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ON FIBERS OF THE TORIC RESOLUTION OF THE EXTENDED PRYM MAP

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ABSTRACT. We study the minimal toric resolution of the extended Prym map. We describe the blowup at certain singular points of the indeterminacy locus of the extended Prym map.

0. INTRODUCTION

The classical Prym map is a morphism from the moduli space R_g of smooth curves C of genus $2g-1$ with a base point free involution ι (so that the quotient curve $C' = C/\iota$ has genus g) to the moduli space of principally polarized abelian varieties A_{g-1} of dimension $g-1$. It associates to each pair (C, ι) its Prym variety $P(C, \iota)$ together with a principal polarization. Recall from [ABH, V] that:

(1) There is an extended rational map $\varphi : \overline{R}_g \to \overline{A}_{g-1}^{\mathrm{Vor}}$. Here, $\overline{R}_g$ is the moduli space of stable curves of genus $2g-1$ with an involution ι whose only fixed points are some of the nodes, and such that the branches at these nodes are not interchanged. $\overline{A}_{g-1}^{\mathrm{Vor}}$ is the toroidal compactification for the 2nd Voronoi fan, an irreducible component in the complete moduli $\overline{\mathrm{AP}}_{g-1}$ of stable semiabelic pairs, [A].
(2) The locus of the indeterminacy of φ is precisely the closure

$$\overline{\bigsqcup_{n\geq 2}} \mathrm{FS}_n,$$

where $\mathrm{FS}_n \subset \overline{R}_g$ is the Friedman-Smith locus, which we will define below.

This raises a basic question:

Question 0.1. *Describe the minimal blowup $p : R \to \overline{R}_g$ resolving the singularities of the rational map φ, so that $\psi : R \to \overline{A}_{g-1}^{\mathrm{Vor}}$ is a morphism.*

An unpublished result by Alexeev answers this question generically, i.e. along each component FS_n. The moduli space $\overline{R}_g$ can be covered locally by finite quotients U/G of toroidal schemes U. In each neighborhood U the locus FS_n is a smooth codimension n subscheme. Along it the morphism p is an ordinary blowup of FS_n with the reduced scheme structure, i.e. a $\mathbb{P}^{n-1}$-fibration.

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For singular points of the indeterminacy locus of φ, resolution is more complicated. In this paper we answer Question 0.1 for (C, ι) whose dual graph is a double ring (as defined below).

1. Double ring

By their constructions, both $\overline{R}_g$ and $\overline{A}_{g-1}^{\mathrm{Vor}}$ have toroidal structures. More precisely, there exist toroidal schemes U'_α, U_β so that U'_α/G', U_β/G are open neighborhoods in $\overline{R}_g, \overline{A}_{g-1}^{\mathrm{Vor}}$ for some finite groups G', G so that the rational map φ is locally described by a rational map $U'_\alpha \to U_\beta$. Each of these maps, in turn, corresponds to a map of fans $\phi\colon (N', \Sigma') \to (N, \Sigma)$. It follows that the Prym map is resolved by performing a blowup of $\overline{R}_g$ associated to the toric blowup $q\colon V_\alpha \to U'_\alpha$ or, equivalently, the subdivision $(N', \Sigma' \cap \phi^{-1}\Sigma)$ of (N', Σ'). That is the required description of the resolution $p : R \to \overline{R}_g$.

Recall that we call a vertex of the dual graph a *bold vertex* if the corresponding component of the curve is fixed by the involution, and we call an edge a *bold edge* if the corresponding node is fixed and such that the branches at these nodes are not interchanged. We call all other vertices and edges *ordinary*. In other words, an involution on a curve induces an involution on the dual graph. The bold vertices and edges are those that are invariant under this involution.

We denote by DR_n, and call *a double ring of length* n, the following graph. The graph has n bold vertices and $2n$ ordinary edges, so that after factoring by the involution the graph, DR_n/ι is a cycle of length n. In this case the involution is base point free; it fixes components of C and interchanges nodes. Figure 1 illustrates the case $n = 6$.

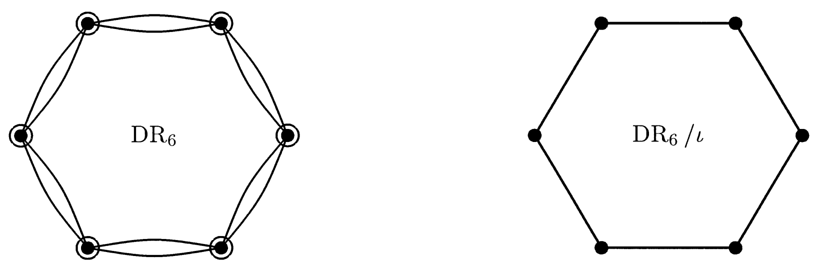

Figure 1.

