A CRITERION FOR SATELLITE 1-GENUS 1-BRIDGE KNOTS

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Abstract. Let $K$ be a knot in a closed orientable irreducible 3-manifold $M$. Suppose $M$ admits a genus 1 Heegaard splitting and we denote by $H$ the splitting torus. We say $H$ is a 1-genus 1-bridge splitting of $(M, K)$ if $H$ intersects $K$ transversely in two points, and divides $(M, K)$ into two pairs of a solid torus and a boundary parallel arc in it. It is known that a 1-genus 1-bridge splitting of a satellite knot admits a satellite diagram disjoint from an essential loop on the splitting torus. If $M = S^3$ and the slope of the loop is longitudinal in one of the solid tori, then $K$ is obtained by twisting a component of a 2-bridge link along the other component. We give a criterion for determining whether a given 1-genus 1-bridge splitting of a knot admits a satellite diagram of a given slope or not. As an application, we show there exist counter examples for a conjecture of Ait Nouh and Yasuhara.

1. Introduction

Let $M$ be a closed orientable irreducible 3-manifold, and $K$ a knot in $M$. We say that $K$ is a 1-genus 1-bridge knot if $(M, K)$ has a 1-genus 1-bridge splitting $H$, that is, there is a Heegaard splitting torus $H$ of $M$ such that $H$ intersects $K$ transversely in two points and $K$ intersects each of the solid tori bounded by $H$ in a trivial arc. (Here, an arc $t$ embedded in a solid torus $V$ with $t \cap \partial V = \partial t$ is called trivial if it is boundary parallel, that is, there is a disc $C$ in $V$ such that $t \subset \partial C$ and $C \cap \partial V = cl(\partial C - t)$. We call such a disc $C$ a cancelling disc of $t$.) The class of 1-genus 1-bridge knots contains all torus knots and 2-bridge knots.

Let $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a 1-genus 1-bridge splitting, and $C_i$ a cancelling disc of $t_i$ in $V_i$ for $i = 1$ and 2. Set $s_i = \partial C_i \cap H$. Then the overstrand $s_1$ and the understrand $s_2$ together give a 1-genus 1-bridge diagram of the splitting. It is a satellite diagram if there is an essential loop $\ell$ in $H$ with $\ell \cap (s_1 \cup s_2) = \emptyset$. We call the isotopy class of such a loop $\ell$ in $H$ (rather than $H - K$) a slope of the satellite diagram. A 1-genus 1-bridge splitting admits a satellite diagram if there is such a pair of cancelling discs. See [1], and also [5]. If the slope of a 1-genus

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1-bridge satellite diagram is meridional in one of the solid tori $V_1$ and $V_2$, then the knot $K$ is trivial. When the slope is longitudinal on $\partial V_i$, $K$ can be obtained from a component of a 2-bridge link by Dehn surgery on the other component, as is essentially shown in [8]. (In fact, $K$ has a 1-bridge diagram on the annulus $A = \text{cl} (\partial V_i - N(\ell))$; that is, shrinking $C_1$ and $C_2$, we can isotope $K$ to be the union of an overstrand very near to $s_1$ and an understrand very close to $s_2$. We can take a core of the other solid torus $V_j$ to be disjoint from $C_1$ and $C_2$. We perform a Dehn surgery on the core so that $\partial A$ bounds two meridian discs $Q$ of the filled solid torus. Thus $K$ is deformed to be in a 1-bridge position with respect to the 2-sphere $A \cup Q$.) When $M = S^3$, the Dehn surgery is the same operation as a twisting.

In this paper, we give a criterion for determining whether a given 1-genus 1-bridge splitting of a knot has a satellite diagram of a given slope or not. Note that every 1-genus 1-bridge splitting has infinitely many diagrams, since a trivial arc in a solid torus has infinitely many isotopy classes of cancelling discs. (In fact, a trivial arc $t$ is isotopic in the solid torus $V$ to every arc $\alpha$ in $\partial V$ such that $\partial \alpha = \partial t$ and such that $\alpha$ is disjoint from a meridian disc $D$ of $V$ with $D \cap t = \emptyset$. See Lemma 2.5 in [8].) A 1-genus 1-bridge diagram of a satellite knot is not always satellite even if the overstrand and the understrand intersect each other in a minimal number of points up to isotopy in $H$ fixing their endpoints. See Figure 1, where a cable knot of the trefoil knot is described. In fact, the projection of the arc $t_1$ is isotopic in $V_1$ to the straight line connecting the two points $H \cap K$.

