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THE IDENTITY IS ISOLATED AMONG COMPOSITION OPERATORS

C-H. CHU, R. V. HÜGLI, AND M. MACKEY

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Abstract. Let $H^\infty(B)$ be the Banach algebra of bounded holomorphic functions on the open unit ball B of a Banach space. We show that the identity operator is an isolated point in the space of composition operators on $H^\infty(B)$. This answers a conjecture of Aron, Galindo and Lindström.

1. Introduction

Let B be the open unit ball of a complex Banach space E, and let $H^\infty(B)$ be the uniform algebra of bounded complex-valued holomorphic functions on B, with the supremum norm $\|f\| = \sup_{x\in B}|f(x)|$. Given any holomorphic self-map ϕ of B, we define the *composition operator* $C_\phi : H^\infty(B) \to H^\infty(B)$ by

$$C_\phi(f) = f\circ\phi \qquad (f \in H^\infty(B)).$$

The collection of such operators with the operator norm topology is denoted by $\mathcal{C}(H^\infty(B))$. This space has been widely studied, and recently, Aron, Galindo and Lindström [1] have determined its path connected components for some special Banach spaces E, thereby extending results of MacCluer, Ohno and Zhao [6] for the case when B is the open unit disc Δ in the complex plane. A main result in [1] is

Theorem 1.1 ([1, Theorem 16])**.** *If $E = C_0(X)$ or E is a Hilbert space, then the composition operators C_ϕ and C_ψ lie in the same path connected component in $\mathcal{C}(H^\infty(B))$ if and only if $\|C_\phi - C_\psi\| < 2$.*

Furthermore, using techniques involving w-strong peak points and determining sets for $H^\infty(B)$ when B belongs to some special Banach spaces, the following result is established.

Theorem 1.2 ([1, Corollary 12])**.** *The identity operator is an isolated point in $\mathcal{C}(H^\infty(B))$ when E is $C_0(X)$ or ℓ_1 or any strictly convex reflexive Banach space.*

Two open questions were raised in [1]. First, does Theorem 1.1 hold when E is a *JB**-triple? A positive answer to this question has been given in [7]. The second conjecture is that Theorem 1.2 holds for every Banach space E. We give a positive

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answer and a simple proof in this paper. We use only the hyperbolic metric, but do not require w-strong peak points.

2. Proof of the conjecture

The space of all complex-valued homomorphisms on $H^\infty(B)$ forms the maximal ideal space of $H^\infty(B)$ and contains, in particular, the point evaluation functionals $\{\delta_x : x \in B\}$. The pseudo-hyperbolic distance on the maximal ideal space is defined by

$$\beta(m,n) = \sup\{|\hat{f}(n)| : f \in H^\infty(B), \|f\| \le 1, \hat{f}(m) = 0\}$$

where $\hat{f}$ is the Gelfand transform of f. We note from [1, Remark 2] that

$$\|C_\phi - C_\psi\| < 2 \qquad \text{if and only if} \qquad \sup_{x\in B} \beta(\delta_{\phi(x)}, \delta_{\psi(x)}) < 1. \tag{1}$$

The *Carathéodory distance* on B is given by

$$C_B(x,y) = \sup\{\gamma(f(x), f(y)) : f \in H(B,\Delta)\}$$

for x, $y \in B$, where γ is the Poincaré metric on the disc Δ and $H(B,\Delta)$ the space of holomorphic maps from B to Δ. Both C_B and β are contracted by holomorphic functions and preserved by biholomorphic functions.

The metric

$$d_B(x,y) := \sup\{|f(x) - f(y)| : f \in H^\infty(B), \|f\| \le 1\} \qquad (x, y \in B)$$

and its relation to the Carathéodory distance C_B are examined in [7] where it is shown that

$$d_B(x,y) = \frac{2 - 2\sqrt{1 - (\tanh C_B(x,y))^2}}{\tanh C_B(x,y)} \qquad (x, y \in B). \tag{2}$$

We have $d_B(x,y) \ge d_\Delta(h(x), h(y))$ for any $h \in H(B,\Delta)$. Also,

$$d_B(x,y) \le 2 \sup\{|f(x)| : f \in H^\infty(B), \|f\| \le 1, f(y) = 0\}. \tag{3}$$

To see this, it suffices to note that for any $f \in H^\infty(B)$ with $\|f\| \le 1$, the function f^y defined by $f^y(x) = \frac{1}{2}(f(x) - f(y))$ is also in the closed unit ball of $H^\infty(B)$.

Let E^* be the dual of a complex Banach space E. We denote the unit spheres of E and E^* by $S(E)$ and $S(E^*)$ respectively. Given $x \in S(E)$, we denote the set of support functionals of x by

$$\operatorname{supp}(x) = \{f \in E^* : \|f\| = f(x) = 1\}.$$

Let $T : E \longrightarrow E$ be a bounded complex linear operator. We recall that the *spatial numerical range* of T is defined by

$$V(T) = \{f(Tx) : x \in S(E), f \in \operatorname{supp}(x)\}$$

(cf. [3]). By [3, Theorem 3.9.4], the numerical radius $v(T)$ of T is given by

$$v(T) = \sup\{|\lambda| : \lambda \in V(T)\}.$$

We also have, by [3, Theorem 1.4.1],

$$\|T\| \ge v(T) \ge \frac{1}{e}\|T\|. \tag{4}$$

We now prove the conjecture that subsumes Theorem 1.2.

