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A COUNTEREXAMPLE TO A LOWER BOUND FOR A CLASS OF PSEUDODIFFERENTIAL OPERATORS

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Abstract. We present a counterexample to a possible improvement of a lower bound for a class of pseudodifferential operators with symplectic characteristic manifold.

1. Introduction

Lower bounds for pseudodifferential operators are usually applied in the study of several problems of existence and uniqueness for partial differential equations. Among the most celebrated inequalities, we recall the sharp Gårding inequality, the Fefferman-Phong inequality and Hörmander's inequality (see, for instance, Chapters XVIII and XXIII in Hörmander [4], and the references therein). Here we are interested in a recent generalization of Hörmander's inequality to pseudodifferential operators with multiple characteristics, obtained by Parenti and Parmeggiani in [7].

Precisely, let $X \subset \mathbb{R}^n$ be an open subset, and let $P = P^* \in \mathrm{OPS}^m(X)$ be a classical and formally self-adjoint pseudodifferential operator. Let $p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$ be its Weyl symbol, with $p_{m-j}(x,\xi)$ positively homogeneous of degree $m-j$ with respect to the variable ξ. We denote by $\Sigma = p_m^{-1}(0) \subset T^*X \setminus 0$ its characteristic manifold, and we assume that P belongs to Boutet de Monvel's classes $\mathrm{OPN}^{m,k}(X)$, i.e., for $j < k/2$, $p_{m-j}(x,\xi)$ vanishes to the order $k-2j$ on Σ, k being an even integer. Moreover, we suppose that the principal symbol p_m of P is non-negative and is transversally elliptic with respect to Σ (i.e., p_m vanishes exactly to the order k on Σ). It is well known that many properties of the operator P (hypoellipticity, spectral lower bound,...) are strictly related to a smooth map $p^{(k)}$, invariantly defined on the normal bundle $N\Sigma$, and obtained by the Taylor expansion of the symbol p on Σ (see Boutet de Monvel [1] or Parenti and Parmeggiani [6] for the precise definition).

Suppose now that Σ is a symplectic submanifold of $T^*X \setminus 0$. Let $\rho \in \Sigma$, and let $\zeta : T^*\mathbb{R}^\nu \to T_\rho\Sigma^\sigma$ ($2\nu = \operatorname{codim}\Sigma$) be any linear symplectomorphism. Setting $p_\zeta(y,\eta) = p^{(k)}(\rho, \zeta(y,\eta))$ for $(y,\eta) \in T^*\mathbb{R}^\nu$, we then consider the Weyl quantization $P_{\rho,\zeta} = \mathrm{Op}^w(p_\zeta)(y, D_y) : \mathcal{S}(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu)$, where $N\Sigma$ is identified with $T\Sigma^\sigma$. As shown in [6], the spectrum of $P_{\rho,\zeta}$, as an unbounded operator on $L^2(\mathbb{R}^\nu)$, is independent of the parametrization ζ, and it turns out to be discrete and bounded

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from below; thus the lowest eigenvalue $\lambda(\rho) := \min \mathrm{Spec}(\mathrm{Op}^w(p_\zeta))$ is a continuous function on Σ, independent of ζ. Similarly, the dimension of the corresponding (finite dimensional) eigenspace $V_{\rho,\zeta} \subset \mathcal{S}(\mathbb{R}^\nu)$ is invariantly defined, as well.

We can now recall the main result of [7] (see Theorem 1.5 and Remark 4.8 and 4.10) we are concerned with.

Theorem 1.1. *Let $P = P^* \in \mathrm{OPN}^{m,k}(X,\Sigma)$, P transversally elliptic, and suppose Σ is symplectic. Moreover, assume that:*

(i) *for any $\zeta : T^*\mathbb{R}^\nu \to T_\rho\Sigma^\sigma$ as above, one has*

$$(\mathrm{Op}^w(p_\zeta)(y,D_y)f,f) \geq 0, \forall f \in \mathcal{S}(\mathbb{R}^\nu), \forall \rho \in \Sigma;$$

(ii) *if $\lambda(\rho_0) = 0$, there exists a conic neighborhood $\Gamma \subset \Sigma$ of ρ_0 such that*

$$\dim V_{\rho,\zeta} = \mathrm{const}, \ \forall \rho \in \Gamma.$$

Then for any compact subset $K \subset X$ there exists $C_K > 0$ such that

$$(Pu,u) \geq -C_K \|u\|^2_{m/2-(k+1)/4}, \quad \forall u \in C_0^\infty(K). \tag{1.1}$$

If we suppose, **in addition**, *that*

(iii) *the eigenspace $V_{\rho,\zeta}$ consists of functions which are all even for every $\rho \in \Gamma$ or all odd for every $\rho \in \Gamma$,*

then the following stronger lower bound holds: for any compact subset $K \subset X$ there exists $C_K > 0$ such that

$$(Pu,u) \geq -C_K \|u\|^2_{m/2-(k+2)/4}, \quad \forall u \in C_0^\infty(K). \tag{1.2}$$

The estimate (1.2) is a generalization of Hörmander's inequality to pseudodifferential operators with characteristics of even order $k \geq 2$.

