be a nonempty space. Denote by $\mathcal{C}(A)$ the smallest closed class that contains A. The relation $X \in \mathcal{C}(A)$ is denoted by $X \gg A$ and X is said to be A-cellular. This defines an order on the class of all the spaces. The class $\mathcal{C}(A)$ can be identified with the image of the composition $Spaces_* \xrightarrow{CW_A} Spaces_* \to Spaces$ of the Dror Farjoun functor CW_A and the forgetful functor (see [8, 5]).

1.4. **Example** (Bousfield, Dror Farjoun, [1, 9])**.** Let A be a nonempty space. Denote by $\overline{\mathcal{C}(A)}$ the smallest closed class that contains A and is closed under extensions by fibrations. The relation $X \in \overline{\mathcal{C}(A)}$ is denoted by $X > A$, and X is said to be A-acyclic. This defines an order on the class of all the spaces. The class $\overline{\mathcal{C}(A)}$ can be identified with the kernel of the Bousfield functor P_A (see [5]).

In this paper we look at certain global features of the orders $(Spaces, \gg)$ and $(Spaces, >)$. Our work was inspired by a result of Bousfield [2], which says that $(Spaces, >)$ is a complete lattice. Recall that an ordered class $(\mathcal{C}, >)$ is called a complete lattice if it satisfies the following condition. For any set $\mathcal{J} \subset \mathcal{C}$, there are objects $\sup(\mathcal{J})$ and $\inf(\mathcal{J})$ in $\mathcal{C}$, such that: if $X \in \mathcal{J}$, then $\sup(\mathcal{J}) > X > \inf(\mathcal{J})$, and for any other objects Y and Z in $\mathcal{C}$ for which $Y > X > Z$, if $X \in \mathcal{J}$, we have $Y > \sup(\mathcal{J})$ and $\inf(\mathcal{J}) > Z$.

The main result of our paper is:

1.5. **Theorem.**

(1) *The order $(Spaces, \gg)$ is a complete lattice.*

(2) *any space A there exists a space B such that $\overline{\mathcal{C}(A)} = \mathcal{C}(B)$. If A is finite, then we can take B to be the wedge of representatives of the isomorphism classes of finite A-acyclic spaces.*

The strategy to prove this theorem is to look for special kinds of generators of a given class $\mathcal{C}$. We say that a space A is a *cellular generator* of $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}(A)$. We say that a set $\mathcal{B} \subset \mathcal{C}$ *strongly generates* $\mathcal{C}$ if any space in $\mathcal{C}$ is weakly equivalent to the homotopy colimit of a diagram $F : I \to Spaces$, where I is contractible and $F(i) \in \mathcal{B}$, for $i \in I$. Strong generation is related to an invariant called "complication", see [6]. Observe that if $\mathcal{B}$ strongly generates $\mathcal{C}$, then, for any choice of basepoints in the elements of $\mathcal{B}$, $\bigvee_{A \in \mathcal{B}} A$ is a cellular generator of $\mathcal{C}$.

The key step in the proof of Theorem 1.5 is a generalization of the following statement to an arbitrary space A:

1.6. **Theorem.** *If A is finite, then a set of representatives of isomorphism classes of finite A-acyclic (cellular) spaces strongly generates the class of A-acyclic (cellular) spaces.*

2. Notation

This paper is written simplicially. The symbols *Spaces* and $Spaces_*$ denote the categories of unpointed and pointed simplicial sets, respectively. We refer to objects of these categories as spaces. All the spaces and maps, unless clearly specified, are assumed to be unpointed.

The n-dimensional simplex is denoted by $\Delta[n]$. Its boundary is denoted by $\partial\Delta[n]$. The symbol $\Delta[n,k]$ denotes the horn in $\Delta[n]$. It is a simplicial subset of $\partial\Delta[n]$ generated by all except the k-th face of $\Delta[n]$. We refer to [7, 10] for the details of how to do homotopy theory in *Spaces* and $Spaces_*$. Symbols: $\hookrightarrow$, $\twoheadrightarrow$, and $\xrightarrow{\sim}$ are used to denote respectively a cofibration, a fibration, and a weak equivalence in *Spaces*.

Let K be a space. The cardinality of its set of all the nondegenerate simplices is denoted by $\#K$.

By $\mathbf{N}$ we denote the natural order on the set of natural numbers $\{0, 1, \ldots\}$.

Let λ be an ordinal number. By $[\lambda]$ we denote a category whose objects are ordinals κ such that $\kappa < \lambda$, and $mor_{[\lambda]}(\kappa, \delta)$ is either empty, if $\kappa > \delta$, or consists of a single element, if $\kappa \leq \delta$.

Let J be a set and λ be a *limit* ordinal. We say that $\#J$ is smaller than the cofinality of λ if the following condition is satisfied. If $F : [\lambda] \to Sets$ is a functor where, for $\kappa < \lambda$, $colim_{[\kappa]}F \to F(\kappa)$ is a monomorphism, then the natural map $colim_{[\lambda]} mor_{Sets}(J, F) \to mor_{Sets}(J, colim_{[\lambda]}F)$ is an isomorphism. For example, the cardinality of a set is smaller than the cofinality of $\mathbf{N}$ if and only if this set is finite. We refer the reader to [6] for a detailed discussion of this condition.

The symbols $hocolim_I F$ and $hocolim_I^* G$ denote respectively the homotopy colimit and the pointed homotopy colimit of $F : I \to Spaces$ and $G : I \to Spaces_*$ as defined in [3].

Let $F : I \to Spaces$ be a functor. Assume that the cardinality of the set of objects of I is smaller than the cofinality of λ. If for any $i \in I$, $\#F(i)$ is also smaller than the cofinality of λ, then the same is true for $\#colim_I F$ and $\#hocolim_I F$.