Recall now that the point $(C, \iota) \in \overline{R}_g$ is in the Friedman-Smith locus FS_n, $n \geq 1$, if $C = C_1 \cup C_2$ is the union of two irreducible components, both invariant under the involution, intersecting at $2n$ points, so that the involution is base point free and interchanges these $2n$ nodes in pairs.

Curves with the graph DR_n are degenerations of curves from FS_2. Points of DR_n for $n \geq 3$ are in the singular locus of $\overline{\mathrm{FS}_2}$.

For DR_n we have the following lattices:

(1) $C_1(\Gamma, \mathbb{Z}) = \bigoplus_{j=1}^{2n} \mathbb{Z}$. Where $\Gamma = \mathrm{DR}_n$ is the dual graph of the curve C, it has vertices $v_1, \ldots, v_n$ and edges $e_1, \ldots e_{2n}$. Denote by z_j the standard

coordinate functions on $C_1(\Gamma,\mathbb{R})$. The involution acts on this group by

$$\begin{cases} \iota(e_j) = e_{j+n}, & \text{for } j \le n, \\ \iota(e_j) = e_{j-n}, & \text{for } n < j \le 2n. \end{cases}$$

(2) $X = H_1(\Gamma,\mathbb{Z}) \subset C_1(\Gamma,\mathbb{Z})$, a group of rank $n+1$.
(3) $\pi^- : X \twoheadrightarrow X^-$, $\pi^-(z) = (z - \iota z)/2$. The group X^- naturally embeds into $\frac{1}{2}C_1(\Gamma,\mathbb{Z})$. In our example, X^- has rank n.

By [A, ABH] the degeneration of Prym varieties is described by two pieces of data:

(1) Discrete: the Delaunay decomposition $\mathrm{Del}(\sum c_j z_j^2)$ of X^-, where c_j can be read off the monodromy action on $H_1(P(C_t,\iota_t),\mathbb{Z})$ and can be arbitrary positive integers, and
(2) Continuous: this is a collection of the data describing two semiabelian group varieties and a trivialization of a certain biextension.

For every element $x \in X^-$ we have $z_i(x) = -z_{n+i}(x)$. It is easy to see that $X^- \to \frac{1}{2}\operatorname{Span}(e_1,\dots,e_n)$ is an embedding identifying X^- with a subgroup of half-integral vectors (x_i) (i.e. vectors with all $x_i \in \frac{1}{2}\mathbb{Z}$) with all x_i being integral or all x_i being non-integral: $X^- = \mathbb{Z}^n \sqcup ((\frac{1}{2},\dots,\frac{1}{2}) + \mathbb{Z}^n)$. The quadratic form $\sum z_j^2$ gives the lattice D_n^*.

Let us consider a positive definite quadratic form $\sum c_j z_j^2$. To describe the corresponding Delaunay decomposition of X^-, we will describe all centers of maximal-dimensional Delaunay cells.

Lemma 1.1. *The center of a maximal-dimensional Delaunay cell cannot have more than one non-half-integral coordinate (i.e. at most one coordinate does not belong to $\frac{1}{2}\mathbb{Z}$).*

Proof. Assume that $\alpha = (\alpha_1,\dots,\alpha_n)$ is a center of the Delaunay cell. Assume α_1 and α_2 are not half-integral. Consider $\beta_1, \beta_2 \in \frac{1}{2}\mathbb{Z}$, such that $\beta_j < \alpha_j < \beta_j + \frac{1}{2}$ for $j = 1,2$. For the points of the lattice X^- closest to α, the first (resp. the second) coordinate can be only β_1 or $\beta_1 + \frac{1}{2}$ (resp. β_2 or $\beta_2 + \frac{1}{2}$). For instance, if for $(x_1,\dots,x_n) \in X^-$ we have $x_1 < \beta_1$, then the point $(x_1+1, x_2,\dots,x_n) \in X^-$ is closer to α. There are four possibilities for the pair of the first two coordinates, but for elements of X^- all coordinates are either all integral or all non-integral. Therefore, for the closest points of X^- to α we have only two possible combinations of the first two coordinates. These points are vertices of the Delaunay cell centered at α. So that cell projects to the plane spanned by the first two coordinates as a segment or a point, and therefore cannot be maximal dimensional. □

Note 1.2. Projections of a maximal-dimensional Delaunay cell on the coordinate axes are of length 1 except for at most one coordinate for which the length is $\frac{1}{2}$. For a maximal-dimensional cell, a coordinate of the center is half-integral if and only if the length of the corresponding projection of the cell is 1. Moreover, in this case the image of the center of the Delaunay cell is the center for the image of the cell. A coordinate of the center is not half-integral if and only if the image of the Delaunay cell has length $\frac{1}{2}$. In this case the center projects in the interior of the image of the Delaunay cell.

