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DESCRIPTIVE PROPERTIES OF THE SET OF EXPOSED POINTS OF COMPACT CONVEX SETS IN $\mathbb{R}^3$

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ABSTRACT. We construct a compact convex subset of $\mathbb{R}^3$ such that the set of its exposed points is not the intersection of an F_σ set and a G_δ set. The existence of such a set answers a question posed by G. Choquet, H.H. Corson and V.L. Klee.

Let us recall that $x \in C$ is called an *exposed point* of the convex set $C \subset \mathbb{R}^n$ if there is a real linear functional f on $\mathbb{R}^n$ such that $f(y) < f(x)$ for all $y \in C,\ y \neq x$. Clearly, x is an exposed point of C if and only if $\{x\} = H \cap C$ for some supporting hyperplane H of C. The set of all exposed points of C is denoted by $\exp C$. We use $\operatorname{conv} A$ to denote the convex hull of A, and $\overline{\operatorname{conv}}\, A$ to denote its closure.

It was shown in [1, Theorem 1.1] that if K is a compact convex set in $\mathbb{R}^3$, then $\exp K$ is the union of a G_δ set and a set which is the intersection of a G_δ set and an F_σ set.

V.L. Klee [3] constructed a compact convex set K in $\mathbb{R}^3$ such that $\exp K$ is not a G_δ set. H.H. Corson [2] gave an example of a compact convex set K in $\mathbb{R}^3$ such that $\exp K$ is not even the union of a G_δ set and an F_σ set.

The aim of our remark is to answer in the negative the following question posed by G. Choquet, H. Corson and V. Klee [1, Problem 1.2]: if K is a three-dimensional compact convex set, must $\exp K$ be the intersection of a G_δ set and an F_σ set?

Theorem 1. *Given a set $E = F \cup G \subset S = \{x \in \mathbb{R}^2;\ \|x\| = 1\}$ such that F is an F_σ subset of S and G is a G_δ subset of S, then there is a compact convex set $C \subset \mathbb{R}^3$ with $\{x \in \mathbb{R}^2;\ (x, 0) \in \exp C\} = F \cup G$.*

The following lemma is probably known; however, we were not able to find a reference. We shall provide the simple proof.

Lemma 2. *There are an F_σ subset F of S and a G_δ subset G of S such that their union $F \cup G$ is not the intersection of an F_σ set and a G_δ set.*

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If we apply Theorem 1 with the set constructed in Lemma 2 we obtain a negative answer to the problem of G. Choquet, H. Corson and V. Klee.

Proof of Lemma 2. As S contains an open subset homeomorphic to $\mathbb{R}$, it is sufficient to look for the required sets in $\mathbb{R}$. Also, as $\mathbb{R}$ contains a homeomorphic copy of the Cantor set $C = \{0,1\}^{\mathbb{N}}$, we may work in C. Finally, as the product $C^2 = C \times C$ is homeomorphic to C, it is enough to find the required sets in C^2.

Let Q be a countable dense subset of C. Then Q^2 is an F_σ set and $(C \setminus Q)^2$ is a G_δ set. We claim that $Q^2 \cup (C \setminus Q)^2$ is not the intersection of an F_σ set and a G_δ set.

Suppose that $Q^2 \cup (C \setminus Q)^2 = E \cap H$, where E is F_σ and H is G_δ in C^2. As E contains the second category set $(C \setminus Q)^2$, it follows that E itself is of second category in C^2 and, being an F_σ set, its interior in not empty. Suppose $U \times V \subset E$, where U and V are nonempty open subsets of C. Let $x \in U \cap Q$ be fixed. Then we have $(\{x\} \times Q) \cap (\{x\} \times V) = H \cap (\{x\} \times V)$, and thus $Q \cap V$ is a G_δ set, which is not the case. □

Proof of Theorem 1. Let $F = \bigcup_{n\in\mathbb{N}} F_n$ and $S \setminus G = \bigcup_{n\in\mathbb{N}} E_n$, where the sets F_n, E_n $(n \in \mathbb{N})$ are closed.

For $x \in S$ and $n \in \mathbb{N}$, let $f_{x,n}$ be the real linear functional on $\mathbb{R}^3$ such that $f_{x,n}(0,0,n) = 1$, $f_{x,n}(x,0) = 1$ and $f_{x,n} \le 1$ on S. Put $H_{x,n} = f_{x,n}^{-1}((-\infty,1])$. Then $H_{x,n}$ is a half-space such that its boundary $\partial H_{x,n}$ touches the circle S.

Using the notation $B_+ = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3 ;\ \|(x,y)\| \le 1,\ y \ge 0\}$ and $B_- = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R} ;\ (x,-y) \in B_+\}$, we define $C = C_+ \cup C_-$, where

$$C_+ = B_+ \cap \bigcap_{n\in\mathbb{N}} \bigcap_{x\in F_n} H_{x,n}$$

and

$$C_- = \overline{\operatorname{conv}} \left(B_- \cup \left(\bigcup_{n=1}^{\infty} E_n \times \left[-\frac{1}{n}, 0 \right] \right) \right).$$

Clearly, C is compact, and it is convex since the convex sets C_+ and C_- satisfy

$$C_+ \subset \{(x,y) \in \mathbb{R}^2 \times \mathbb{R} ;\ \|x\| \le 1,\ y \ge 0\},$$
$$C_- \subset \{(x,y) \in \mathbb{R}^2 \times \mathbb{R} ;\ \|x\| \le 1,\ y \le 0\},$$

and

$$C_+ \cap C_- \supset \{(x,0) \in \mathbb{R}^2 \times \mathbb{R} ;\ \|x\| \le 1\}.$$

It remains to show that $\exp C \cap (S \times \{0\}) = (F \cup G) \times \{0\}$.