In the subsequent paper [8], the authors relaxed hypothesis (ii), but it was an open question if hypothesis (iii) is really a necessary condition. In this note we give a counterexample which shows that hypothesis (iii) cannot be omitted in order to obtain (1.2). Note that, in the double characteristic case treated by Hörmander [3], the eigenspace $V_{\rho,\zeta}$ has dimension 1; hence conditions (ii) and (iii) are automatically satisfied (see [7]).

Actually we shall show that, under the hypotheses (i) and (ii) only, if $k > 2$ the following estimate is in general false:

For every $\epsilon > 0$, every $\mu < m/2-(k+1)/4$, and for every compact subset $K \subset X$ there exists $C_{\epsilon,\mu,K}$ such that

$$(Pu,u) \geq -\epsilon\|u\|^2_{m/2-(k+1)/4} - C_{\epsilon,\mu,K}\|u\|^2_\mu, \quad \forall u \in C_0^\infty(K). \tag{1.3}$$

Therefore estimate (1.1) is sharp, since inequality (1.3) immediately follows from (1.2).

2. The counterexample

Let $P = P^* \in \mathrm{OPN}^{m,k}(\mathbb{R}^n,\Sigma)$, with $\Sigma = \{(x,\xi) \in T^*\mathbb{R}^n \setminus 0 : x' = \xi' = 0\}$, $\xi = (\xi',\xi'')$ being the dual variables to $x = (x',x'') \in \mathbb{R}^\nu \times \mathbb{R}^{n-\nu}$ (this is actually the microlocal model for any symplectic submanifold). We set

$$p^{(k)}(\rho = (x'',\xi''),x',\xi') = \sum_{|\alpha|+|\beta|+2j=k} (\alpha!\beta!)^{-1}\partial^\alpha_{x'}\partial^\beta_{\xi'}p_{m-j}(x'=0,x'',\xi'=0,\xi'')x'^\alpha\xi'^\beta$$

and

$$p^{(k+1)}(\rho=(x'',\xi''),x',\xi') = \sum_{|\alpha|+|\beta|+2j=k+1} (\alpha!\beta!)^{-1}\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}p_{m-j}(x'=0,x'',\xi'=0,\xi'')x'^{\alpha}\xi'^{\beta}.$$

Consider now the operators $p^{(k)}(\rho,x',D_{x'})$ and $p^{(k+1)}(\rho,x',D_{x'})$ on $\mathcal{S}(\mathbb{R}^{\nu})$, obtained from $p^{(k)}$ and $p^{(k+1)}$ by quantizing with respect to the variables x',ξ'.

Remark 2.1. Here we consider the usual symbol p of P, since the corresponding localized operator $p^{(k)}(\rho,x',D_{x'})$ turns out to coincide with the localized operator defined in the Introduction by Weyl quantization of the Weyl symbol of P (see Proposition 2.3 in [6] for the proof of this fact).

Proposition 2.1.1. *Suppose that the total symbol $p(x,\xi)$ of P does not depend on the x''-variables and that inequality* (1.3) *holds. Then, for any $\rho_0\in\Sigma$ such that $\lambda(\rho_0)=0$, one has*

$$(p^{(k+1)}(\rho_0,x',D_{x'})v,v)\geq 0,\ \forall v\in \operatorname{Ker} p^{(k)}(\rho_0,x',D_{x'}). \tag{2.1}$$

It will be clear from the proof that the left hand side of (2.1) is real.