A category is called filtered if, for any two objects a and b, there are morphisms $a \to c$ and $b \to c$, and, for any two morphisms $\alpha : a \to b$ and $\beta : a \to c$, there are morphisms $\alpha' : b \to d$ and $\beta' : c \to d$ such that $\alpha'\alpha = \beta'\beta$. If I is filtered, then I is contractible.

3. Approximations

Let $\mathcal{B}$ be a set of spaces. To see that $\mathcal{B}$ strongly generates $\mathcal{C}$, we need a criterion which guarantees that a space can be expressed as the homotopy colimit of a diagram indexed by a contractible category with values in $\mathcal{B}$.

Let X be a space. Define $\mathcal{B}\!\downarrow\! X$ to be a category whose objects are maps $Y \to X$ where $Y \in \mathcal{B}$, and a set of morphisms between $\alpha_0 : Y_0 \to X$ and $\alpha_1 : Y_1 \to X$ consists of *cofibrations* $i : Y_0 \hookrightarrow Y_1$ such that $\alpha_1 i = \alpha_0$. By $\omega_{\mathcal{B}} : \mathcal{B}\!\downarrow\! X \to Spaces$ let us denote the forgetful functor $(Y \to X) \mapsto Y$.

3.1. **Proposition.** *Let $\mathcal{B}$ be a set of connected spaces. Assume:*

- *The category $\mathcal{B}\!\downarrow\! X$ is filtered.*
- *If K is a finite complex, then any map $K \to X$ can be expressed as a composition $K \to Y \to X$ where $Y \in \mathcal{B}$. Moreover, any commutative diagram of the*

form:

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ L \arrow[r] & X \end{tikzcd}$$

where K, L are finite, $Y \in \mathcal{B}$, and $K \hookrightarrow L$ is a cofibration, can be extended to a commutative diagram:

$$\begin{tikzcd} K \arrow[rr] \arrow[dd, hook] & & Y \arrow[dl, hook] \arrow[dd] \\ & Y' \arrow[dr] & \\ L \arrow[ur] \arrow[rr] & & X \end{tikzcd}$$

where $Y' \in \mathcal{B}$ and $Y \hookrightarrow Y'$ is a cofibration.

Then the map $hocolim_{\mathcal{B}\downarrow X}(\omega_{\mathcal{B}}) \to X$ is a weak equivalence.

Proof. Let us denote the map $hocolim_{\mathcal{B}\downarrow X}(\omega_{\mathcal{B}}) \to X$ by f. Since $\mathcal{B}\downarrow X$ is connected (it is even contractible) and since all the values of $\omega_{\mathcal{B}}$ are connected, then so is the space $hocolim_{\mathcal{B}\downarrow X}(\omega_{\mathcal{B}})$. Thus to show the proposition it is enough to prove that f induces an isomorphism on homotopy groups π_*, for $* \geq 0$, for some choice of basepoints.

Observe that any pointed map $\alpha : K \to X$, where K is finite, factors through f. To see this let $K \to Y \to X$ be a factorization of α given by the assumption. Let $Y \to hocolim_{\mathcal{B}\downarrow X}(\omega_{\mathcal{B}})$ be the map induced by $Y \to X$ in $\mathcal{B}\downarrow X$. It is clear that the composition $K \to Y \to hocolim_{\mathcal{B}\downarrow X}(\omega_{\mathcal{B}}) \to X$ equals α.

An element in $\pi_* X$ is represented by a map $\alpha : sd^k\partial\Delta[n] \to X$ where $sd^k\partial\Delta[n]$ denotes the k-th barycentric subdivision of $\partial\Delta[n]$. Since $sd^k\partial\Delta[n]$ is finite, the above remark shows that α factors through f. It follows that f induces an epimorphism on π_*. In particular, X is connected.

Since $\mathcal{B}\downarrow X$ is filtered, any element of $\pi_* hocolim_{\mathcal{B}\downarrow X}(\omega_{\mathcal{B}})$ can be represented by a map $\alpha : sd^k\partial\Delta[n] \to Y$, for some $Y \to X$ in $\mathcal{B}\downarrow X$. Assume that α represents an element in the kernel of $\pi_* f$. This means that after possibly further subdividing there is a commutative diagram:

$$\begin{tikzcd} sd^l\partial\Delta[n] \arrow[r] \arrow[d, hook] & sd^k\partial\Delta[n] \arrow[r] & Y \arrow[d] \\ L \arrow[rr] & & X \end{tikzcd}$$

where L is finite and contractible, for example the cone on $sd^l\partial\Delta[n]$. By the assumptions we can then find $Y' \in \mathcal{B}$, for which the above diagram can be extended to:

$$\begin{tikzcd} sd^l\partial\Delta[n] \arrow[r] \arrow[d, hook] & sd^k\partial\Delta[n] \arrow[r] & Y \arrow[dl, hook] \arrow[d] \\ L \arrow[r] & Y' \arrow[r] & X \end{tikzcd}$$

It follows that α represents a trivial class in the homotopy groups of $hocolim_{\mathcal{B}\downarrow X}(\omega_{\mathcal{B}})$ and thus $\pi_* f$ is also a monomorphism. $\square$

4. "Small" $\mathcal{C}$-spaces

In order to find strong generators of closed classes we are going to consider spaces for which a relative Quillen's small object argument holds. Let $\mathcal{C}$ be a class of spaces and λ be a limit ordinal. Note that the collection of isomorphism classes of spaces in $\mathcal{C}$, whose cardinality is smaller than the cofinality of λ, forms a set. We denote this set by $\mathcal{C}_{\ll\lambda}$. Sometimes, by slight abuse of language, we say that X belongs to $\mathcal{C}_{\ll\lambda}$ if X is isomorphic to a space in $\mathcal{C}$ whose cardinality is smaller than the cofinality of λ. For example, $\mathcal{C}_{\ll\mathbf{N}}$ consists of finite complexes that belong to $\mathcal{C}$.

