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AN EXTREMAL PROBLEM OF QUASICONFORMAL MAPPINGS

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Abstract. In this paper, the following problem is studied. Let Ω_1 and Ω_2 be two domains in the complex plane with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Suppose that $f_j : \Omega_j \to f_j(\Omega_j)$ $(j = 1, 2)$ are two quasiconformal mappings satisfying $f_1|_{\Omega_1 \cap \Omega_2} = f_2|_{\Omega_1 \cap \Omega_2}$. Let F be the mapping in $\Omega_1 \cup \Omega_2$ defined by $F|_{\Omega_j} = f_j$ $(j = 1, 2)$. If both f_1 and f_2 are uniquely extremal, is F always uniquely extremal? It is shown in this paper that the answer to this problem is no.

§1. Introduction

Let Ω and D be two domains in the complex plane $\mathbb{C}$ and let $f : \Omega \to D$ be a quasiconformal mapping from Ω onto D. This means that f is an orientation-preserving homeomorphism of Ω onto D with locally L^2−generalized derivatives $\partial_z f$ and $\partial_{\overline{z}} f$ which satisfy the Beltrami equation

$$\partial_{\overline{z}} f(z) = \mu(z)\partial_z f(z), \quad z \in \Omega,$$

where μ is a bounded measurable function with $\|\mu\|_\infty < 1$. The function μ is called the Beltrami coefficient of f, and

$$K[f] = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$$

is called the maximal dilatation of f. It is well known that a quasiconformal mapping can be continuously extended to the accessible boundary points of Ω. So the boundary values of a quasiconformal mapping between two domains whose boundaries consist of Jordan arcs and isolated points are well defined. In what follows we always assume that the domains under consideration have such boundaries.

A quasiconformal mapping $f : \Omega \to D$ is said to be extremal if $K[f] \leq K[g]$ for any quasiconformal mapping $g : \Omega \to D$ with $g|_{\partial\Omega} = f|_{\partial\Omega}$. If f is an extremal quasiconformal mapping and $K[f] < K[g]$ for any quasiconformal mapping $g : \Omega \to D$ with $g|_{\partial\Omega} = f|_{\partial\Omega}$ and $g \not\equiv f$, then f is said to be uniquely extremal.

A basic problem in the theory of quasiconformal mappings is to determine whether a given quasiconformal mapping is extremal or uniquely extremal, and to characterize the uniquely extremal mapping (see [Ah], [BLM], [LV], [RS1], [RS2]

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and so on). There has been important progress in characterizing uniquely extremality in recent years (see [BLM] and [Re]).

In his paper [Re1], E. Reich studied the following problem. Let Ω_1 and Ω_2 be two domains with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\partial\Omega_1 \cap \partial\Omega_2 = \gamma$, where γ is a Jordan arc. Let f_j be a quasiconformal mapping of Ω_j $(j = 1, 2)$ with $f_1|_\gamma = f_2|_\gamma$ and let F be the quasiconformal mapping in $\Omega = \Omega_1 \cup \Omega_2 \cup \gamma$ defined by $F|_{\Omega_j} = f_j$ $(j = 1, 2)$. If both f_1 and f_2 are uniquely extremal, is F uniquely extremal? Reich provided a counterexample to this problem in [Re1].

In this paper, we shall study the following problem, posed by Chen Jixiu and Shen Yuliang [CS], which is an improvement of the above problem. Let Ω_1 and Ω_2 be two domains with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Suppose that $f_1 : \Omega_1 \to D_1$ and $f_2 : \Omega_2 \to D_2$ are two quasiconformal mappings satisfying $f_1|_{\Omega_1\cap\Omega_2} = f_2|_{\Omega_1\cap\Omega_2}$. Let F be the mapping in $\Omega_1 \cup \Omega_2$ defined by $F|_{\Omega_j} = f_j$ $(j = 1, 2)$. If both f_1 and f_2 are uniquely extremal, is F always uniquely extremal?

