m-ADIC p-BASIS AND REGULAR LOCAL RING

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Abstract. We introduce the concept of m-adic p-basis as an extension of the concept of p-basis. Let \((S, m)\) be a regular local ring of prime characteristic \(p\) and \(R\) a ring such that \(S \supset R \supset S^p\). Then we prove that \(R\) is a regular local ring if and only if there exists an m-adic p-basis of \(S/R\) and \(R\) is Noetherian.

1. Introduction

Some forty years ago, E. Kunz conjectured the following: If \((S, m)\) is a regular local ring of prime characteristic \(p\), and if \(R\) a ring with \(S \supset R \supset S^p\) such that \(S\) is finite as an \(R\)-module, then the following are equivalent:

1. \(R\) is a regular local ring.
2. There exists a p-basis of \(S/R\).

This was proved by T. Kimura and the second author ([KN2 or K 15.7]). Without the finiteness assumption, however, this result is not true anymore. In this paper we generalize this result to the non-finite situation by introducing a topological generalization of the concept of p-basis, which we call the m-adic p-basis (see Definition 2.1).

Then we prove the following main theorem:

Theorem 3.4. Let \((S, m)\) be a regular local ring of prime characteristic \(p\), and \(R\) a ring such that \(S \supset R \supset S^p\). Then the following conditions are equivalent:

1. \(R\) is a regular local ring.
2. There exists an m-adic p-basis of \(S/R\), and \(R\) is Noetherian.

This theorem covers the following situation: Let \(k\) be a field of characteristic \(p\) with \([k : k^p] = \infty\), and let \(S = k[[X]]\). Then \(S\) is regular, but \(S/S^p\) does not have a p-basis (cf. [KN1 Example 3.8]). However \(S/R\) has an m-adic p-basis. That is, \(C \cup \{X\}\) is an m-adic p-basis of \(S/S^p\), where \(m := (X)S\) and \(C\) is a p-basis of \(k/k^p\).

If \(S\) is finite as an \(R\)-module, then any m-adic p-basis of \(S/R\) is a p-basis of \(S/R\), and \(R\) is Noetherian (see Remark 3.5).

2. Preliminaries

All rings in this paper are commutative rings with identity elements. We always denote by \(p\) a prime number.

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Let $R$ be a ring and $S$ an $R$-algebra with $\text{char}(S) = p$. Let $m$ be an ideal of $S$.
Let $S^p$ denote the subring $\{x^p; x \in S\}$ of $S$, $m^{(p)}$ the ideal $\{x^p; x \in m\}$ of $S^p$ and $RS^p$ the subring of $S$ generated by the set $\{ax^p; a \in R, x \in S\}$. If $(S, m)$ is a local ring and $R$ is a ring such that $S \supset R \supset S^p$, then $R$ is a local ring with the maximal ideal $m \cap R$. For any subset $A$ of $S$, we denote by $\overline{A}$ the set of residue classes of the elements of $A$ modulo $m$. When we say “$\overline{A}$ is a $p$-basis”, we tacitly assume that $A$ maps injectively to $\overline{A}$.

As an extension of the concept of $p$-basis, we introduce the concept of $m$-adic $p$-basis as follows:

**Definition 2.1.** A subset $B$ of $S$ is called an $m$-adic $p$-basis of $S/R$ if the following conditions are satisfied:
1) $B$ is $p$-independent over $RS^p$.
2) $S$ is the closure of the subring $T := RS^p[B]$ in $S$ for the $m$-adic topology; that is, $S = \overline{T} = \bigcap_{r=0}^\infty (T + m^r)$.
3) $m^r \cap T = m^r_T$ for every $r \geq 1$, where $m_T := m \cap T$.

**Lemma 2.2.** Suppose that $(S, m, L)$ is a local ring and $R$ a ring such that $S \supset R \supset S^p$. Then the following assertions are true:
1) There are quasi coefficient fields $k$ and $k_R$ of $S$ and $R$ respectively such that $k \supset k_R \supset k^p$.
2) Let $k$ and $k_R$ be fields satisfying the conditions of (1), and let $S^*$ and $R^*$ be the $m$-adic and $m_R$-adic completions of $S$ and $R$ respectively, where $m_R := m \cap R$. Then there are unique coefficient fields $K$ and $K_R$ of $S^*$ and $R^*$ respectively such that $K \supset k$ and $K_R \supset k_R$ (cf. [M Theorem 28.3]). If $S^* \supset R^* \supset (S^*)^p$, then the following assertions are true:
1) $K \supset K_R \supset K^p$.
2) Any $p$-basis of $k/k_R$ is a $p$-basis of $K/K_R$, and any $p$-basis of $k_R/k^p$ is a $p$-basis of $K_R/K^p$.