4.1. **Definition.** Let $\mathcal{C}$ be a class of spaces and λ be a limit ordinal. A space X is called λ-$\mathcal{C}$-small if, for any map $K \to X$ such that $\#K$ is smaller than the cofinality of λ, there exists a factorization $K \to Y \to X$ where $Y \in \mathcal{C}_{\ll\lambda}$.

In the case that $\mathcal{C}$ is closed under weak equivalences we can always arrange so that the map $K \to Y$, in the above definition, is a cofibration. To do that just take $Y' = colim(K \times \Delta[1] \hookleftarrow K \to Y)$.

The above definition is motivated by Proposition 3.1:

4.2. **Lemma.** *Let $\mathcal{C}$ be a closed class and λ be a limit ordinal. Assume that $\mathcal{C}$ consists of connected spaces. If X is λ-$\mathcal{C}$-small, then the set $\mathcal{C}_{\ll\lambda}$ satisfies the assumptions of Proposition 3.1.*

Proof. Since $\mathcal{C}$ consists of connected spaces, any λ-$\mathcal{C}$-small space is connected. Let $\alpha : A \to X$ and $\beta : B \to X$ be elements of $\mathcal{C}_{\ll\lambda} \downarrow X$. Since X is connected, there is a finite contractible space L and a commutative diagram:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & L & * \arrow[l] \arrow[r] & B \arrow[d, "\beta"] \\ A \arrow[rrr, "\alpha"] & & & X \end{tikzcd}$$

The map $hocolim(A \leftarrow * \to L \leftarrow * \to B) \to X$ belongs to $\mathcal{C}_{\ll\lambda} \downarrow X$. Moreover, since $\mathcal{C}$ is closed, for any morphisms $A \hookrightarrow B$ and $A \hookrightarrow C$ in $\mathcal{C}_{\ll\lambda} \downarrow X$, the space $D = colim(B \hookleftarrow A \hookrightarrow C)$ belongs to $\mathcal{C}_{\ll\lambda}$. This shows that $\mathcal{C}_{\ll\lambda} \downarrow X$ is filtered.

Consider now a commutative diagram:

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ L \arrow[r] & X \end{tikzcd}$$

where K, L are finite, $Y \in \mathcal{C}_{\ll\lambda}$, and $K \hookrightarrow L$ is a cofibration.

The push-out $Z = colim(L \hookleftarrow K \to Y)$ clearly has cardinality smaller than the cofinality of λ. Since X is λ-$\mathcal{C}$-small, the induced map $Z \to X$ can be factored as $Z \hookrightarrow Y' \to X$, where $Y' \in \mathcal{C}_{\ll\lambda}$. This shows that the second condition of Proposition 3.1 is also satisfied. $\square$

Being λ-$\mathcal{C}$-small is not a homotopy invariant condition. To make it so we need a modification:

4.3. **Proposition.** *Let $\mathcal{C}$ be a closed class and λ be a limit ordinal whose cofinality is bigger than $\mathbf{N}$ or $\lambda = \mathbf{N}$. Then the following is a closed class:*

$$\mathcal{C}[\lambda] = \{X \mid \text{there exists } X \xrightarrow{\sim} X' \text{ such that } X' \text{ is } \lambda\text{-}\mathcal{C}\text{-small}\}.$$

4.4. **Proposition.** *Let λ be a limit ordinal whose cofinality is bigger than $\mathbf{N}$. If $\{\mathcal{C}_i\}_{i\in I}$ is a family of closed classes such that $\#I$ is smaller than the cofinality of λ, then:*

$$\big(\bigcap_{i\in I}\mathcal{C}_i\big)[\lambda]=\bigcap_{i\in I}\big(\mathcal{C}_i[\lambda]\big).$$

Propositions 4.3 and 4.4 will be proved at the end of the section.

4.5. **Corollary.** *Let $\mathcal{C}$ be a closed class and λ be a limit ordinal whose cofinality is bigger than $\mathbf{N}$ or $\lambda=\mathbf{N}$. Then $\mathcal{C}_{\ll\lambda}$ strongly generates $\mathcal{C}[\lambda]$ and, for any choice of basepoints in the elements of $\mathcal{C}_{\ll\lambda}$, $\mathcal{C}[\lambda]=\mathcal{C}(\bigvee_{A\in\mathcal{C}_{\ll\lambda}}A)$.*

Proof. It is clear that $\mathcal{C}_{\ll\lambda}\subset\mathcal{C}[\lambda]$.

If $\mathcal{C}$ contains a non-connected space, then $\mathcal{C}$ consists of all the spaces (since $\mathcal{C}$ is closed). In particular, all discrete spaces belong to $\mathcal{C}$. In this case the corollary is clear since any λ-$\mathcal{C}$-small space can be expressed as the realization (the homotopy colimit) of a simplicial space whose values are discrete spaces of cardinality smaller than the cofinality of λ.