Proof. Let $\chi\in C_0^{\infty}(\mathbb{R}^n)$ be a real valued function, identically equal to 1 in a neighborhood of $(0,x_0'')$, with $\rho_0=(0,x_0'',0,\xi_0'')$. We then observe that the operator $\chi P\chi$ satisfies the following global version of (1.3):

For every $\epsilon>0$, and every $\mu<m/2-(k+1)/4$, there exists $C_{\epsilon,\mu}$ such that

$$(\chi P\chi u,u)\geq -\epsilon\|u\|^2_{m/2-(k+1)/4}-C_{\epsilon,\mu}\|u\|^2_{\mu},\quad \forall u\in\mathcal{S}(\mathbb{R}^n). \tag{2.2}$$

Now let $v\in\operatorname{Ker} p^{(k)}(\rho_0,x',D_{x'})$, and let $\phi\in\mathcal{S}(\mathbb{R}^{n-\nu})$ be such that $\|\phi\|_0=1$. We set $u=\phi v\in\mathcal{S}(\mathbb{R}^n)$ and $u_t(x)=e^{it^2\langle x,\xi_0\rangle}u(tx',t^{1/2}(x''-x_0''))$, $x=(x',x'')\in\mathbb{R}^{\nu}\times\mathbb{R}^{n-\nu}$. We now proceed as in [6].

A direct computation yields, for any $s\in\mathbb{R}$,

$$\|u_t\|_s^2=t^{4s-(n+\nu)/2}\big(\|v\|^2_{L^2(\mathbb{R}^{\nu})}+o(1)\big),\ \text{as } t\to+\infty.$$

Moreover, since the total symbol p does not depend on the x''-variables, one gets

$$Pu_t(x)=e^{it^2\langle x,\xi_0\rangle}\psi_t\big(tx',t^{1/2}(x''-x_0'')\big)$$

with

$$\psi_t(y)=(2\pi)^{-n}\int e^{i\langle y,\eta\rangle}p\Big(\frac{y'}{t},x_0'',t\eta',t^{1/2}\eta''+t^2\xi_0''\Big)\widehat{u}(\eta)d\eta.$$

An application of Taylor expansion gives

$$p\big(\frac{y'}{t},x_0'',t\eta',t^{1/2}\eta''+t^2\xi_0''\big)=t^{4(m/2-k/4)}p^{(k)}(\rho_0,y',\eta') + t^{4(m/2-(k+1)/4)}p^{(k+1)}(\rho_0,y',\eta')+o(t^{4(m/2-(k+1)/4)}).$$

Hence,

$$\begin{aligned}(\chi P\chi u_t,u_t)&=t^{4(m/2-k/4)-(n+\nu)/2}(\underbrace{p^{(k)}(\rho_0,x',D_{x'})v}_{=0},v)\\ &+t^{4(m/2-(k+1)/4)-(n+\nu)/2}(p^{(k+1)}(\rho_0,x',D_{x'})v,v)+o(t^{4(m/2-(k+1)/4)-(n+\nu)/2})\\ &\geq -\epsilon t^{4(m/2-(k+1)/4)-(n+\nu)/2}(\|v\|_0^2+o(1))-C_{\epsilon,\mu}t^{4\mu-(n+\nu)/2}(\|v\|_0^2+o(1)).\end{aligned}$$

Upon dividing by $t^{4(m/2-(k+1)/4)-(n+\nu)/2}$ and letting $t \to +\infty$, $\epsilon \to 0^+$, we conclude the proof. □

Remark 2.2. If the total symbol of P depends on all the (x,ξ)-variables, Proposition 2.1.1 is, in general, false. However, by using the localization function

$$u_t(x) = e^{it^2\langle x,\xi_0\rangle} u(t(x-x_0))$$

with $\rho_0 = (x_0,\xi_0) \in \Sigma$ and by arguing as above, it turns out that the following differential operator in $\mathbb{R}^n$:

$$\sum_{|\alpha|+|\beta|+2j=k+1} (\alpha!\beta!)^{-1} (\partial_x^\alpha \partial_\xi^\beta p_{m-j})(\rho_0) x^\alpha D^\beta,$$

is positive if it is restricted on the space $\operatorname{Ker} p^{(k)}(\rho_0, x', D_{x'}) \otimes \mathcal{S}(\mathbb{R}^{n-\nu})$.

It is now easy to produce the counterexample by using the necessary condition (2.1).

For any even integer $k \geq 4$, we can write $k = 4k_1 + 2k_2$ for suitable $k_1 \in \mathbb{N}$, $k_2 \in \{0,1\}$. Consider the following differential operator in $\mathbb{R}^3$:

$$\begin{aligned} P = {} & (D_{x_1}^2 + x_1^2 D_{x_3}^2 + D_{x_2}^2 + x_2^2 D_{x_3}^2)^{k_2} (2(D_{x_1}^2 + x_1^2 D_{x_3}^2) + D_{x_2}^2 + x_2^2 D_{x_3}^2 - 7 D_{x_3})^{2k_1} \\ & + x_1 x_2^2 D_{x_3}^{k/2+1}. \end{aligned}$$

