PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 11, Pages 3417–3421
S 0002-9939(04)07416-7
Article electronically published on June 17, 2004

THIN POSITION AND ESSENTIAL PLANAR SURFACES

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(Communicated by Ronald A. Fintushel)

Abstract. Abby Thompson proved that if a link K is in thin position but not in bridge position, then the knot complement contains an essential meridional planar surface, and she asked whether some thin level surface must be essential. This note is to give a positive answer to this question, showing that if a link is in thin position but not bridge position, then a thinnest level surface is essential. A theorem of Rieck and Sedgwick follows as a consequence, which says that thin position of a connected sum of small knots comes in the obvious way.

The concept of thin position was introduced by David Gabai in [G], and has been used successfully in attacking some very difficult problems; see, for example, [G], [GL], [ST], and [T1]. In [T2] Abby Thompson proved that if a knot K is in thin position but not in bridge position, then some thin level surface can be compressed to produce an essential planar surface in the complement of K with meridional boundary slope; in particular, by a theorem of Culler, Gordon, Luecke and Shalen [CGLS] this implies that the knot is large in the sense that its complement contains some closed essential surfaces. This was further explored by Heath and Kobayashi [HK], who showed that a certain thin level surface of a nonsplit link L in thin position but not bridge position can be compressed to give some natural tangle decomposition of L, and the decomposing spheres then give rise to essential meridional planar surfaces in the link complement. In both [T2] and [HK] the essential planar surfaces come from compression of a thin level surface. There are examples in [HK] showing that some essential meridional planar surfaces are not level surfaces of a knot in thin position. This leads to a question raised by Thompson [T3], which asks whether a link L in thin position but not in bridge position has a level surface that is essential. The purpose of this paper is to give a positive solution to this problem.

Theorem 1. *If a link L in S^3 is in thin position but not in bridge position, then a thinnest level surface Q of L is an essential surface in $S^3 - \mathrm{Int}N(L)$.*

We give some definitions. Consider S^3 as $\mathbb{R}^3 \cup \{\infty\}$, and let ρ be the height function $\rho : \mathbb{R}^3 \to \mathbb{R}$ defined by $\rho(x, y, z) = z$. For each $t \in \mathbb{R}$, let $P(t) = \rho^{-1}(t)$ be the horizontal plane in $\mathbb{R}^3$ at height t. When t is not a critical level of ρ, define $Q(t)$ to be the punctured sphere $(P(t) \cup \{\infty\}) - \mathrm{Int}N(L)$, called the *level surface* at level

Received by the editors February 27, 2003 and, in revised form, June 16, 2003.
2000 *Mathematics Subject Classification.* Primary 57M25.
Key words and phrases. Thin position, knots and links, essential planar surfaces.
Partially supported by NSF grant #DMS 0203394.

t, where $N(L)$ is a small regular neighborhood of L intersecting P in meridional disks. If I is an interval on $\mathbb{R}$, denote by $Z_I = \mathbb{R}^2 \times I$ the set of points in $\mathbb{R}^3$ whose z-coordinate is in I.

Let L be a link in $\mathbb{R}^3$ such that the restriction of ρ to L is a Morse function, and let $a_0, ..., a_n$ be the critical points of L, labeled so that the corresponding critical values $t_i = \rho(a_i)$ satisfy $t_{i-1} < t_i$ for all i. Let $s_i \in (t_{i-1}, t_i)$. Thus $P_i = P(s_i)$ is a plane between a_{i-1} and a_i, called a level plane corresponding to the critical point a_{i-1}. The *width of* $P(t)$ (with respect to L) is defined as $w(P(t)) = |P(t) \cap L|$, where $|A|$ denotes the number of elements in A. The *width of* L is $w(L) = \sum_1^n w(P_i)$. A link L is in *thin position* if $w(L)$ is minimal up to isotopy of L.

A plane $P(t)$ with $t_{i-1} < t < t_i$ is called a *thin level plane* of L, and t a thin level, if a_{i-1} is a local maximum and a_i a local minimum. Similarly, $t \in (t_{i-1}, t_i)$ is a *thick level* and $P(t)$ a thick level plane if a_{i-1} is a local minimum and a_i a local maximum. A thin level plane $P(t)$ is a *thinnest level plane* and the corresponding planar surface $Q(t) = (P(t) \cup \{\infty\}) - \mathrm{Int}N(L)$ a *thinnest level surface* if $w(P(t))$ is minimal among all thin planes. The link L is *in bridge position* if it has no thin level.