If $\mathcal{C}$ consists of connected spaces, the corollary follows from Lemma 4.2 □

We can now prove the "cellular" part of Theorem 1.6:

4.6. **Corollary.** *Let A be a space. Assume that either $\lambda=\mathbf{N}$, if A is finite, or λ is a limit ordinal whose cofinality is bigger than $\#A$, if $\#A$ is infinite. Then $\mathcal{C}(A)=\mathcal{C}(A)[\lambda]$ and $\mathcal{C}(A)$ is strongly generated by the set $\mathcal{C}(A)_{\ll\lambda}$.*

Proof. The inclusion $\mathcal{C}(A)[\lambda]\subset\mathcal{C}(A)$ is obvious. Since the cardinality of A is smaller than the cofinality of λ, it is also clear that $A\in\mathcal{C}(A)[\lambda]$. Since $\mathcal{C}(A)[\lambda]$ is closed, we have $\mathcal{C}(A)\subset\mathcal{C}(A)[\lambda]$. □

As a corollary we also get part 1 of Theorem 1.5:

4.7. **Corollary.** *The order $(Spaces,\gg)$ is a complete lattice.*

Proof. Let $\mathcal{J}$ be a set of spaces. For any choice of basepoints in the spaces of $\mathcal{J}$, $\inf(\mathcal{J})=\bigvee_{A\in\mathcal{J}}A$.

Note that $\sup(\mathcal{J})$ is represented by a cellular generator of $\bigcap_{A\in\mathcal{J}}\mathcal{C}(A)$. To show the existence of such a generator choose a limit ordinal λ whose cofinality is bigger than the cardinalities of spaces in $\mathcal{J}$, $\#\mathcal{J}$, and $\mathbf{N}$. According to Proposition 4.4 and Corollary 4.6 we have equalities:

$$\big(\bigcap_{A\in\mathcal{J}}\mathcal{C}(A)\big)[\lambda]=\bigcap_{A\in\mathcal{J}}\big(\mathcal{C}(A)[\lambda]\big)=\bigcap_{A\in\mathcal{J}}\mathcal{C}(A).$$

Since any class of the form $\mathcal{C}[\lambda]$ has a cellular generator (see Corollary 4.5), then so does $\bigcap_{A\in\mathcal{J}}\mathcal{C}(A)$. □

To prove Propositions 4.3 and 4.4 we start with showing that, in the case $\mathcal{C}$ is closed, λ-$\mathcal{C}$-small spaces are preserved by many operations. We would like to understand to what extent weak equivalences preserve λ-$\mathcal{C}$-smallness.

4.8. **Lemma.** *Let $\mathcal{C}$ be closed and λ be a limit ordinal whose cofinality is bigger than $\mathbf{N}$ or $\lambda=\mathbf{N}$.*

(1) *Let $X\to Y\to X$ be maps whose composition is the identity. If Y is λ-$\mathcal{C}$-small, then so is X.*

(2) *If* $E \xrightarrow{\sim} B$ *is an acyclic fibration, then* E *is* λ-$\mathcal{C}$*-small if and only if* B *is.*
(3) *Let* I *be an arbitrary set and* $X = colim(X_0 \leftarrow \coprod_I \Delta[n,k] \hookrightarrow \coprod_I \Delta[n])$*. If* X_0 *is* λ-$\mathcal{C}$*-small, then so is* X*.*
(4) *Let* $X = colim(X_0 \hookleftarrow X_1 \hookrightarrow X_2)$*, where the indicated arrows are cofibrations. If, for all* i*,* X_i *is* λ-$\mathcal{C}$*-small, then so is* X*.*
(5) *Let* $X = colim(X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots)$*, where the indicated arrows are cofibrations. If, for all* i*,* X_i *is* λ-$\mathcal{C}$*-small, then so is* X*.*
(6) *Let* I *be an arbitrary set. If, for all* $i \in I$*,* X_i *is* λ-$\mathcal{C}$*-small, then, for any choice of basepoints in* X_i*, so is* $\bigvee_{i\in I} X_i$*.*
(7) *If* X *is* λ-$\mathcal{C}$*-small, then any map* $X \to Y$ *can be expressed as a composition* $X \overset{\sim}{\hookrightarrow} X' \twoheadrightarrow Y$*, where* X' *is* λ-$\mathcal{C}$*-small and the indicated maps are an acyclic cofibration and a fibration.*
(8) *Let* $X \xrightarrow{\sim} Y$ *be a weak equivalence. If* X *is* λ-$\mathcal{C}$*-small, then so is* Y*.*
(9) *Let* $X \xrightarrow{\sim} Y$ *be a weak equivalence. If* Y *is* λ-$\mathcal{C}$*-small and* X *is fibrant, then* X *is also* λ-$\mathcal{C}$*-small.*
(10) *If* $X \in \mathcal{C}[\lambda]$ *and* X *is fibrant, then* X *is* λ-$\mathcal{C}$*-small.*

Proof of (1)*.* This follows directly from the definitions.

Proof of (2)*.* This part follows easily from the fact that any map $K \to B$ factors as $K \to E \overset{\sim}{\twoheadrightarrow} B$.

Proof of (3)*.* Let $K \to X$ be a map where $\#K$ is smaller than the cofinality of λ. Note that there is $J \subset I$ for which $K \to X$ factors through

$$X' = colim(X_0 \leftarrow \coprod_J \Delta[n,k] \hookrightarrow \coprod_J \Delta[n])$$

and $\#J$ is smaller than the cofinality of λ. Clearly $\#(\coprod_J \Delta[n,k])$ and $\#(\coprod_J \Delta[n])$ are also smaller than the cofinality of λ.

By pulling back $K \to X'$ along the appropriate maps we can form the following commutative diagram where the squares on the right are pull-backs:

$$\begin{tikzcd}
K \arrow[d] & = & colim & (& K_0 \arrow[d] & K_1 \arrow[l] \arrow[r, hook] \arrow[d] & K_2 \arrow[d] &) \\
X' & = & colim & (& X_0 & \coprod_J \Delta[n,k] \arrow[l] \arrow[r, hook] & \coprod_J \Delta[n] &)
\end{tikzcd}$$

Since $\#K$ and $\#J$ are smaller than the cofinality of λ, the same is true for $\#K_i$, and thus also for the cardinality of $P = colim(K_0 \leftarrow K_1 \to \coprod_J \Delta[n,k])$. Let $P \to X_0$ be the map induced by the above diagram. By the assumption we can express $P \to X_0$ as $P \to Y \to X_0$, where $Y \in \mathcal{C}_{\ll\lambda}$. To get the desired factorization of $K \to X$, take $Y' = colim(Y \leftarrow \coprod_J \Delta[n,k] \hookrightarrow \coprod_J \Delta[n])$. This space has cardinality smaller than the cofinality of λ and is weakly equivalent to Y, thus belongs to $\mathcal{C}_{\ll\lambda}$.

