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ON PERTURBATIONS OF THE GROUP OF SHIFTS ON THE LINE BY UNITARY COCYCLES

G. G. AMOSOV AND A. D. BARANOV

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ABSTRACT. It is shown that the class of perturbations of the semigroup of shifts on $L^2(\mathbb{R}_+)$ by unitary cocycles V with the property $V_t - I \in s_2,\ t \geq 0$ (where s_2 is the Hilbert-Schmidt class) contains strongly continuous semigroups of isometric operators, whose unitary parts possess spectral decompositions with the measure being singular with respect to the Lebesgue measure. Thus, we describe also the subclass of strongly continuous groups of unitary operators that are perturbations of the group of shifts on $L^2(\mathbb{R})$ by Markovian cocycles W with the property $W_t - I \in s_2,\ t \in \mathbb{R}$.

1. INTRODUCTION

Consider the one-parameter group of shifts $S = \{S_t,\ t \in \mathbb{R}\}$ acting on the Hilbert space $H = L^2(\mathbb{R})$ by the formula $(S_t f)(x) = f(x+t),\ f \in H$. The group of unitary operators S determines a group of *-automorphisms $\alpha = \{\alpha_t,\ t \in \mathbb{R}\}$ acting on the algebra $B(H)$ consisting of all bounded operators on H such that $\alpha_t(x) = S_t x S_{-t},\ x \in B(H),\ t \in \mathbb{R}$. Let $H_0 = L^2(\mathbb{R}_+)$ and assume that H_0 is naturally embedded into H. Then, given $t \geq 0$, it is possible to define correctly the restrictions $T_t = S_{-t}|_{H_0}$ and $\beta_t = \alpha_{-t}|_{B(H_0)}$ forming the semigroup of right shifts on the semi-line $T = \{T_t,\ t \in \mathbb{R}_+\}$ and the semigroup $\beta = \{\beta_t,\ t \in \mathbb{R}_+\}$ consisting of $*$-endomorphisms of $B(H_0)$. As shown in [1], β satisfies the property $\bigcap_{n\in\mathbb{N}} \beta_{tn}(B(H_0)) = \{\mathbb{C}\mathbf{1}\},\ t > 0$. The semigroup β is said to be *the flow of Powers shifts.*

Let $\mathcal{U}(H)$ and $\mathcal{U}(H_0)$ denote the groups of all unitary operators on the spaces H and H_0. A set $W = \{W_t \in \mathcal{U}(H),\ t \in \mathbb{R}\},\ W_0 = I$, is said to be a *multiplicative unitary* 1*-cocycle* of the group S (or, equivalently, of the group α), if

$$W_{t+s} = W_t \alpha_t(W_s),\ t, s \in \mathbb{R}. \tag{1}$$

It follows from (1) and the condition $W_0 = I$ that

$$W_{-t} = \alpha_{-t}(W_t^*),\ t \in \mathbb{R}. \tag{2}$$

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Thus, it suffices to determine the cocycle only for positive or only for negative values of the parameter t. Analogously, it is possible to define a *multiplicative unitary* 1-*cocycle* $V = \{V_t \in \mathcal{U}(H_0),\ t \in \mathbb{R}_+\},\ V_0 = I$, of the semigroup T (or, what is the same, β) consisting of the operators V_t satisfying the property

$$V_{t+s} = V_t\beta_t(V_s),\ t, s \geq 0.$$

For short, usually one refers to α-cocycles and β-cocycles if one needs W and V respectively.