Proof of Theorem 1. Let P be a thinnest level plane, and let Q be the corresponding planar surface defined above. Moving L up or down if necessary, we may assume without loss of generality that $P = P(0)$, i.e., it is the (x, y)-plane in $\mathbb{R}^3$. Our goal is to show that Q is an essential surface in $E(L)$. Recall that a properly embedded compact orientable surface in a 3-manifold M is *essential* if (i) it is incompressible (in particular it is not a 2-sphere bounding a 3-ball), and (ii) it is not boundary parallel. Since P is a thin level plane, it is easy to show that Q is never a 2-sphere bounding a 3-ball, and it is not boundary parallel. Therefore, we need only show that it is not a compressible punctured sphere.

Assume to the contrary that Q is compressible, and let D be a compressing disk. Without loss of generality we may assume that D is in the upper half space $\mathbb{R}^3_+ = Z_{[0,\infty)}$. We will show below that either L is not in thin position or P is not a thinnest level plane, which will contradict the assumption and complete the proof of the theorem.

The (x, z)-coordinate plane cuts $\mathbb{R}^3_+$ into Y_- and Y_+, where Y_- is the left half space of $\mathbb{R}^3_+$, consisting of points in $\mathbb{R}^3_+$ with negative y-coordinate, and Y_+ the right half space of $\mathbb{R}^3_+$. Let D' be the disk on P with $\partial D = \partial D'$, and let B be the 3-ball bounded by $D \cup D'$. Define $\alpha = L \cap B$ and $\beta = L \cap (\mathbb{R}^3_+ - B)$. Since D is a compressing disk of Q, both α and β are nonempty.

An isotopy ϕ_t of $\mathbb{R}^3$ is called an *h-isotopy* if $\phi_0 = id$ and ϕ_1 is a level-preserving map on L, i.e., $\rho \circ \phi_i = \rho$ on L. Note that ϕ_t does not have to be level-preserving on L when $t \neq 0, 1$. Since ϕ_1 is level-preserving, the link $\phi_1(L)$ has the same width as L.

Lemma 2. *Up to h-isotopy we may assume that $\alpha \subset Y_-$ and $\beta \subset Y_+$.*

Proof. By a level-preserving isotopy that shifts the whole link L to the left we may assume that the 3-ball B lies in Y_-. Let $I = [0, \epsilon]$ be an interval containing no critical value of ρ. So $L \cap Z_I$ can be assumed to be a set of vertical arcs from $\mathbb{R}^2 \times 0$ to $\mathbb{R}^2 \times \epsilon$. We may also assume that $B \cap Z_I = D' \times I$. Let f_t be an isotopy supported in B that shrinks α into $D' \times I$, i.e., $f_0 = id$ and $f_1(\alpha) \subset D' \times I$. (Note that f_t is not level-preserving.) There is now a level-preserving isotopy g_t of $f_1(L)$, supported outside of $D' \times I$, moving β into Y_+. Let h_t be the reverse isotopy of f_t, i.e., $h_t = f_{1-t}$. Then the union of these three isotopies is the required h-isotopy. $\square$

The separation of α and β by the (x, z)-coordinate plane allows us to modify α by "vertical isotopy" without intersecting the rest of L. Let ϕ_t be an isotopy of the positive half of the z-axis. Then $id \times \phi_t$ is an isotopy of $Y_- = \mathbb{R}^2_- \times \mathbb{R}_+$, which can be extended to an isotopy f_t of $\mathbb{R}^3$ supported in a small neighborhood of Y_- in $\mathbb{R}^3_+$, and hence is the identity on $L - \beta$. This f_t is called a vertical isotopy on Y_- determined by ϕ_t.

Denote by m_α the maximum value of $\rho(\alpha)$. Let n_α be the first local minimum level of α, counted from the top down, and $n_\alpha = 0$ if α has no local minimum. (It can be shown that α must have some local minima, but this is not necessary.) Define m_β and n_β similarly.

Lemma 3. *Either $n_\alpha > m_\beta$ or $n_\beta > m_\alpha$.*

Proof. Since P is a thin level plane, at least one of α or β contains some local minima; so n_α and n_β cannot both be 0. Without loss of generality we may assume that $n_\alpha < n_\beta$. Then above the level n_β all the critical points of L are local maxima. Note that lowering a local maximum of α through the level of a local maximum of β will not change $w(L)$. If $m_\alpha > n_\beta$, then deforming a local maximum of α downward to a level just below n_β would reduce $w(L)$, contradicting the minimality of $w(L)$. $\square$

By Lemma 3, we may assume without loss of generality that $m_\alpha < n_\beta$. In particular, β must have some minima. Let $r \in (0, n_\beta)$ be such that $|P(r) \cap \beta|$ is minimal among all $|P(z) \cap \beta|$, $z \in [0, n_\beta]$. Let $I = [a, b]$ be a maximal interval in $[0, \infty)$ containing r and having no critical value of β in its interior. By the definition of r, one can see that b is a local minimum level of β, and a is either 0 or a local maximum level of β.