Theorem 2.1. *Let E be a Banach space with open unit ball B. Then the identity operator is an isolated point in the space of composition operators on $H^\infty(B)$.*

Proof. Let $I : H^\infty(B) \longrightarrow H^\infty(B)$ be the identity operator, and suppose that C_ϕ is in the component of I for some holomorphic self-map ϕ of B. We show that ϕ is the identity map on B. We have $\|C_\phi - I\| < 2$ by [5] and $\sup_{x\in B} \beta(\delta_{\phi(x)}, \delta_x) < 1$ by (1). Since

$$\begin{aligned}\beta(\delta_{\phi(x)}, \delta_x) &= \sup\{|\hat{f}(\delta_{\phi(x)})| : f \in H^\infty(B), \|f\| \le 1, \hat{f}(\delta_x) = 0\}\\ &= \sup\{|f(\phi(x))| : f \in H^\infty(B), \|f\| \le 1, f(x) = 0\}\\ &\ge \frac{1}{2} \sup\{|f(\phi(x)) - f(x)| : f \in H^\infty(B), \|f\| \le 1\} \qquad \text{(by (3))}\end{aligned}$$

we see that

$$\sup_{x\in B} d_B(\phi(x), x) < 2$$

and hence (2) gives

$$\sup_{x\in B} C_B(\phi(x), x) < \infty$$

or that

$$\sup_{x\in B, f\in H(B,\Delta)} \gamma(f(x), f(\phi(x))) < \infty.$$

Let $\lambda \in S(E^*)$ be norm-attaining; that is, there exists $x_\lambda \in S(E)$ with $\lambda(x_\lambda) = 1$. Define $\psi : \Delta \to \Delta$ by $\psi(\zeta) = \lambda(\phi(\zeta x_\lambda))$. Then ψ is holomorphic, and we have

$$\begin{aligned}\sup_{\zeta\in\Delta} \gamma(\zeta, \psi(\zeta)) &= \sup_{\zeta\in\Delta} \gamma(\lambda(\zeta x_\lambda), \lambda(\phi(\zeta x_\lambda)))\\ &\le \sup_{x\in B} \gamma(\lambda(x), \lambda(\phi(x)))\\ &\le \sup_{x\in B, f\in H(B,\Delta)} \gamma(f(x), f(\phi(x))) \ < \infty.\end{aligned}$$

Since $\gamma(\zeta, \psi(\zeta)) = \tanh^{-1} \beta(\zeta, \psi(\zeta))$ on Δ, we have $\sup_\Delta \beta(\zeta, \psi(\zeta)) < 1$ and it follows from the one-dimensional result that $\psi = id_\Delta$. Hence we have

$$\zeta = \lambda(\phi(\zeta x_\lambda)) \qquad (\zeta \in \Delta). \tag{5}$$

In particular, we have $\lambda(\varphi(0)) = 0$. By the Bishop-Phelps theorem [2], the norm-attaining functionals in E^* are norm-dense in E^*. Therefore $\phi(0) = 0$. Let us write (5) in the form

$$id_\Delta = \lambda \circ \phi \circ i_{x_\lambda}$$

where $i_{x_\lambda} : \Delta \to B$ is the map $i_{x_\lambda}(\zeta) = \zeta x_\lambda$. Taking the derivative at $\zeta \in \Delta$ of both sides we obtain

$$1 = \lambda(\phi'(\zeta x_\lambda)(x_\lambda))$$

which gives

$$1 = \lambda(\phi'(0)x_\lambda).$$

The above arguments imply that, for any $x \in S(E)$ and $f \in \operatorname{supp}(x)$, we have $1 = f(\phi'(0)x)$. Let $T = \phi'(0) - I$. We obtain

$$\begin{aligned}V(T) &= \{f(Tx) : x \in S(E), f \in \operatorname{supp}(x)\}\\ &= \{f(\phi'(0)x) - f(x) : x \in S(E), f \in \operatorname{supp}(x)\}\\ &= \{0\}.\end{aligned}$$

It follows from (4) that $\|T\| = 0$. Hence $\phi'(0) = I$. Since we have already established that $\phi(0) = 0$, Cartan's uniqueness theorem asserts that ϕ itself is the identity map on B as required (see [4, Proposition 6.6]). $\square$

Corollary 2.2. *Let E be a Banach space and ψ a biholomorphic self-map of the open unit ball B of E. Then C_ψ is isolated in $\mathcal{C}(H^\infty(B))$.*

Proof. The result is true for $\psi = id$ from above. Now observe that C_ψ is a homeomorphism of $\mathcal{C}(H^\infty(B))$ that takes the identity to the composition operator C_ψ. $\square$

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School of Mathematical Sciences, Queen Mary College, University of London, London E1 4NS, England

E-mail address: c.chu@qmul.ac.uk

Department of Mathematics, University of California, Irvine, California 92697

E-mail address: remo.huegli@stat.unibe.ch

Department of Mathematics, University College, Dublin 4, Ireland

E-mail address: michael.mackey@ucd.ie