We say that a semigroup of *-endomorphisms $\tilde{\beta}$ is a *cocycle perturbation* of the flow of Powers shifts β if there is a β-cocycle V such that

$$\tilde{\beta}_t(x) = V_t\beta_t(x)V_t^*,\ t \geq 0,\ x \in B(H_0).$$

This notion was introduced by W. Arveson [2]. In the same paper it was proved that the so-called *completely spatial* semigroups of endomorphisms are cocycle perturbations of the flow of Powers shifts. On the other hand, already in [1] the problem was posed to investigate cocycle perturbations of the flow of Powers shifts on arbitrary von Neumann factors that are not isomorphic to the algebra $B(H)$ of all bounded operators. In the present paper we continue the investigation of cocycle perturbations initiated in [1, 2]. It is shown in [3] (and in [4], where the group of shifts with the discrete parameter $t \in \mathbb{Z}$ is considered) that, for quasifree semigroups on the hyperfinite von Neumann factors, to solve this problem one needs to describe the set of all β-cocycles V with the property

$$V_t - I \in s_2,\ t \geq 0,$$

where s_2 denotes the Hilbert-Schmidt class of operators. Here we give new examples of such cocycles V.

Note that each β-cocycle $V_t \in \mathcal{U}(H_0)$ can be extended to an α-cocycle $W_t \in \mathcal{U}(H)$ if one continues the action of each V_t by the formula $V_tf = f,\ f \in H \ominus H_0$, and puts $W_{-t} = V_t,\ t \geq 0$. Furthermore, for positive values of t the action of the cocycle W_t is determined by means of the equality (2). Denote by $H_t \subset H$ the subspace of the past before the moment t, which is generated by functions with supports in the segment $[-t, +\infty)$. Note that $H_t = S_tH_0,\ t \in \mathbb{R}$. The cocycle W obtained by the extension procedure of V given above satisfies the property of "localization of the action to the past" $W_t|_{H_t^\perp} = I$ and is said to be *Markovian* (see [5, 6]). We shall say that the Markovian α-cocycle W and the β-cocycle V are associated. Thus, to describe all β-cocycles V with the property $V_t - I \in s_2,\ t \geq 0$, one needs to describe all Markovian α-cocycles W with the property $W_t - I \in s_2,\ t \in \mathbb{R}$.

In [5, 6] a model is constructed describing all Markovian α-cocycles. The Markovian α-cocycles are completely determined by the β-cocycles associated with them. Every β-cocycle V defines a semigroup of isometric operators

$$\tilde{T} = \{\tilde{T}_t = V_tT_t,\ t \in \mathbb{R}_+\}.$$

It turns out (see [6]) that the deficiency index of the generator of $\tilde{T}_t$ equals one and, vice versa, every semigroup with unital deficiency index of the generator is a cocycle perturbation of T. Let $H_0 = H_u \oplus H_{is}$ be the Wold decomposition of the semigroup $\tilde{T}$. Here the subspaces H_u and H_{is} reduce $\tilde{T}$ to the semigroups of unitary operators and completely non-unitary operators respectively. The techniques introduced in [3]–[7] allow one to prove (see [5, 3]) that for every unitary semigroup $U = \{U_t,\ t \in \mathbb{R}\}$ on the Hilbert space K it is possible to choose a β-cocycle V with $V_t - I \in s_2$

such that the cocycle perturbation of T by V leads to the semigroup $\tilde{T}$, whose unitary part $\tilde{T}|_{H_u}$ is unitarily equivalent to U if U has a pure point spectrum or a bounded generator. In the present paper, using the results of [8], we prove that the class of perturbations of the semigroup of shifts T by the β-cocycles V with the property $V_t - I \in s_2,\ t \geq 0$, contains the semigroups of isometric operators with the unitary parts U, which admit the spectral representations with the measures being singular with respect to the Lebesgue measure.

2. Main result

In what follows we shall use the methods of the theory of invariant subspaces of the semigroup of shifts T (see [9]). Suppose that the subspace $K \subset H_0$ and $T_t K \subset K,\ t \geq 0$. Then there is an inner function Θ in the upper half-plane $\mathbb{C}_+$ (i.e., a bounded function, analytic in $\mathbb{C}_+$, such that the modulus of its non-tangential boundary values is equal to one a.e. with respect to the Lebesgue measure on the real line) such that $K = M_\Theta H_0$, where the isometric operator M_Θ is defined by the formula $M_\Theta = \mathcal{F}^{-1}\Theta\mathcal{F}$. Here and in what follows we denote by Θ and $\mathcal{F}$ the operator of multiplication by the function $\Theta(z)$ and the Fourier transform respectively. By P_L we denote the orthogonal projection onto the subspace L.