Proof of 4, 5, *and* 6*.* Since the proofs are analogous we show only part 4. Let us point out however that part 5 requires the assumption that either $\lambda = \mathbf{N}$ or the cofinality of λ is bigger than $\mathbf{N}$.

Let $K \to X$ be a map where $\#K$ is smaller than the cofinality of λ. By pulling back $K \to X$ along the appropriate maps, we can form the following commutative diagram, where the squares on the right are pull-backs and the indicated arrows are cofibrations:

$$\begin{tikzcd} K \arrow[d] & = & colim & (& K_0 \arrow[d] & K_1 \arrow[l, hook'] \arrow[r, hook] \arrow[d] & K_2 \arrow[d] &) \\ X & = & colim & (& X_0 & X_1 \arrow[l, hook'] \arrow[r, hook] & X_2 &) \end{tikzcd}$$

Since $K_i \subset K$, $\#K_i$ is smaller than the cofinality of λ. Let $K_1 \to Y_1 \to X_1$ be a factorization of $K_1 \to X_1$ where $Y_1 \in C_{\ll\lambda}$. Define $\overline{Y_i} = colim(Y_1 \leftarrow K_1 \hookrightarrow K_i)$. There are natural maps $\overline{Y_i} \to X_i$. Since the cardinalities of $\overline{Y_i}$ are smaller than the cofinality of λ, we can find appropriate factorizations $\overline{Y_i} \hookrightarrow Y_i \to X_i$. The space $Y = colim(Y_0 \hookleftarrow Y_1 \hookrightarrow Y_2)$ gives the desired factorization of $K \to X$.

Proof of (7). This follows from parts 3 and 5 and the standard way of factoring a map in *Spaces* into an acyclic cofibration followed by a fibration (the acyclic cofibrations are generated by the inclusions of horns into simplexes).

Proof of (8). According to part 7 we can factor $X \xrightarrow{\sim} Y$ as $X \overset{\sim}{\hookrightarrow} X' \overset{\sim}{\twoheadrightarrow} Y$ where X' is λ-$\mathcal{C}$-small. Part 8 follows now from part 2.

Proof of (9). Factor $X \xrightarrow{\sim} Y$ by an acyclic cofibration followed by an acyclic fibration $X \overset{\sim}{\hookrightarrow} Y' \overset{\sim}{\twoheadrightarrow} Y$. According to part 2, Y' is λ-$\mathcal{C}$-small. Since X is fibrant, it is a retract of Y'. Part 9 follows now from part 1. □

Proof of Proposition 4.3. In view of Lemma 4.8, to prove the proposition it is enough to show that $\mathcal{C}[\lambda]$ is closed under weak equivalences.

Let $X \xrightarrow{\sim} Y$ be a weak equivalence. We need to prove that $X \in \mathcal{C}[\lambda]$ if and only if $Y \in \mathcal{C}[\lambda]$. Assume that $X \in \mathcal{C}[\lambda]$. By possibly factoring a given map into a cofibration followed by an acyclic fibration, using part 2 of Lemma 4.8, we can assume that there is an acyclic cofibration $X \overset{\sim}{\hookrightarrow} X'$, where X' is λ-$\mathcal{C}$-small. Let $Y' = colim(X' \overset{\sim}{\hookleftarrow} X \xrightarrow{\sim} Y)$. Since X' is λ-$\mathcal{C}$-small, part 8 of Lemma 4.8 implies that so is Y'. It follows that $Y \in \mathcal{C}[\lambda]$.

Assume that $Y \in \mathcal{C}[\lambda]$. Let $Y \xrightarrow{\sim} Y'$ be a weak equivalence where Y' is λ-$\mathcal{C}$-small. The composition $X \xrightarrow{\sim} Y \xrightarrow{\sim} Y'$ gives the desired weak equivalence. □

Proof of Proposition 4.4. The inclusion $(\bigcap_{i\in I} \mathcal{C}_i)[\lambda] \subset \bigcap_{i\in I}(\mathcal{C}_i[\lambda])$ is clear since, for any $i \in I$, $(\bigcap_{i\in I} \mathcal{C}_i)[\lambda] \subset \mathcal{C}_i[\lambda]$.

Let $X \in \bigcap_{i\in I}(\mathcal{C}_i[\lambda])$. This means that, for any $i \in I$, there is an acyclic cofibration $X \overset{\sim}{\hookrightarrow} X_i$ where X_i is λ-$\mathcal{C}_i$-small. Define $Z := \bigcup_X X_i$. Observe that the inclusions $X_i \hookrightarrow \bigcup_X X_i$ are weak equivalences. Part 8 of Lemma 4.8 implies that, for all $i \in I$, Z is λ-$\mathcal{C}_i$-small. Since $X \to Z$ is also a weak equivalence, to show that $X \in (\bigcap_{i\in I} \mathcal{C}_i)[\lambda]$ it is enough to prove that Z is λ-$(\bigcap_{i\in I} \mathcal{C}_i)$-small.