This problem is also connected with many other studies in characterizing unique extremality of quasiconformal mappings (see Theorems 2.3, 3.1, 4.1 and Example 5.3.1 in [Re]).

In this paper, we will construct some counterexamples where F is not uniquely extremal or even not extremal.

§2. Some counterexamples

We look at the quadratic differential

$$\varphi(z)dz^2 = \frac{dz^2}{(z^2-1)^2}.$$

Then φdz^2 is a holomorphic quadratic differential in $\overline{\mathbb{C}} \setminus \{1, -1\}$ and has poles of order two at $z = 1$ and $z = -1$. It is easy to see that

$$\varphi(x)dx^2 = \frac{dx^2}{(x^2-1)^2} > 0, \qquad \text{for all } x \in \mathbb{R} \setminus \{1, -1\},$$

and

$$\varphi(iy)d(iy)^2 = \frac{-dy^2}{(y^2+1)^2} < 0, \qquad \text{for all } y \in \mathbb{R}.$$

So the intervals $(-\infty, -1), (-1, 1)$ and $(1, +\infty)$ are horizontal trajectories of φdz^2 and the imaginary axis is a vertical trajectory of φdz^2.

To study the trajectory structure of the quadratic differential φdz^2, we look at the function

$$w = \Phi(z) := \frac{1}{2}\log\frac{z-1}{z+1} - i\frac{1}{2}\pi,$$

which is a single-valued holomorphic function in $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ with $\Phi(i) = -i\pi/4$. The function $w = \Phi(z)$ maps $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ to the strip

$$\Sigma := \{u + iv \mid -\infty < u < +\infty;\ -\tfrac{1}{2}\pi < v < \tfrac{1}{2}\pi\}.$$

On the other hand,

$$\{\Phi'(z)\}^2 = \varphi(z).$$

So each horizontal line in Σ corresponds to a horizontal trajectory of φdz^2 and each vertical segment in Σ corresponds to a vertical trajectory of φdz^2.

Let

$$u(z) := \operatorname{Re}\Phi(z) = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right|$$

and let

$$v(z) := \operatorname{Im}\Phi(z) = \frac{1}{2}[\arg(z-1) - \arg(z+1)] - \frac{1}{2}\pi.$$

Then $u(z) = C$ $(C \in \mathbb{R})$ is a horizontal trajectory of φdz^2 and $v(z) = C$ $(C \in (-\frac{1}{2}\pi, \frac{1}{2}\pi))$ is a vertical trajectory of φdz^2.

It is easy to check that $u(z) = C$ is a circle with the radius

$$r = \left(\frac{1+e^C}{1-e^C}\right)^2 - 1$$

and the center $z = (1+e^C)/(1-e^C)$. One can easily see that, for each $C \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, $v(z) = C$ is a circle (or a straight line) that passes through 1 and -1, for each point at which, the circumference angle with respect to 1 and -1 is $C + \pi/2$ or $\pi/2 - C$.

Let $f : \mathbb{C} \to \mathbb{C}$ be the quasiconformal mapping of $\mathbb{C}$ onto $\mathbb{C}$ with the Beltrami coefficient

$$\mu = k\frac{\overline{\varphi}}{|\varphi|}, \quad \text{where } k \in (0,\, 1),$$

keeping the points 0 and i fixed.

The quasiconformal mapping $w = f(z)$ can also be got by the following construction.