Proposition 1.3. *Let $\phi : (N',\Sigma') \to (N,\Sigma)$ be a toric model of the Prym map in a neighborhood of a point (C,ι) with the dual graph $\Gamma(C,\iota) = \mathrm{DR}_n$. Then Σ'*

consists of a single positive quadrant in $\mathbb{Z}^n$ and its faces, and the image $\phi(\Sigma')$ is a single simplicial cone consisting of elements $\sum_{j=1}^n c_j\sigma_j$ with $c_j \geq 0$ subdivided by the Voronoi cones by cutting along the hyperplanes

$$\sum_{j\in J} c_j = \sum_{j\notin J} c_j$$

for every proper subset $J \subset \{1,\dots,n\}$.

Proof. Generators σ_j of the cone $\phi(\Sigma')$ correspond to quadratic forms z_j^2. Two elements $\sum_{j=1}^n c_j\sigma_j$ and $\sum_{j=1}^n c_j'\sigma_j$ corresponding to the quadratic forms $\sum c_j z_j^2$ and $\sum c_j' z_j^2$ are in the same Voronoi cone if their quadratic forms give the same Delaunay decomposition.

Consider sets of vertices of X^- closest to half-integral points. After shifting by an element of $\mathbb{Z}^n \subset X^-$ we can assume that all coordinates of a half-integral point $\alpha = (\alpha_1,\dots,\alpha_n)$ are either 0 or $\frac{1}{2}$. We call $J_\alpha \subset \{1,\dots,n\}$ a subset of coordinates of α that are equal to $\frac{1}{2}$. The square of the distance from α to $(0,\dots,0)$ is $\frac{1}{4}\sum_{j\in J_\alpha} c_j$ and the square of the distance from α to $(\frac{1}{2},\dots,\frac{1}{2})$ is $\frac{1}{4}\sum_{j\notin J_\alpha} c_j$. If $\sum_{j\in J_\alpha} c_j < \sum_{j\notin J_\alpha} c_j$, then the set of points closest to α consists of $(x_1,\dots,x_n) \in X^-$ with

$$\begin{cases} x_j = 0 \text{ or } 1, \text{ for } j \in J, \\ x_j = 0, \text{ for } j \notin J. \end{cases}$$

If $\sum_{j\in J_\alpha} c_j > \sum_{j\notin J_\alpha} c_j$, then the set of closest points consists of $(x_1,\dots,x_n) \in X^-$ with

$$\begin{cases} x_j = \frac{1}{2}, \text{ for } j \in J, \\ x_j = \pm\frac{1}{2}, \text{ for } j \notin J. \end{cases}$$

If $\sum_{j\in J_\alpha} c_j = \sum_{j\notin J_\alpha} c_j$, then the set of closest points is the union of the two sets described above. In each case we obtain a Delaunay cell that contains α. If for two weights $c = (c_1,\dots,c_n)$ and $c' = (c_1',\dots,c_n')$ there exists J_α so that the two corresponding inequalities differ, then the two obtained cells containing α differ and c and c' are in different Voronoi cones. Therefore, we have proved that subdivision of $\phi(\Sigma)$ by the Voronoi cones is a refinement of subdivision by cutting along the hyperplanes

$$\sum_{j\in J} c_j = \sum_{j\notin J} c_j$$

for every subset $J \subset \{1,\dots,n\}$.

Now assume that two weights c' and c'' are in the same subset after cutting along the hyperplanes $\sum_{j\in J} c_j = \sum_{j\notin J} c_j$. Assume that the Delaunay decomposition $\mathrm{Del}(c')$ has a maximal-dimensional cell with the center $(\alpha_1,\dots,\alpha_n)$. Immediately, $(\alpha_1,\dots,\alpha_n) \notin X^-$. We will show that the Delaunay decomposition $\mathrm{Del}(c'')$ has the same maximal-dimensional cell, though coordinates of its center may be different. By Lemma 1.1 we have two cases.

Case 1. All coordinates of the center $(\alpha_1,\dots,\alpha_n)$ are half-integral.