**Proof.** Put $L_R := R/m_R$. (1) Let $A = \{a_i; i \in I\}$ and $B = \{b_j; j \in J\}$ be subsets of $S$ and $R$ respectively such that $A := \overline{a_i}; i \in I$ and $B := \overline{b_j}; j \in J$ are $p$-bases of $L/L_R$ and $L_R/L^p$, respectively, where $a_i := a_i + m$, $b_j := b_j + m_R$. For the prime field $h$ of $S$, we consider the subring $h[A, B]$. Since $h[A, B] \cap m = (0)$ in $S$, it follows that $S$ contains the field of quotients $k := h(A, B)$ of $h[A, B]$ and $k$ is a quasi coefficient field of $S$. Furthermore we see that $h(A^p, B^p) := k_R$ is a quasi coefficient field of $R$, where $A^p := \{a_i^p; i \in I\}$. Then we have $k \supset k_R \supset k^p$.

(2) Let $K$ and $K_R$ be the coefficient fields of $S^*$ and $R^*$ such that $K \supset k$ and $K_R \supset k_R$ respectively. Assume that $S^* \supset R^* \supset (S^*)^p$. Since $K$ is integral over the subring $k_R[K^p]$, then $k_R[K^p] := K'$ is a field contained in $R^*$. Let $f : R^* \to L_R$ be the canonical mapping, and $C$ a $p$-basis of $k_R/k^p$. Then $K' = K^p(C)$ and $f(K') = L^p(C)$, where $C := \{f(c); c \in C\}$. Since $C$ is a $p$-basis of $L_R/L^p$, $K'$ is a coefficient field of $R^*$. By the uniqueness of the coefficient field of $R^*$ containing $k_R$, we see that $K' = K_R$, and hence $K \supset K_R \supset K^p$.

Assertion 2) of (2) is clear.

**Lemma 2.3.** Let $L$ be a field with $\text{char}(L) = p$, $K$ a field such that $L \supset K \supset L^p$ and $C$ a $p$-basis of $L/K$. Put

$$S := L[[X_1, \cdots, X_n]] \supset R := K[[X_1^p, \cdots, X_r^p, X_{r+1}, \cdots, X_n]]$$
for some \( r \) \((0 \leq r \leq n)\). Then \( C \cup \{X_1, \cdots, X_r\} \) is an \( m \)-adic \( p \)-basis of \( S/R \), where \( m := (X_1, \cdots, X_n)S \).

**Proof.** We have \( R[X_1, \cdots, X_r] = K[[X_1, \cdots, X_n]] \). Put \( T := R[C, X_1, \cdots, X_r] = K[[X_1, \cdots, X_n]][C] \) and \( m_T := m \cap T \). Then \( S \supset T \supset R \supset S^p \). By [M Theorem 22.3], \( S \) is faithfully flat over \( T \), and it follows from \( m = m_T \) that \( m^r \cap T = m_T^r \) for every \( r \geq 1 \). A routine computation shows that \( C \cup \{X_1, \cdots, X_r\} \) is \( p \)-independent over \( R \). Furthermore, since \( S/m = T/m_T \) and \( m = m_T S \), we have that \( S \) is the closure of \( T \) in \( S \) for the \( m \)-adic topology. \( \square \)

3. **Main result**

In this section, we use the following notation:
- \( S \) is a regular local ring with \( \text{char}(S) = p \),
- \( m \) is the maximal ideal of \( S \),
- \( L = S/m \),
- \( R \) is a ring such that \( S \supset R \supset S^p \),
- \( n = m \cap R \).

For the proof of the main result, we need several lemmas.

**Lemma 3.1.** Suppose that \( R \) is Noetherian. Then the following statements hold:

1. \( S \) is faithfully flat over \( R \) if and only if \( R \) is regular.
2. Let \( S^* = R^* \) and \( (S^p)^* \) be the \( m \)-adic, \( n \)-adic, and \( m^{(p)} \)-adic completions of \( S/R \) and \( S^p \) respectively. If \( R \) is regular, then \( S^* \supset R^* \supset (S^p)^* = (S^*)^p \).