Let $K \to Z$ be a map, where $\#K$ is smaller than the cofinality of λ. We need to prove that this map factors as $K \to Y \to Z$, where $\#Y$ is smaller than the cofinality of λ and $Y \in \bigcap_{i\in I} \mathcal{C}_i$. We will construct Y as the "telescope" of a diagram indexed by a certain ordinal number. Let us choose a well ordering on the set I and denote by e its smallest element. Define κ to be the following order on $I \times \mathbf{N}$. Set $(i, m) < (j, n)$ if $m < n$ in $\mathbf{N}$ or $m = n$ and $i < j$ in I. Our strategy is to define, by induction, a diagram $F : [\kappa] \to Spaces$ together with a natural transformation $F \to Z$ such that:

- the map $K \to Z$ factors as $K \to F(e,0) \to Z$;
- for any $(i,n) \in \kappa$, $\#F(i,n)$ is smaller than the cofinality of λ;
- for any $(i,n) \in \kappa$, $F(i,n) \in \mathcal{C}_i$.

Assume that we have such a functor. Observe that the imposed assumptions on I imply that $\#hocolim_{[\kappa]}F$ is smaller than the cofinality of λ. Moreover, for any $i \in I$, $hocolim_{[\kappa]}F \in C_i$ since $hocolim_{[\kappa]}F \simeq hocolim_{n\in\{0,1,\dots\}}F(i,n)$. This shows that $K \to hocolim_{[\kappa]}F \to Z$ is the desired factorization.

Since Z is λ-C_e-small we can find a factorization $K \to Y \to Z$, where $\#Y$ is smaller than the cofinality of λ and $Y \in C_e$. Define $F(e,0) := Y$ and $F(e,0) \to Z$ to be the map $Y \to Z$.

Assume that we have defined F for all $(i,m) < (j,n)$. Consider the map $hocolim_{[j,n]}F \to Z$. Since $\#hocolim_{[j,n]}F$ is smaller than the cofinality of λ, we can factor this map as $hocolim_{[j,n]}F \to Y \to Z$, where $Y \in \mathcal{C}_j$ and $\#Y$ is smaller than the cofinality of λ. Define $F(j,n) := Y$. It is clear that F satisfies the above requirements. $\square$

5. Parameterized cellular spaces

To prove part 2 of Theorem 1.5 we need to discuss parameterized families of cellular spaces. Let $f : E \to B$ be a map. For any simplex $\sigma : \Delta[n] \to B$, define $df(\sigma) := \lim(\Delta[n] \xrightarrow{\sigma} B \xleftarrow{f} E)$.

5.1. **Definition.** Let $\mathcal{C}$ be a class of spaces. A map $f : E \to B$ is called a relative $\mathcal{C}$-complex if for any simplex $\sigma : \Delta[n] \to B$, $df(\sigma) \in \mathcal{C}$.

Let $\mathcal{C}$ be a closed class. Since $\mathcal{C}$ consists of nonempty spaces, a relative $\mathcal{C}$-complex $f : E \to B$ is always an *onto* map.

A key property of closed classes is given by:

5.2. **Theorem** ([4]). *Let $\mathcal{C}$ be a closed class. Let $H, G : I \to Spaces$ be diagrams and $\Psi : H \to G$ be a natural transformation. If for any $i \in I$, $\Psi_i : H(i) \to G(i)$ is a relative $\mathcal{C}$-complex, then for any basepoint in $hocolim_I G$, the homotopy fiber of $hocolim_I \Psi : hocolim_I H \to hocolim_I G$ is in $\mathcal{C}$.* $\square$

The above implies:

5.3. **Corollary.** *Let $\mathcal{C}$ be a closed class.*

(1) *If $f : E \to B$ is a relative $\mathcal{C}$-complex, then for any basepoint, the homotopy fiber of f is in $\mathcal{C}$.*
(2) *Let $H, G : I \to Spaces$ be diagrams and $H \to G$ be a natural transformation. If for any $i \in I$, $H(i) \to G(i)$ is a relative $\mathcal{C}$-complex and $hocolim_I G$ is contractible, then $hocolim_I H \in \mathcal{C}$.* $\square$

Relative complexes enjoy the following properties:

5.4. **Proposition.** *Let $\mathcal{C}$ be a closed class and λ be a limit ordinal.*

(1) *If $E \to B$ is a relative $\mathcal{C}_{\ll\lambda}$-complex, then, for any $h : X \to B$, so is the pull-back $h^*E \to X$.*

(2) *Consider the following commutative diagram where the indicated arrows are cofibrations:*

$$\begin{array}{ccccc} E & = & colim & (& E_0 \longleftarrow E_1 \hookrightarrow E_2 &) \\ \downarrow f & & & & \downarrow f_0 \qquad \downarrow f_1 \qquad \downarrow f_2 & \\ B & = & colim & (& B_0 \longleftarrow B_1 \hookrightarrow B_2 &) \end{array}$$

If f_i are relative $\mathcal{C}_{\ll\lambda}$-complexes, then so is f.

(3) *Let δ be an ordinal number smaller than the cofinality of λ. Let $\Psi : H \to G$ be a natural transformation between functors $H, G : [\delta] \to Spaces$. Assume that for any $l < \delta$, $colim_{i<l} H(i) \hookrightarrow H(l)$ and $colim_{i<l} G(i) \hookrightarrow G(l)$ are cofibrations and $\Psi_l : H_l \to G_l$ is a relative $\mathcal{C}_{\ll\lambda}$-complex. Then $colim_{[\delta]}\Psi : colim_{[\delta]} H \to colim_{[\delta]} G$ is also a relative $\mathcal{C}_{\ll\lambda}$-complex.*

Proof. Part 1 is clear.

Since the proofs of parts 2 and 3 are analogous, we show only part 2. We have to prove that $df(\sigma) \in \mathcal{C}_{\ll\lambda}$, for any simplex $\sigma : \Delta[n] \to B$.