Let $\zeta = g(w)$ be a stretch mapping of Σ onto itself defined by

$$w = u + iv \mapsto \zeta = Ku + iv,$$

where $K = (1+k)/(1-k)$. Then the mapping $\tilde{f} : z \mapsto \Phi^{-1} \circ g \circ \Phi(z)$ is a quasiconformal mappings of $\mathbb{C}\setminus\{(-\infty,\, -1]\cup[1,\, +\infty)\}$ onto itself with the Beltrami coefficient $\mu = k\overline{\varphi}/|\varphi|$. Obviously, $\tilde{f}$ can be extended to the slits $(-\infty, -1]$ and $[1, +\infty)$ so that $\tilde{f}$ is a quasiconformal mapping of the whole plane. By the above construction we see that every point on the imaginary axis is fixed by $\tilde{f}$. From the uniqueness theorem of quasiconformal mappings, we see that $\tilde{f} \equiv f$.

Now we give our first counterexample, as follows.

Example 1. Let

$$\Omega_1 = \{z = x + iy \mid x < 0,\ x^2 + (y-1)^2 < 2;\ \text{or } x < 0,\ \ x^2 + (y+1)^2 < 2\}$$

and let

$$\Omega_2 = \{z = x + iy \mid (x+2)^2 + y^2 < 1\}.$$

Let f_j be the restriction of f to Ω_j $(j = 1,\, 2)$. Note that $\Phi \circ f_1 \circ \Phi^{-1}$ is a stretch mapping of the trip

$$\{w = u + iv|\ u > 0;\ -3\pi/8 < v < 3\pi/8\}.$$

By a result of Strebel [St], $\Phi \circ f_1 \circ \Phi^{-1}$ is uniquely extremal, and hence so is f_1.

Since φ is meromorphic on $\overline{\Omega}_2$ and has only one pole of order 2 at $z = -1$, it follows from a result of Sethares [Se] that f_2 is also uniquely extremal.

Let $F := f|_{\Omega_1\cup\Omega_2}$. Then F is not uniquely extremal. In fact, $\Omega = \Omega_1 \cup \Omega_2$ is bounded by the imaginary axis and 3 circles with a puncture $z = -1$, and the mapping F is a Teichmüller mapping associated with a quadratic differential φdz^2 that has a pole at $z = -1$. Taking a sufficiently small disk D around $z = -1$, then F is smooth on ∂D and hence the boundary dilatation of $F|_{D\setminus\{-1\}}$ is one. Then there is a Teichmüller mapping $\tilde{F}$ of $D\setminus\{-1\}$, associated with a holomorphic quadratic differential of finite norm, such that

$$\tilde{F}|_{\partial D\cup\{-1\}} = F|_{\partial D\cup\{-1\}}.$$

As the quadratic differential associated with $\tilde{F}$ is regular on D or has a pole of order one at $z=-1$, we see that $\tilde{F}$ is different from $F|_{D\setminus\{-1\}}$. Therefore $F = f|_\Omega$ is not uniquely extremal.

This example shows that the union of two uniquely extremal quasiconformal mappings is still extremal, but it is not uniquely extremal. We shall give another example which shows that the union of two uniquely extremal quasiconformal mappings need not even be extremal.

Example 2. Let $K>1$ be a real number and let $f_K(z) := |z|^{K-1}z$. Then f_K is a quasiconformal mapping of $\mathbb{C}$ onto itself with 0 and 1 fixed and with the Beltrami coefficient $\mu = kz/\overline{z}$, where $k=(K-1)/(K+1)$. The Beltrami coefficient μ can be expressed as $\mu = k\overline{\phi}/|\phi|$, where

$$\phi(z) = \frac{1}{z^2}.$$

We consider

$$\Omega_1 := \{x+iy \mid x^2+y^2<1 \text{ and } y>x,\ \textit{or}\ y<-x\}$$

and

$$\Omega_2 := \{x+iy \mid x^2+y^2<1 \text{ and } x>0\}.$$

Let $f_j = f_K|_{\Omega_j}$ $(j=1,2)$ and let $F = f_K|_\Omega$, where $\Omega = \Omega_1\cup\Omega_2$.