**Proof.** (1) The assertion follows from [M Theorem 23.1 and Theorem 23.7].

(2) Suppose that \( S \) and \( R \) are regular. Since \( S \) is faithfully flat over \( R \) in virtue of (1), the \( n \)-adic topology of \( R \) coincides with the topology induced by the \( m \)-adic topology of \( S \) on the subspace \( R \subset S \). Hence the \( m \)-adic completion \( S^* \) contains the \( n \)-adic completion \( R^* \). The rest of assertion (2) is clear. \( \square \)

**Lemma 3.2.** If \( R \) is regular, then \( n = m^{(p)} R \) or \( n \notin m^2 \).

**Proof.** By almost the same reasoning as that of the proof of [KN2 Lemma 5], we have the assertion. \( \square \)

**Lemma 3.3.** Let \( R \) be regular, and let \( \{a_1, \cdots, a_r\} \) \((a_i \in m, a_i = a_i + (nS + m^2))\) and \( \{b_1, \cdots, b_s\} \) \((b_i \in n, b_i = b_i + m^2)\) be bases of the \( L \)-vector spaces

\[
m/(nS + m^2) \quad \text{and} \quad (nS + m^2)/m^2
\]

respectively. Then \( \dim S = \dim R = r + s \). Furthermore the maximal ideals of \( S \) and \( R \) are expressed respectively as follows:

\[
m = (a_1, \cdots, a_r, b_1, \cdots, b_s)S \quad \text{and} \quad n = (a_1^p, \cdots, a_r^p, b_1, \cdots, b_s)R.
\]

**Proof.** We have \( \dim S = \dim_R m/m^2 = r + s \), and \( \{a_1, \cdots, a_r, b_1, \cdots, b_s\} \) is a regular system of parameters of \( S \). Furthermore we see that \( \{b_1, \cdots, b_s\} \) \((b_i := b_i + n^2 \in n/n^2)\) is linearly independent over \( K := R/n \). Thus there exists a subset \( \{y_1, \cdots, y_r\} \) of \( n \) such that \( \{b_1, \cdots, b_s, y_1, \cdots, y_r\} \) is a regular system of parameters of \( R \). Put \( a := (b_1, \cdots, b_s)R \). Then we have that \( aS \cap R = a \), since \( S/R \) is faithfully flat by Lemma 3.1. Put \( S' := S/aS, m' := m/aS, R' := R/a \) and \( n' := n/a \). Then \( (S', m') \) and \( (R', n') \) are regular local rings, with \( S' \supset R' \supset (S'^p) \), \( \dim S' = \dim R' = r \), and each maximal ideal has the following form respectively:

\[
m' = (a_1', \cdots, a_r')S' \quad \text{and} \quad n' = (y_1', \cdots, y_r')R', \quad \text{where} \quad a_i' := a_i + aS
\]
and $y'_i := y_i + a$. Furthermore we have $r = \dim_L m'/(m')^2 = \dim_L m/(m^2 + aS)$ and $\dim_L m'/((m')^2 + (m')^2) = \dim_L m/(mS + m^2) = r$, so that $\dim_L m'/((m')^2) = \dim_L n'/((n')^2 + (m')^2)$. On the other hand, since

$$\dim_L m'/((m')^2) = \dim_L (n'S' + (m')^2)/(m')^2 + \dim_L n'/((n')^2 + (m')^2),$$

we have

$$\dim_L n'S' + (m')^2/(m')^2 = 0,$$

and thus we get $n'S' + (m')^2 = (m')^2$. By Lemma 3.2, we see that $n' = m^{(p)}R'$. This implies that $n = (a_1^p, \ldots, a_r^p, b_1, \ldots, b_s)R$.

The main result of this paper is the following theorem:

**Theorem 3.4.** Let $(S, m)$ be a regular local ring with $\text{char}(R) = p$ and $R$ a ring such that $S \supset R \supset S^p$. Then the following conditions are equivalent:

1. $R$ is a regular local ring.
2. There exists an $m$-adic $p$-basis of $S/R$, and $R$ is Noetherian.