Denote by K the image of the map σ. It is a finite subspace of B. Since the f_i are relative $\mathcal{C}_{\ll\lambda}$-complexes, the subspace $f^{-1}(K) \subset E$ has cardinality smaller than the cofinality of λ. It follows that so does $df(\sigma)$.

Let us consider the pull-backs:

$$K_i = \lim(\Delta[n] \xrightarrow{\sigma} B \leftarrow B_i), \qquad X_i = \lim(K_i \to B_i \xleftarrow{f_i} E_i).$$

These spaces fit into the following commutative diagram where the indicated arrows are cofibrations:

$$\begin{array}{ccccc} df(\sigma) & = & colim & (& X_0 \longleftarrow X_1 \hookrightarrow X_2 &) \\ \downarrow & & & & \downarrow \qquad \downarrow \qquad \downarrow & \\ \Delta[n] & = & colim & (& K_0 \longleftarrow K_1 \hookrightarrow K_2 &) \end{array}$$

Since f_i is a relative $\mathcal{C}$-complex, then so is $X_i \to K_i$, by part 1. We can therefore use part 2 of Corollary 5.3 to conclude that $df(\sigma)$ belongs to $\mathcal{C}$. □

6. Relatively "small"-$\mathcal{C}$-spaces

6.1. **Theorem.** *Let λ be a limit ordinal whose cofinality is bigger than $\mathbf{N}$ or $\lambda = \mathbf{N}$. If $\mathcal{C}$ is a closed class that is closed under extensions by fibrations, then so is $\mathcal{C}[\lambda]$.*

As a corollary we get both the second part of Theorem 1.5 and Theorem 1.6.

6.2. **Corollary.** *Let A be a space. If the cofinality of λ is bigger than $\#A$, then $\overline{\mathcal{C}(A)} = \overline{\mathcal{C}(A)}[\lambda]$. Moreover, $\overline{\mathcal{C}(A)}$ is strongly generated by the set $\overline{\mathcal{C}(A)}_{\ll\lambda}$ and, for any choice of basepoints in the elements of $\overline{\mathcal{C}(A)}_{\ll\lambda}$, $\overline{\mathcal{C}(A)} = \mathcal{C}\big(\bigvee_{X \in \overline{\mathcal{C}(A)}_{\ll\lambda}} X\big)$.*

Proof. The inclusion $\overline{\mathcal{C}(A)}[\lambda] \subset \overline{\mathcal{C}(A)}$ is clear. Since $\#A$ is smaller than the cofinality of λ, $A \in \overline{\mathcal{C}(A)}[\lambda]$. Since $\overline{\mathcal{C}(A)}[\lambda]$ is a closed class that is closed under extensions by fibrations (see Theorem 6.1), $\overline{\mathcal{C}(A)} \subset \overline{\mathcal{C}(A)}[\lambda]$. The rest of the corollary follows from Corollary 4.5. □

6.3. **Corollary** (Bousfield [2]). *The order $(Spaces, >)$ is a complete lattice.*

Proof. Let $\mathcal{J}$ be a set of spaces. For any choice of basepoints in the spaces of $\mathcal{J}$, $\inf(\mathcal{J}) = \bigvee_{A\in\mathcal{J}} A$.

Note that $\sup(\mathcal{J})$ is represented by a cellular generator of the class $\bigcap_{A\in\mathcal{J}} \overline{\mathcal{C}(A)}$. Since, by Corollary 6.2, any class of the form $\overline{\mathcal{C}(A)}$ has a cellular generator, the existence of $\sup(\mathcal{J})$ follows from Corollary 4.7. $\square$

To prove Theorem 6.1 we need to introduce a relative notion of λ-$\mathcal{C}$-smallness.

6.4. **Definition.** Let $\mathcal{C}$ be a class of spaces and λ be a limit ordinal. A map $p : X \to B$ is called λ-$\mathcal{C}$-small if, for any $\alpha : K \to X$ where $\#K$ is smaller than the cofinality of λ, there exists a relative $\mathcal{C}_{\ll\lambda}$-complex $f : E \to B$ and a commutative diagram:

$$\begin{tikzcd} K \arrow[r] \arrow[rr, bend left, "\alpha"] \arrow[dr] & E \arrow[r] \arrow[d, "f"] & X \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

6.5. **Remark.** Observe that if C is closed under weak equivalences, we can always arrange so that the map $K \to E$, in the above definition, is a cofibration. If not, let $E' = colim(E \leftarrow K \hookrightarrow K \times \Delta[1])$ and $E' \to E$ be induced by id_E and $K \times \Delta[1] \overset{pr_K}{\to} K \to E$. By a direct verification one can show that the composition $E' \to E \overset{f}{\to} B$ is also a relative $\mathcal{C}_{\ll\lambda}$-complex

6.6. **Lemma.** *Let $\mathcal{C}$ be a closed class and λ be a limit ordinal whose cofinality is bigger than $\mathbf{N}$ or $\lambda = \mathbf{N}$. Let $p : X \twoheadrightarrow B$ be a fibration. If $\#B$ is smaller than the cofinality of λ and, for any vertex $b \in B$, $p^{-1}(b)$ is λ-$\mathcal{C}$-small, then p is a λ-$\mathcal{C}$-small map.*

Proof. We first prove the lemma in the case when B is contractible. Let $\alpha : K \to X$ be a map, where $\#K$ is smaller than the cofinality of λ. Since in this case $p^{-1}(b) \hookrightarrow X$ is a weak equivalence, by part 8 of Lemma 4.8, X is λ-$\mathcal{C}$-small. We can find therefore a factorization $K \overset{g}{\to} Y \overset{h}{\to} X$ of α where $Y \in \mathcal{C}_{\ll\lambda}$. Consider now the following commutative diagram:

$$\begin{tikzcd} & Y \arrow[r, "h"] \arrow[d, "id\times p\,h"] & X \arrow[d, two heads, "p"] \\ K \arrow[ur, "g"] \arrow[r, "g\times p\,\alpha"'] & Y \times B \arrow[r, "pr_B"'] \arrow[ur, "s"] & B \end{tikzcd}$$

where s is any lifting in the above square. Such a lifting does exists since $id \times p\,h : Y \to Y \times B$ is an acyclic cofibration. It is clear that $pr_B : Y \times B \to B$ is the desired relative $\mathcal{C}_{\ll\lambda}$-complex.