We will describe all vertices of the cell. Coordinates of the center α are divided into subsets of integral and non-integral. As we know, $X^- = \mathbb{Z}^n \sqcup ((\frac{1}{2},\dots,\frac{1}{2}) + \mathbb{Z}^n)$. If β is one of the closest points of the poset $\mathbb{Z}^n$ to α, then β has coordinates

$$\beta_j = \begin{cases} \alpha_j, \text{ if } \alpha_j \in \mathbb{Z}, \\ \alpha_j \pm \frac{1}{2}, \text{ if } \alpha_j \notin \mathbb{Z}. \end{cases}$$

The square of the distance from α to these points is $\frac{1}{4}\sum_{j,\ \text{s.t.}\alpha_j\notin\mathbb{Z}} c'_j$. If β is one of the closest points of the poset $(\frac{1}{2},\dots,\frac{1}{2})+\mathbb{Z}^n$ to α, then β has coordinates

$$\beta_j = \begin{cases} \alpha_j \pm \frac{1}{2},\ \text{if } \alpha_j \in \mathbb{Z}, \\ \alpha_j,\ \text{if } \alpha_j \notin \mathbb{Z}. \end{cases}$$

The square of the distance from α to these points is $\frac{1}{4}\sum_{j,\ \text{s.t.}\alpha_j\in\mathbb{Z}} c'_j$. This Delaunay cell should have vertices from both the posets in order to be maximal-dimensional, which means $\sum_{j,\ \text{s.t.}\alpha_j\notin\mathbb{Z}} c'_j = \sum_{j,\ \text{s.t.}\alpha_j\in\mathbb{Z}} c'_j$.

We should have the same equality for the weights of the second form

$$\sum_{j,\ \text{s.t.}\alpha_j\notin\mathbb{Z}} c''_j = \sum_{j,\ \text{s.t.}\alpha_j\in\mathbb{Z}} c''_j.$$

The closest points of X^- to α for the metrics given by the form $\sum c''_j z_j^2$ will be the same as the points closest for the form $\sum c'_j z_j^2$ described above. So we have the same Delaunay cell, and in this case its center is also the same half-integral point.

Case 2. One coordinate of the center is not half-integral and all others are half-integral.

Coordinates of α are of three types: integral; non-integral, but half-integral; non-half-integral. The last type consists of one coordinate only, let it be $\alpha_{j_0} \notin \frac{1}{2}\mathbb{Z}$. Let $\gamma \in \frac{1}{2}\mathbb{Z}$, such that $\gamma < \alpha_{j_o} < \gamma + \frac{1}{2}$. The j_0's coordinate of any vertex of the Delaunay cell is either γ or $\gamma+\frac{1}{2}$. Denote $J_1 = \{j \in \{1,\dots,n\}|\ \text{s.t. } (\alpha_j-\gamma)\in\mathbb{Z}\}$ and $J_2 = \{j\in\{1,\dots,n\}|\ \text{s.t. } (\alpha_j-\gamma-\frac{1}{2})\in\mathbb{Z}\}$; so $\{j_0\}\sqcup J_1\sqcup J_2 = \{1,\dots,n\}$. The points of X^- that are closest to the center α consist of two subsets S_1 and S_2. S_1 consists of points with coordinates

$$\beta_j = \begin{cases} \gamma,\ \text{if } j = j_0, \\ \alpha_j,\ \text{if } j \in J_1, \\ \alpha_j \pm \frac{1}{2},\ \text{if } j\in J_2. \end{cases}$$

S_2 consists of points with coordinates

$$\beta_j = \begin{cases} \gamma+\frac{1}{2},\ \text{if } j = j_0, \\ \alpha_j \pm \frac{1}{2},\ \text{if } j \in J_1, \\ \alpha_j,\ \text{if } j\in J_2. \end{cases}$$

The square of the distance from α to S_1 is $c'_{j_0}(\alpha_{j_0}-\gamma)^2+\frac{1}{4}\sum_{j\in J_2}c'_j$. The square of the distance from α to S_2 is $c'_{j_0}(\alpha_{j_0}-(\gamma+\frac{1}{2}))^2+\frac{1}{4}\sum_{j\in J_1}c'_j$. The two inequalities follow:

$$\begin{cases} \sum_{j\in J_1} c'_j < c'_{j_0} + \sum_{j\in J_2} c'_j, \\ \sum_{j\in J_1} c'_j + c'_{j_0} > \sum_{j\in J_2} c'_j. \end{cases}$$

Hence, we should have the same inequalities for the weights of the second form:

$$\begin{cases} \sum_{j\in J_1} c''_j < c''_{j_0} + \sum_{j\in J_2} c''_j, \\ \sum_{j\in J_1} c''_j + c''_{j_0} > \sum_{j\in J_2} c''_j. \end{cases}$$