Let $U = \{U_t,\ t \geq 0\}$ be an arbitrary strongly continuous semigroup of unitary operators on $K^\perp$. Define a strongly continuous semigroup of isometric operators $\tilde{T}_t$ on H_0 by the following formula:

$$\tilde{T}_t = U_t P_{K^\perp} + T_t P_K,\ t \geq 0.$$

By means of the operator M_Θ one can construct a dilation of the semigroup $\tilde{T}$ to a strongly continuous group of unitary operators $\tilde{S}$ on H such that

$$\tilde{S}_{-t} = S_{-t} P_{H_t^\perp} + M_\Theta S_{-t} P_{H_t \ominus H_0} + \tilde{T}_t P_{H_0},\ t \geq 0, \tag{3}$$

and for $t > 0$ we should put $\tilde{S}_t = \tilde{S}^*_{-t}$. Now let us introduce the α-cocycle W by the formula $W_t = \tilde{S}_t S_{-t},\ t \in \mathbb{R}$. Then, by construction, one can see that $W_t|_{H_t^\perp} = I,\ t \geq 0$, or, what is equivalent, by virtue of the equality (2), we get $W_{-t}|_{H_0^\perp} = I,\ t \geq 0$. Thus, W is a Markovian cocycle, the subspace H_0 is invariant with respect to W_t for $t \leq 0$, and it is possible to determine the β-cocycle $V_t = W_{-t}|_{H_0},\ t \geq 0$ (see [6]). Notice that the semigroup U is the unitary part of $\tilde{T}$. In [3]–[7] it was shown that if U has a pure point spectrum or a bounded generator, then there is an inner function Θ such that the cocycles W and V determined by the group (3) satisfy the properties $W_t - I \in s_2,\ t \in \mathbb{R}$, and $V_t - I \in s_2,\ t \geq 0$, respectively. To prove the inclusion in the Hilbert-Schmidt class, an estimation of Θ was used for U with a pure point spectrum, and the von Neumann theorem was used to extend the result to U with a bounded generator. In [3, 7] only β-cocycles were investigated, while in [5, 6] the notion of Markovianity was involved.

Suppose that for a strongly continuous group of unitary operators $\tilde{U}$ on a Hilbert space $\tilde{K}$ there exists a measure μ on the real line $\mathbb{R}$, $\mu(\mathbb{R}) < +\infty$, such that $\tilde{U}$ is unitarily equivalent to the group of unitary operators $\overline{U}$ acting on the Hilbert space $L^2(\mu)$ by the formula

$$(\overline{U}_t f)(x) = e^{itx} f(x),\ f \in L^2(\mu). \tag{4}$$

The representation (4), which is said to be spectral, surely exists if there is a vector $e \in \tilde{K}$ cyclic with respect to the group $\tilde{U} = e^{itH}$ (i.e., a vector e such that for the generator H the vectors $\{f(H)e \mid f \in L^\infty\}$ form a dense set in $\tilde{K}$). An arbitrary group of unitary operators $\tilde{U}$ is unitarily equivalent to a no more than countable orthogonal sum of unitary groups of the form (4) (see [10]). Note that a group of unitary operators possessing both bounded generator and pure point spectrum admits the spectral decomposition (4). So the following theorem gives also an alternative proof of the results obtained in [3]–[7].