For a general B, we prove the lemma by induction on the cell decomposition of B. Let $B = colim(B_0 \leftarrow \partial\Delta[n] \hookrightarrow \Delta[n])$. Let $\alpha : K \to X$ be a map, where $\#K$ is smaller than the cofinality of λ. By pulling-back α and p along the appropriate maps we get the following commutative diagram where the squares on the right are pull-backs and the indicated arrows are respectively fibrations and cofibrations :

$$\begin{tikzcd} K \arrow[d, "\alpha"'] & = & colim & (& K_0 \arrow[d, "\alpha_0"] & K_1 \arrow[l] \arrow[r, hook] \arrow[d, "\alpha_1"] & K_2 \arrow[d, "\alpha_2"] &) \\ X \arrow[d, two heads, "p"'] & = & colim & (& X_0 \arrow[d, two heads, "p_0"] & X_1 \arrow[l] \arrow[r, hook] \arrow[d, two heads, "p_1"] & X_2 \arrow[d, two heads, "p_2"] &) \\ B & = & colim & (& B_0 & \partial\Delta[n] \arrow[l] \arrow[r, hook] & \Delta[n] &) \end{tikzcd}$$

By the inductive assumption, the map $\alpha_0 : K_0 \to X_0$ yields an appropriate relative $\mathcal{C}_{\ll\lambda}$-complex $g_0 : E_0 \to B_0$. Define $E_1 = \lim(E_0 \xrightarrow{g_0} B_0 \leftarrow \partial\Delta[n])$. Let $g_1 : E_1 \to \partial\Delta[n]$ be the induced map. According to part 1 of Proposition 5.4, g_1 is also a relative $\mathcal{C}_{\ll\lambda}$-complex. Since $\partial\Delta[n]$ is finite, it follows that $\#E_1$ is smaller than the cofinality of λ. Let $P = colim(E_1 \leftarrow K_1 \hookrightarrow K_2)$. It is clear that $\#P$ is also smaller than the cofinality of λ. This space fits into a commutative diagram:

$$\begin{tikzcd} K_2 \arrow[r] \arrow[rr, bend left, "\alpha_2"] \arrow[dr] & P \arrow[r] \arrow[d] & X_2 \arrow[dl, "p_2"] \\ & \Delta[n] & \end{tikzcd}$$

Since $\Delta[n]$ is contractible, $P \to X_2$ yields an appropriate relative $\mathcal{C}_{\ll\lambda}$-complex $g_2 : E_2 \to \Delta[n]$ where $P \hookrightarrow E_2$ is a cofibration (see Remark 6.5). Define $g : E \to B$ to be:

$$\begin{tikzcd} E \arrow[d, two heads, "g"'] & := & colim & (& E_0 \arrow[d, two heads, "g_0"] & E_1 \arrow[l] \arrow[r, hook] \arrow[d, two heads, "g_1"] & E_2 \arrow[d, two heads, "g_2"] &) \\ B & = & colim & (& B_0 & \partial\Delta[n] \arrow[l] \arrow[r, hook] & \Delta[n] &) \end{tikzcd}$$

According to part 2 of Proposition 5.4, g is a relative $\mathcal{C}_{\ll\lambda}$-complex. This concludes the lemma in the case when B is finite. The case when B is infinite follows from part 3 of Proposition 5.4. $\square$

Proof of Theorem 6.1. Let $p : X \to B$ be a map such that $B \in \mathcal{C}[\lambda]$ and, for any choice of a basepoint in B, $Fib(X \to B) \in \mathcal{C}[\lambda]$. We have to show that $X \in \mathcal{C}[\lambda]$.

Since $\mathcal{C}[\lambda]$ is closed under weak equivalences, we can arrange so that p is a fibration and B is fibrant. It follows from part 10 of Lemma 4.8 that spaces B and $p^{-1}(b)$, for any vertex b in B, are λ-$\mathcal{C}$-small. To prove the theorem it is enough to show that X is also λ-$\mathcal{C}$-small. Let $\alpha : K \to X$ be a map where $\#K$ is smaller than the cofinality of λ. We can factor the composition $p\,\alpha : K \to B$ into $K \to B' \to B$, where $B' \in \mathcal{C}_{\ll\lambda}$. Let us pull back p along $B' \to B$ and form the following commutative diagram:

$$\begin{tikzcd} K \arrow[r, "\alpha'"] \arrow[rr, bend left, "\alpha"] & X' \arrow[r] \arrow[d, two heads, "p'"'] & X \arrow[d, two heads, "p"] \\ & B' \arrow[r] & B \end{tikzcd}$$

The fibration $p' : X' \to B'$ satisfies the assumptions of Lemma 6.6. Let $g : E \to B'$ be the induced relative $\mathcal{C}_\lambda$-complex for $\alpha' : K \to X'$. It is clear that $E \in \mathcal{C}$ since $\mathcal{C}$ is closed under extensions by fibrations. It is also clear that $\#E$ is smaller than the cofinality of λ. $\square$

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