**Proof.** (1) $\Rightarrow$ (2). With the same notation as for Lemma 3.3, we have $\dim_S = \dim_R = r + s$, $m = (a_1, \ldots, a_r, b_1, \ldots, b_s)S$ and $n = (a_1^p, \ldots, a_r^p, b_1, \ldots, b_s)R$. Let $S^*$ and $R^*$ be the $m$-adic and $n$-adic completions of $S$ and $R$ respectively. Then we have $S^* \supset R^* \supset (S^*)^p$ by Lemma 3.1. Hence there are coefficient fields $K$ and $K_R$ of $S^*$ and $R^*$ respectively such that $K \supset K_R \supset K^p$ and $K/K_R$ has a $p$-basis $C$ with $C \subset S$, by Lemma 2.2. Thus the complete local rings $S^*$ and $R^*$ have the forms $S^* = K[[a_1, \ldots, a_r, b_1, \ldots, b_s]]$ and $R^* = K[[a_1^p, \ldots, a_r^p, b_1, \ldots, b_s]]$, whence $A := C \cup \{a_1, \ldots, a_r\}$ is an $m^*$-adic $p$-basis of $S^*/R^*$ by Lemma 2.3, where $m^* := mS^*$. Then $S \supset A$, and $A$ is also $p$-independent over $R$. Put $T := R[A]$ and $m_T = m \cap T$. Then $(T, m_T)$ is a Noetherian local ring (cf. [KN1] Lemma 2.3). Furthermore, $m_T = (a_1, \ldots, a_r, b_1, \ldots, b_s)T$, and so $(T, m_T)$ is regular. It follows that $S$ is faithfully flat over $T$ by Lemma 3.1. Therefore $m^* \cap T = m_T$ for every $r \geq 1$. Since $T/m_T = S/m$ and $m = m_TS$, we have $S = T$, where $T$ is the closure of $T$ in $S$ for the $m$-adic topology. Consequently, $A$ is an $m$-adic $p$-basis of $S/R$.

(2) $\Rightarrow$ (1). Let $A$ be an $m$-adic $p$-basis of $S/R$. Put $T := R[A]$ and $m_T := m \cap T$. Then $S \supset T \supset S^p$, and $T/m_T = L$. Thus there exists a subset $C$ of $A$ such that $C := \{c; c \in C\}$ ($c := c + m$) is a $p$-basis of $L/K$, where $K := R/n$. Put $A_1 := A - C$ and $P := R[C]$. Then $T = P[A_1]$. Since the L-vector space $m_T/m_T^2$ is a subspace of $m/m^2$, we have $\dim_L m_T/m_T^2 \leq \dim_L m/m^2 = \dim S < \infty$. The module of differentials $\Omega_{T/p}$ is the free $T$-module with the free basis $\{da; a \in A_1\}$, where $d$ is the associated derivation of $\Omega_{T/p}$. On the other hand, we have the following exact sequence:

$$0 \to m_T/(m_T^2 + m_P T) \to \Omega_{T/p}/m_T \Omega_{T/p} \to \Omega_{L/L} \to 0,$$

where $m_P := m \cap P$ (cf. [K] 6.7]). Therefore we get

$$|A_1| = \dim_L \Omega_{T/p}/m_T \Omega_{T/p} = \dim_L m_T/(m_T^2 + m_P T) \leq \dim_L m_T/m_T^2 < \infty,$$

whence $A_1$ is a finite set. Since the ring $P$ is Noetherian (cf. [KN2] Lemma 2.3), $T$ is also Noetherian and hence

$$\dim T \leq \dim_L m_T/m_T^2 \leq \dim_L m/m^2 = \dim S = \dim T,$$

so we have $\dim T = \dim_L m_T/m_T^2$. It follows that $T$ is a regular local ring. By Lemma 3.1, then $S$ is flat over $T$, and so $S$ is flat over $R$. Consequently, $R$ is regular by the same lemma.

\[\square\]
Remarks 3.5. Let \((S, \mathfrak{m})\) be a Noetherian local ring with \(\text{char}(S) = p\), and \(R\) a ring such that \(S \supset R \supset S^p\). Assume that \(S\) is finitely generated as an \(R\)-module. Then it is known that \(R\) is Noetherian (see [M, Theorem 3.7]). Furthermore, a subset \(B\) of \(S\) is an \(\mathfrak{m}\)-adic \(p\)-basis of \(S/R\) if and only if \(B\) is a \(p\)-basis. For, let \(B\) be an \(\mathfrak{m}\)-adic \(p\)-basis of \(S/R\). Then \(T := R[B]\) is an \(R\)-submodule of the \(R\)-module \(S\), and \((R, n) (n := \mathfrak{m} \cap R)\) is a Zariski ring. Since \(\mathfrak{m}^s \subset nS\) for some \(s \geq 1\), the \(\mathfrak{m}\)-adic topology of \(S\) coincides with the \(n\)-adic topology of the \(R\)-module \(S\). Thus \(T\) is closed in \(S\) for the \(\mathfrak{m}\)-adic topology, and so we have \(S = T\). Therefore \(B\) is a \(p\)-basis of \(S/R\).

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