Both f_1 and f_2 are uniquely extremal. In fact, f_1 can be expressed as

$$f_1 = \Psi^{-1}\circ g\circ\Psi(z), \quad z\in\Omega_1,$$

where Ψ is a single-valued branch of $-\log z$ on $\mathbb{C}$ with a slit $[0,+\infty)$ and g is a stretch mapping $u\mapsto Ku,\ v\mapsto v$ of a strip. Making use of the result in [St] again, we see that f_1 is uniquely extremal. Similarly, we can prove that f_2 is also uniquely extremal.

On the other hand, it is easy to see that $\Omega = \Delta\setminus\{0\}$, where Δ is the unit disc, and the boundary correspondence of $F:\Omega\to\Omega$ is the identity. Obviously, F is not extremal.

In the above examples, the union of domains Ω is doubly connected and one component of its boundary is an isolated point. Now we are going to give other examples in which the union of domains is a simply connected domain and its boundary consists of Jordan arcs.

To construct such examples, we need a new result [Ma] obtained by V. Marković. It says that an affine stretch of the plane $\mathbb{C}$ punctured at integer lattices is uniquely extremal.

Example 3. Let $\mathbb{Z}$ be the set of integer numbers. Define

$$\Lambda_1 := \{m+ni\in\mathbb{C} \mid m\in\mathbb{Z};\ n\in\mathbb{Z}\}$$

and

$$\Lambda_2 := \{m+(n+\tfrac{1}{2})i\in\mathbb{C} \mid m\in\mathbb{Z};\ n\in\mathbb{Z}\}.$$

Let

$$\Omega = \{x+yi \mid y>\max\{C,\ x^2\}\},$$

where $C > 0$ is a constant. Now we consider the stretch mapping

$$g_K : x \mapsto Kx;\ y \mapsto y,$$

where $K > 1$. It is known that $g_K|_\Omega$ is extremal but not uniquely extremal (see [AH] or [RS2]).

Let

$$\Omega_1 := \Omega \setminus \{x + ni \in \Omega \mid n \in \mathbb{Z};\ x \geq x'_n\},$$

where x'_n the smallest number of the set $A_n := \{m \mid m \in \mathbb{Z},\ m + ni \in \Omega\}$, and let

$$\Omega_2 := \Omega \setminus \{x + (n + \tfrac{1}{2})i \mid n \in \mathbb{Z};\ x \geq x''_n\},$$

where x''_n is the smallest number of the set $B_n := \{m \mid m \in \mathbb{Z},\ m + (n + \frac{1}{2})i \in \Omega\}$.

Now define $f_j = g_K|_{\Omega_j}$ $(j = 1, 2.)$. Then both f_1 and f_2 are uniquely extremal. In fact, any quasiconformal mapping g of Ω_1 with $g|_{\partial\Omega_1} = f_1|_{\partial\Omega_1}$ can be extended to the whole plane by defining $g = g_K$ outside of Ω. The resulting mapping g has the same values on Λ_1 as g_K. It follows from the result in [Ma] that g_K is extremal with respect to the boundary correspondence

$$\Lambda_1 \to g_K(\Lambda_1) : m + ni \mapsto Km + ni,\ m, n \in \mathbb{Z}.$$

So $K[g] \geq K = K[f_1]$, and hence $f_1 = g_K|_{\Omega_1}$ is extremal with respect to its boundary correspondence. The uniqueness of the extremal mapping g_K with respect to the stretch of Λ_1 also implies the uniqueness of the extremal mapping f_1 with respect to its boundary correspondence. Thus f_1 is uniquely extremal. Similarly, we can also conclude that f_2 is uniquely extremal.

However, it is known that $F = g_K|_\Omega$ is extremal but not uniquely extremal.

If we consider the domain

$$\{x + yi \mid y > \max\{C,\ |x|\}\}$$

instead of the parabolic domain in Example 3, and use a result of Reich and Strebel (cf. [AH], [Re2] and [RS3]), then we can get another example which shows that the union of two uniquely extremal mappings is not extremal ([RS3] or [AH]).

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