Let f_t be a vertical isotopy supported in $Z_{[a,\infty)}$ that pushes α downward to $\alpha' = f_1(\alpha)$ lying below the level b. Let L' be the presentation of L obtained this way. We will show below that $w(L') < w(L)$, which then contradicts the assumption, completing the proof of Theorem 1.

Since the isotopy f_t is supported in $Z_{[a,\infty)}$, the critical points of L below level a and the widths of the corresponding level planes remain unchanged. So we need only calculate the sum of the widths for those level planes corresponding to critical points of L and L' above level a. Denote by $P_1, ..., P_k$ (resp. $P'_1, ..., P'_k$) the level planes of L corresponding to (i.e., lying just above) the critical points of α (resp. α') above the level a, and by $R_1, ..., R_h$ those corresponding to critical points of β above level a, labeled according to their height. Since β has no critical values between a and b, all R_j are above level b. So we have

$$|R_j \cap L| = |R_j \cap (\alpha \cup \beta)| \geq |R_j \cap \beta| = |R_j \cap L'|.$$

Also, since the top level of α is below n_β, by the choice of r and the fact that $|P_i \cap \alpha| = |P'_i \cap \alpha|$ we have

$$\begin{aligned} |P_i \cap (\alpha \cup \beta)| &= |P_i \cap \alpha| + |P_i \cap \beta| \geq |P_i \cap \alpha| + |P(r) \cap \beta| \\ &= |P'_i \cap \alpha| + |P'_i \cap \beta| = |P'_i \cap L'|. \end{aligned}$$

It follows that $w(L) \geq w(L')$, and equality holds if and only if it holds in all the above inequalities. On the other hand, since P is a thinnest level of L, the level plane just below b cannot be disjoint from α, because otherwise it would be a thin level plane with width $|P(r) \cap \beta| \leq |P \cap \beta| < |P \cap L|$, contradicting the choice of P. Therefore, α must intersect the level plane R_1 lying just above level b. So $|R_1 \cap L| > |R_1 \cap L'|$, and $w(L) > w(L')$, which contradicts the assumption that L is in thin position. $\square$

Let K_1 and K_2 be knots in S^3. Putting them in thin position, with K_1 above K_2, then taking the obvious connected sum, we have a projection of $K = K_1 \# K_2$ with width $w(K_1) + w(K_2) - 2$. Hence we have $w(K) \leq w(K_1) + w(K_2) - 2$. It was conjectured that $w(K) = w(K_1) + w(K_2) - 2$ for all $K = K_1 \# K_2$. See [RS, SS] for some work concerning this conjecture.

Recall that a knot K is a *small knot* if its complement contains no closed essential surface. An essential surface F in the knot exterior $E(K) = S^3 - \mathrm{Int}N(K)$ is a *meridional essential surface* if ∂F is a nonempty set of meridional curves on $\partial E(K)$. By [CGLS], if K is small, then its exterior contains no meridional essential planar surface. The following result is due to Rieck and Sedgwick [RS]. It proves the above conjecture for the connected sum of small knots K_1 and K_2.

Corollary 4 ([RS]). *Let K_1 and K_2 be nontrivial knots in S^3 such that $E(K_i) = S^3 - \mathrm{Int}N(K_i)$ contains no meridional essential planar surfaces. Let K be a thin position embedding of $K_1 \# K_2$. Then there is a level sphere S in S^3 intersecting K in two points, decomposing K into K_1 and K_2. In particular, $w(K_1 \# K_2) = w(K_1) + w(K_2) - 2$.*

Proof. Let F be a sphere in S^3 that realizes the connected sum $K_1 \# K_2$. Then $P = F \cap E(K)$ is a meridional essential planar surface. Since the exterior of K_i contains no meridional essential planar surface, one can show that P is the only meridional essential planar surface in $E(K)$.

It is easy to see that K cannot be in bridge position: By Schubert's theorem $b(K) = b(K_1) + b(K_2) - 1$. So a bridge position projection of K can be obtained by putting K_i in bridge position, with K_1 above K_2, taking the connected sum, then raising the maxima of K_2 over the minima of K_1. But the last operation would increase width; hence, if $K = K_1 \# K_2$ is in bridge position, then it cannot be in thin position. (See [RS] for an alternative proof.)

It now follows from Theorem 1 that if K is in thin position, then some thinnest level surface Q of K is an essential planar surface in $S^3 - \mathrm{Int}N(K)$. Since a meridional essential planar surface of K is unique, Q is the same as the surface P above up to isotopy. Hence the result follows. $\square$

Acknowledgment

I would like to thank Abby Thompson for raising the question, and for some interesting conversations about the proof of Theorem 1. Thanks also go to Jennifer Schultens for some interesting discussions about thin position and width of the connected sum of knots.

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