Theorem 1. *Suppose $\tilde{U}$ is a strongly continuous semigroup on a Hilbert space $\tilde{K}$ that admits a spectral representation* (4)*, where μ is singular with respect to the Lebesgue measure. Then there are an inner function Θ and a semigroup of unitary operators U on the Hilbert space $K^\perp = H_0 \ominus M_\Theta H_0$, which is unitarily equivalent to $\tilde{U}$, such that the Markovian α-cocycle $W_t = \tilde{S}_t S_{-t},\ t \in \mathbb{R}$, determined by the group* (3) *and its restriction $V_t = W_{-t}|_{H_0},\ t \ge 0$, which is a β-cocycle, satisfy the properties $W_t - I \in s_2,\ t \in \mathbb{R}$ and $V_t - I \in s_2,\ t \ge 0$.*

Proof. The measure of the spectral representation μ defines the singular inner function Θ:

$$\Theta(z) = \exp\left(i \int\limits_{\mathbb{R}} \frac{d\mu(t)}{t-z}\right),\ z \in \mathbb{C}_+.$$

It is shown in [8] that the condition $\mu(\mathbb{R}) < +\infty$ implies that

$$1 - \Theta \in H^2(\mathbb{C}_+). \tag{5}$$

Essentially, this is a half-plane version of the results of P. R. Ahern and D. N. Clark [11] for the unit disk. The Ahern-Clark theorem states that if an inner function θ in the unit disk has a finite angular derivative at a point $\zeta,\ |\zeta| = 1$, then the nontangential boundary value $\theta(\zeta)$ is well-defined and the function

$$g_\zeta(z) = \frac{1 - \overline{\theta(\zeta)}\theta(z)}{1 - \overline{\zeta} z}$$

belongs to the Hardy class H^2 in the disk. Now the inclusion (5) follows by means of a conformal mapping of the unit disk onto the half-plane.

It follows from (5) that the operator $(1-\Theta)\mathcal{F} : L^2([0,t]) \to H$ belongs to the Hilbert-Schmidt class s_2, since its kernel $k(x,y) = (1-\Theta(y))e^{-ixy}$ satisfies the condition $k \in L^2([0,t] \times \mathbb{R})$. Thus,

$$P_{H_0 \ominus H_{-t}} M_\Theta P_{H_0 \ominus H_{-t}} - P_{H_0 \ominus H_{-t}} \in s_2,\ t \ge 0. \tag{6}$$

By virtue of the theorem on triangulation of the truncated shift (see [9]), the group $\overline{U}$ defined by the formula (4) is unitarily equivalent to a certain group $U = \{U_t \mid t \in \mathbb{R}\}$ acting on the space $K^\perp$ such that

$$U_t = P_{K^\perp} T_t|_{K^\perp} + R,\ R \in s_2. \tag{7}$$

Thus, the operator U_t is a normal part of the operator of the truncated shift $P_{K^\perp} T_t|_{K^\perp}$.

Let us define the operators $\tilde{S}_t$ by (3). Then

$$\tilde{S}_{-t} - S_{-t} = (T_t - U_t) P_{K^\perp} + (M_\Theta - I) S_{-t} P_{H_t \ominus H_0}. \tag{8}$$

Considering the cocycle $W_t = \tilde{S}_t S_{-t}$, we get that (8) is equivalent to

$$W_{-t} - I = (T_t - U_t) P_{K^\perp} S_t P_{H_{-t}} + (M_\Theta - I) P_{H_0 \ominus H_{-t}},\ t \ge 0. \tag{9}$$

The first term in the right-hand side of (9) belongs to s_2 by virtue of the theorem on triangulation (7). The second term is in s_2 due to (6). □

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Department of Higher Mathematics, Moscow Institute of Physics and Technology, Dolgoprudni 141700, Russia

E-mail address: `amosov@fizteh.ru`

Department of Mathematics and Mechanics, St. Petersburg State University, 28, Universitetski pr., St. Petersburg, 198504, Russia

Current address: Laboratoire de Mathématiques Pures, Université Bordeaux 1, 351, Cours de la Libération, 33405 Talence, France

E-mail address: `antonbaranov@netscape.net`