The points of X^- closest to point $\beta_1 = (\alpha_1,\dots,\gamma,\dots,\alpha_n)$ with the metrics given by the form $\sum c''_j z_j^2$ are points of S_1. The points of X^- closest to point $\beta_2 = (\alpha_1,\dots,\gamma+\frac{1}{2},\dots,\alpha_n)$ are points of S_2. There is a point β somewhere between β_1 and β_2 with the two distances from β to S_1 and to S_2 being equal. Then we have

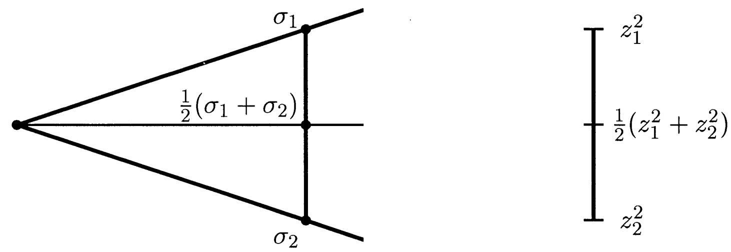

FIGURE 2.

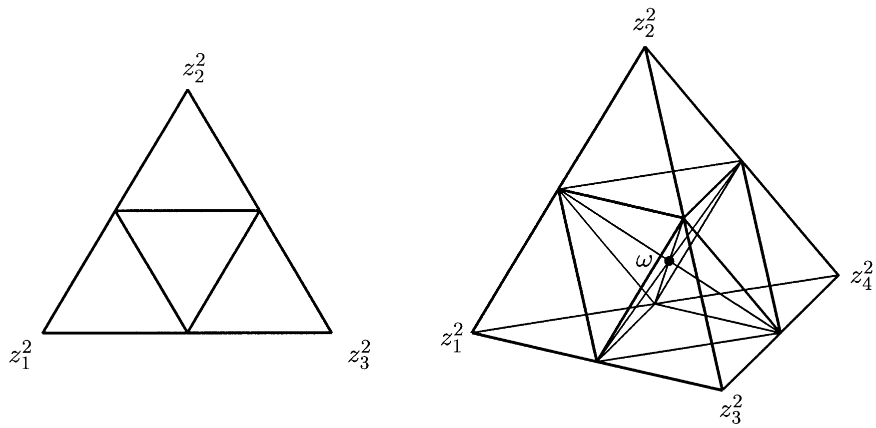

FIGURE 3.

the Delaunay cell with the center β for the form $\sum c_j'' z_j^2$ the same as the cell with the center α for the form $\sum c_j' z_j^2$.

We have proved that all maximal-dimensional Delaunay cells for the weights c' and c'' are the same. Every non-maximal-dimensional cell is a face of some maximal-dimensional cell. Therefore, for the weights c' and c'' non-maximal-dimensional cells also are the same. This proves that c' and c'' are in the same Voronoi cone. The subdivision of $\phi(\Sigma')$ by the Voronoi cone is the same as the subdivision by cutting along the hyperplanes $\sum_{j\in J} c_j = \sum_{j\notin J} c_j$. □

Let us use Proposition 1.3 to analyze the fibers of q for small n. Note that fibers of p may differ from fibers of q by an action of a finite group.

From Proposition 1.3 it follows that for $\mathrm{DR}_2 = \mathrm{FS}_2$ fibers of q are isomorphic to $\mathbb{P}^1$. Figure 2 shows the decomposition of the 2-dimensional simplicial cone $\phi(\Sigma')$ as well as the decomposition of the 1-simplex Δ_1 (so that $\phi(\Sigma')$ is the cone over Δ_1).

The decompositions of Δ_2 for DR_3 and Δ_3 for DR_4 are shown in Figure 3.

For DR_3 Voronoi cones simply cut the corners of Δ_2. Each fiber of q is 3 copies of $\mathbb{P}^1$, all intersecting at one point.

For DR_4 Voronoi cones cut off the corners of Δ_3 first, then cut the remaining octahedron into 8 simplexes. The main component of each fiber of q is $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, corresponding to the ray passing through the center $\omega = \frac{1}{4}\sum z_i^2$. In addition, each fiber contains 4 copies of $\mathbb{P}^1$ attached to $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.

When $n \geq 5$, the fibers of q and the fan $(N', \Sigma' \cap \phi^{-1}\Sigma)$ become even more complicated. The subdivision of the simplicial cone $\phi(\Sigma')$ has several 1-dimensional Voronoi cones in the interior (e.g. rays $(c\sigma_1 + c\sigma_2 + \cdots + (n-3)c\sigma_i + \cdots + c\sigma_n)$, for $c \geq 0$), which means that fibers of p have several components of the maximal possible dimension.

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