However, the Heegaard diagram of a 1-genus 1-bridge splitting is unique up to homeomorphism for the homeomorphism class of the splitting. Here the Heegaard diagram is a pair of isotopy classes of meridian loops $m_1$ and $m_2$ of $V_1$ and $V_2$ in the twice punctured torus $H - K$ such that $m_i$ bounds a meridian disc disjoint from $t_i$ in $V_i$. (Note that such a meridian disc is unique up to isotopy of the pair $(V_i, t_i)$. See Lemma 2.4 in [8].) We can easily obtain the Heegaard diagram from a 1-genus 1-bridge diagram. We use Heegaard diagrams in our criterion instead of 1-genus 1-bridge diagrams.

**Theorem 1.1.** Let $(M, K) = (V_1, t_1) \cup_{H} (V_2, t_2)$ be a 1-genus 1-bridge splitting, and $D_i$ a meridian disc of $V_i$ with $D_i \cap t_i = \emptyset$ for $i = 1$ and 2. Suppose that $\partial D_1 \cap \partial D_2$
is minimized up to isotopy in $H - K$. This splitting has a satellite 1-genus 1-bridge diagram of slope $\ell_0$ if and only if there is a simple closed curve $\ell$ isotopic to $\ell_0$ in $H$ such that $\ell \cup \partial D_i$ does not separate the two points $H \cap K$ for $i = 1$ and 2.

**Addendum 1.2.**

(1) In the “only if part”, we can take $\ell$ so that $|\ell \cap \partial D_i| = |\ell \cdot D_i|$ for $i = 1$ and 2, where $\ell \cdot D_i$ denotes the algebraic intersection number.

(2) The “if part” holds even if $\partial D_1$ and $\partial D_2$ do not intersect each other in a minimal number of points up to isotopy in $H - K$.

The proof is given in Section 2. We apply this result to torus knots in Section 3, and obtain the next result.

**Corollary 1.3.** A 1-genus 1-bridge splitting of a torus knot $T(p,q)$ with $q = p + 2$ has a satellite diagram of slope $(1,1)$. In particular, $T(p,p + 2)$ is obtained from a component of a 2-bridge link in $S^3$ by twisting along the other component.

This corollary gives counterexamples for a conjecture by Ait Nouh and Yasuhara [1], which says if a torus knot $T(p,q)$ is obtained by twisting a trivial knot, then $q = kp \pm 1$ for some integer $k$. The type of the 2-bridge link will be given in [2]. Such a 2-bridge link admits infinitely many exceptional Dehn surgeries, since an $n$-twisting is realized also by a $(-1/n)$-Dehn surgery, and no Dehn surgery on a torus knot yields a hyperbolic 3-manifold.

In Section 4, applying Theorem [1.1], we show that any 1-genus 1-bridge splitting of the torus knot $T(5,12)$ does not admit a satellite diagram of longitudinal slope of one of the solid tori bounded by the splitting torus. A similar argument works for $T(p,q)$ with $p = 4k + 1$, $q = np + 2$ for $k \geq 1$ and $n \geq 2$.

**2. Proof of Theorem [1.1]**

We prove Theorem [1.1] and its addendum in this section.

First we prove the “if part”. Since $\ell \cup \partial D_i$ does not separate the two points $\partial t_i$, there is an arc $\alpha_i$ (in $H$) such that $\partial \alpha_i = \partial t_i$ and $\alpha_i \cap (\ell \cup \partial D_i) = \emptyset$. Lemma 2.5 in [6] allows us to take a cancelling disc $C_i$ of $t_i$ in $V_i$ so that $C_i \cap H = \alpha_i$. This is because we obtain a ball by cutting $V_i$ along $D_i$. Thus the discs $C_1$ and $C_2$ give a satellite 1-genus 1-bridge diagram disjoint from $\ell$.