To show that $F \times \{0\} \subset \exp C$, suppose that $z_0 = (x_0, 0) \in F_n \times \{0\}$ for some $n \in \mathbb{N}$. Then the supporting plane $\{(x, 2y) \in \mathbb{R}^2 \times \mathbb{R} ;\ f_{x_0,n}(x,y) = 1\}$ meets C in the single point z_0, thus $z_0 \in \exp C$.

Let $x_0 \in G$ and $z = (x_0, y) \in C$. We shall show that necessarily $y = 0$, and so the point $z_0 = (x_0, 0)$ is an exposed point of C (it is even an exposed point of the set $\{(x,y) \in \mathbb{R}^2 \times \mathbb{R} ;\ \|x\| \le 1,\ x \ne x_0\} \cup \{(x_0,0)\}$ which contains C). Clearly $y \le 0$, as $C_+ \subset B_+$. In what follows we are using the fact that the convex hull of a compact subset of $\mathbb{R}^3$ is compact (and thus closed) [4, Theorem 17.2]. For an arbitrary $n \in \mathbb{N}$ we have

$$C_- = \operatorname{conv} \left(\operatorname{conv} \left(B_- \cup \bigcup_{k=1}^{n} (E_k \times [-\tfrac{1}{k}, 0]) \right) \cup \overline{\operatorname{conv}} \left(\bigcup_{k=n}^{\infty} (E_k \times [-\tfrac{1}{k}, 0]) \right) \right).$$

Thus $z = (x_0, y) = \alpha_1 \cdot (x_1, y_1) + \alpha_2 \cdot (x_2, y_2)$, where α_1, α_2 are nonnegative reals with $\alpha_1 + \alpha_2 = 1$, $(x_1, y_1) \in \operatorname{conv}(B_- \cup \bigcup_{k=1}^n (E_k \times [-\frac{1}{k}, 0]))$ and $(x_2, y_2) \in \overline{\operatorname{conv}} \bigcup_{k=n}^\infty (E_k \times [-\frac{1}{k}, 0])$. Since $\|x_0\| = 1$ and $\|x_i\| \leq 1$ $(i = 1, 2)$, it follows that $x_0 = x_1 = x_2$. Thus $y_1 = 0$, as $B_- \cap (S \times (-\infty, 0)) = \emptyset$, and $x_1 = x_0 \in G \subset S \setminus \bigcup_{k=1}^n E_k$. Therefore $y = \alpha_2 y_2 \geq \alpha_2 \cdot (-\frac{1}{n}) \geq -1/n$, and, as $n \in \mathbb{N}$ was arbitrary, we have $y = 0$.

To prove the other inclusion, we suppose that $x \in S \setminus (F \cup G)$ is such that $(x, 0) \in \exp C$, and we proceed to a contradiction. As $x \notin G$, it belongs to E_{n_0} for some $n_0 \in \mathbb{N}$. Thus $\{x\} \times [-\frac{1}{n_0}, 0] \subset C$, and necessarily $\partial H_{x,n_1} \cap C = f_{x,n_1}^{-1}(1) \cap C = \{(x, 0)\}$ for some sufficiently large $n_1 \in \mathbb{N}$.

On the other hand, we shall prove that for any $n \in \mathbb{N}$ (we shall use it for $n = n_1$) there is an $\varepsilon > 0$ such that

$$B_+ \cap \operatorname{conv}\Big((S \times \{0\}) \cup \{(0, 0, n)\}\Big) \cap \operatorname{conv}\Big\{(x, 0), (x, \varepsilon), (0, 0, 1)\Big\} \subset C. \tag{1}$$

Suppose for a while that (1) is proved. Then we make the observation that there is an element distinct from $(x, 0)$ in the set

$$\partial H_{x,n_1} \cap B_+ \cap \operatorname{conv}\Big((S \times \{0\}) \cup \{(0, 0, n_1)\}\Big) \cap \operatorname{conv}\Big\{(x, 0), (x, \varepsilon), (0, 0, 1)\Big\} \cap C,$$

which contradicts the fact that $f_{x,n_1}^{-1}(1) \cap C = \{(x, 0)\}$.

It remains to prove (1). As $\operatorname{dist}(x, \bigcup_{k=1}^n F_k) > 0$, there is an $\varepsilon > 0$ such that

$$\bigcap_{k=1}^n \bigcap_{x' \in F_k} H_{x',k} \supset \bigcap_{k=1}^n \bigcap_{x' \in F_k} H_{x',1} \supset \operatorname{conv}\{(x, 0), (x, \varepsilon), (0, 0, 1)\}.$$

Also

$$\bigcap_{k=n}^\infty \bigcap_{x' \in F_k} H_{x',k} \supset \bigcap_{k=n}^\infty \bigcap_{x' \in F_k} H_{x',n} \supset \operatorname{conv}\left((S \times \{0\}) \cup \{(0, 0, n)\}\right).$$

Now the inclusion (1) follows from the definition of C_+. □

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