We have $P = P^* \in \mathrm{OPN}^{k,k}(\mathbb{R}^3, \Sigma)$ with

$$\Sigma = \{x_1 = \xi_1 = x_2 = \xi_2 = 0,\ \xi_3 \neq 0\}.$$

Moreover, P is transversally elliptic with respect to Σ and its total symbol does not depend on the variable x_3. For $\rho_0 = (\overline{x}_3, \overline{\xi}_3) \in \Sigma$, and $x' = (x_1, x_2)$, it turns out that

$$\begin{aligned} & p^{(k)}(\rho_0, x', D_{x'}) \\ & \quad = (D_{x_1}^2 + x_1^2 \overline{\xi}_3^2 + D_{x_2}^2 + x_2^2 \overline{\xi}_3^2)^{k_2} (2(D_{x_1}^2 + x_1^2 \overline{\xi}_3^2) + D_{x_2}^2 + x_2^2 \overline{\xi}_3^2 - 7\overline{\xi}_3)^{2k_1}, \end{aligned}$$

and

$$p^{(k+1)}(\rho_0, x', D_{x'}) = x_1 x_2^2 \overline{\xi}_3^{k/2+1}$$

as a multiplication operator. It is easily seen that the spectrum of $p^{(k)}(\rho_0, x', D_{x'})$ is given by

$$\operatorname{Spec} p^{(k)}(\rho_0, x', D_{x'}) = \{F(\rho_0, \beta);\ \beta \in \mathbb{Z}_+^2\},$$

with

$$F(\rho_0, \beta) := |\overline{\xi}_3|^{k_2} (2|\beta| + 2)^{k_2} ((4\beta_1 + 2\beta_2 + 3)|\overline{\xi}_3| - 7\overline{\xi}_3)^{2k_1}.$$

Setting

$$\phi_h(t) := \pi^{-1/4} (2^h h!)^{-1/2} \left(\frac{d}{dt} - t\right)^h e^{-t^2/2},\ h = 0, 1, \ldots,$$

for the h-th Hermite function, it is immediate to check that the function

$$\phi_{\beta_1}(|\overline{\xi}_3|^{1/2} x_1) \phi_{\beta_2}(|\overline{\xi}_3|^{1/2} x_2)$$

is an eigenfunction for $p^{(k)}(\rho_0, x', D_{x'})$ corresponding to the eigenvalue $F(\rho_0, \beta)$. Since tensor products of Hermite functions define a Hilbert basis for $L^2(\mathbb{R}^2)$, we obtain in this way the whole spectrum of $p^{(k)}(\rho_0, x', D_{x'})$. In particular, we see that when $\overline{\xi}_3 < 0$ the localized operator $p^{(k)}(\rho_0, x', D_{x'})$ is invertible, whereas when $\overline{\xi}_3 > 0$ its lowest eigenvalue is zero. In this case we have

$$\operatorname{Ker} p^{(k)}(\rho_0, x', D_{x'}) = \operatorname{span}\{u_1, u_2\},$$

where $u_1(x_1, x_2) = \phi_0(\overline{\xi}_3^{1/2} x_1)\phi_2(\overline{\xi}_3^{1/2} x_2)$ and $u_2(x_1, x_2) = \phi_1(\overline{\xi}_3^{1/2} x_1)\phi_0(\overline{\xi}_3^{1/2} x_2)$, so that hypotheses (i) and (ii) in Theorem 1.1 are fulfilled but not (iii), because u_1 is even and u_2 is odd.

Let us now verify that estimate (1.3) does not hold for P. By taking $u := u_1 + u_2$, we have

$$(p^{(k+1)}(\rho_0, x', D_{x'})u, u) = 2\overline{\xi}_3^{k/2+1} \int x_1 x_2^2 u_1(x_1, x_2) u_2(x_1, x_2)\, dx_1 dx_2$$
$$= 2\overline{\xi}_3^{k/2-3/2} \int x_1 \phi_0(x_1)\phi_1(x_1)\, dx_1 \int x_2^2 \phi_0(x_2)\phi_2(x_2)\, dx_2 < 0,$$

since

$$\int x_1 \phi_0(x_1)\phi_1(x_1)\, dx_1 = -\pi^{-1/2} 2^{1/2} \int x_1^2 e^{-x_1^2}\, dx_1 < 0,$$

and

$$\int x_2^2 \phi_0(x_2)\phi_2(x_2)\, dx_2 = (2\pi)^{-1/2} \int x_2^2 (2x_2^2 - 1) e^{-x_2^2}\, dx_2$$
$$= 2(2\pi)^{-1/2} \int e^{-x_2^2}\, dx_2 > 0.$$

This contradicts (2.1).

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