Now we prove the “only if part”. Suppose that the 1-genus 1-bridge splitting $(M,K) = (V_1,t_1) \cup_H (V_2,t_2)$ has a satellite diagram with slope $\ell$. Then there is a cancelling disc $C_i$ of $t_i$ in $V_i$ such that the arc $s_i = C_i \cap H$ is disjoint from $\ell$ for $i = 1$ and 2. We can assume that $C_1$ and $C_2$ are isotoped so that $s_1$ and $s_2$ intersect each other in a minimal number of points. Let $A$ be the annulus obtained by cutting $H$ along $\ell$. For $i = 1$ and 2, we will find a meridian disc $D_i$ of $V_i$ with $D_i \cap t_i = \emptyset$ and with the following properties:

1. $\partial D_i$ either intersects $A$ in essential arcs, or is entirely contained in int $A$.
2. $\ell \cup \partial D_i$ is disjoint from the arc $s_i$ for $i = 1$ and 2.
3. $\partial D_i \cup s_i$ has no bigon in $H - K$ for $(i,j) = (1,2)$ and $(2,1)$.
4. $\partial D_1 \cap \partial D_2$ is minimized up to isotopy of $D_1$ and $D_2$ in $(V_i,t_i)$. 

Condition (2) implies that $\ell \cup \partial D_i$ does not separate the two points $H \cap K$, since $s_i$ connects them and is disjoint from $\ell \cup \partial D_i$. If such discs are found, then Theorem [1.1] follows from uniqueness of the isotopy class of meridian discs disjoint from the
trivial arc (see Lemma 2.4 in [5]). Note that the union \( \partial D_1 \cup \partial D_2 \) is also unique up to isotopy in \((H, K \cap H)\) if \( \partial D_1 \) and \( \partial D_2 \) intersect each other minimally. Hence, it is sufficient to find such a pair of discs \( D_1 \) and \( D_2 \), although Theorem [14] is stated for every pair of discs \( D_1 \) and \( D_2 \) such that their boundary circles intersect each other minimally. Condition (1) implies Addendum [1.2] (1).

First, we take a meridian disc \( D'_i \) of \( V_i \) so that it satisfies condition (1). \( D'_i \) may intersect \( t_i \) and \( s_i \). Then we isotope \( D'_i \) near \( \partial D'_i \) along subarcs of \( s_i \) so that it satisfies condition (2). Condition (1) is kept during this operation, because \( s_i \) is disjoint from \( \ell \). Next we will isotope \( D'_i \) so that it satisfies condition (3). Suppose that \( \partial D'_i \cup s_j \) has a bigon in \( H - K \). Note that the bigon face \( Q \) is disjoint from \( \ell \) by the conditions \( s_j \cap \ell = \emptyset \) and (1). The disc \( Q \) is disjoint also from \( s_i \), because of condition (2) and the condition that \( s_i \cap s_j \) is minimal. Hence we can isotope \( D'_i \) near its boundary in \((V_i, t_i)\) along the disc \( Q \) slightly beyond the arc \( Q \cap s_j \). This reduces the number of intersection points \( \partial D'_i \cap s_j \) by two. By repeating such operations, we can deform \( D'_i \) so that it satisfies condition (3). Finally, we isotope \( D'_i \) and \( D'_2 \) so that they satisfy condition (4). If their boundary circles do not intersect each other minimally, then \( \partial D'_i \cup \partial D'_2 \) has a bigon \( R \) in \( H - K \). See [2]. For \( i = 1 \) and 2, \( R \) is disjoint from \( s_i \) by conditions (2) and (3). The circle \( \ell \) intersects \( R \) in subarcs. Each such subarc connects the arcs \( \partial D'_i \cap R \) and \( \partial D'_2 \cap R \) because of condition (1). We isotope \( D'_i \) near its boundary along the disc \( R \) slightly beyond the arc \( \partial D'_2 \cap R \). During the isotopy, conditions (1), (2) and (3) are kept. Repeating this process, we can deform \( D'_1 \) and \( D'_2 \) so that it satisfies condition (4). We isotope \( t_i \) along \( C_i \) to be very close to \( s_i \) and disjoint from \( D'_i \). Thus we have obtained the desired pair of discs \( D_1 \) and \( D_2 \).

### 3. Proof of Corollary 1.3

We prove Corollary 1.3 in this section.

By Theorem 3 in [7], all the 1-genus 1-bridge splitting tori of a torus knot are isotopic. Hence it is enough to show that, for a certain 1-genus 1-bridge splitting, the \((p, p + 2)\)-torus knot has a satellite diagram of slope \((1, 1)\).

Let \( K \) be the \((p, p + 2)\)-torus knot in \( S^3 \). It is entirely contained in a standard torus \( H \) which divides \( S^3 \) into two solid tori \( V_1 \) and \( V_2 \) such that \( K \) goes around \( p \) times longitudinally in \( V_1 \) and \( p + 2 \) times in \( V_2 \). There is a circle \( \ell \) of slope \((1, 1)\) in \( H \) such that \( \ell \) intersects \( K \) in precisely two points \( x \) and \( y \). Let \( D_i \) be a meridian disc of \( V_i \) for \( i = 1 \) and 2. We can take \( D_1 \) so that its boundary intersects \( \ell \) only in the point \( x \) and \( K \) in \( p \) points, one of which is \( x \). We can take \( D_2 \) so that its boundary is away from \( x \) and \( y \) and intersects \( \partial D_1 \) in one point, \( \ell \) in one point \( z \), and \( K \) in \( p + 2 \) points. \( x \) is the only triple intersection point of \( \partial D_1, \partial D_2, K, \) and \( \ell \). See Figure 2. Let \( s_2 \) be a very short subarc of \( K \) near \( x \), and \( s_1 \) the complementary arc \( \text{cl}(K - s_2) \). Because \( \partial D_2 \) is away from \( x \), it is disjoint from the arc \( s_2 \). Among the \( p + 1 \) subarcs of \( s_1 \) obtained by cutting \( s_1 \) at the \( p \) points \((s_1 \cap \partial D_1) \cup y \), there is an arc \( \alpha \) connecting a point of \( s_1 \cap \partial D_1 \) and a point of \( \partial s_1 \). Note that \( \alpha \) is disjoint from \( y \). We isotope \( D_1 \) near its boundary along the arc \( \alpha \) on the torus \( H \). Repeating this operation, we obtain \( D'_1 \) from \( D_1 \) such that \( \partial D'_1 \) is disjoint from \( s_1 \). Since \( s_1 \) is disjoint from \( x, \partial D'_1 \) intersects \( \ell \) only in the single point \( x \). Hence \( \ell \cup \partial D'_1 \) does not separate the two points \( H \cap K \) (though \( s_1 \) intersects \( \ell \) in the point \( y \)). We call this \( D'_1 \) simply \( D_1 \) again. Moreover, \( \partial D_2 \) intersects \( \ell \) only in the single point \( z \), so \( \ell \cup \partial D_2 \) does not separate the two points \( H \cap K \) (though \( s_2 \) intersects
\[ \ell \text{ in the point } x \). We push the interior of the arc \( s_i \) into the interior of \( V_i \), to form a trivial arc \( t_i \) in \( V_i \) for \( i = 1 \) and 2. Note that \( K \) is isotopic to \( t_1 \cup t_2 \), and that \( t_i \) is disjoint from \( D_i \). Thus \( H \) gives a 1-genus 1-bridge splitting of \( K = t_1 \cup t_2 \), and \( D_1 \) and \( D_2 \) together give a Heegaard diagram of this splitting such that \( \ell \cup \partial D_i \) does not separate the two points \( H \cap K = \partial t_i \) for \( i = 1 \) and 2. Theorem 1.1 with Addendum \( \text{II} \, (2) \) implies that \( K \) has a satellite diagram of slope \( \ell \). 

4. PROOF OF HAVING NO SATellite DIAGRAM OF LONGITudINAL SLOPE

In this section, we show that the torus knot \( K = T(5, 12) \) does not have a 1-genus 1-bridge splitting which admits a satellite diagram of longitudinal slope of one of the solid tori.

Let \( (S^3, K) = (V_1, t_1) \cup_H (V_2, t_2) \) be a 1-genus 1-bridge splitting. By Theorem 3 in \( [7] \), there are cancelling discs \( C_1, C_2 \) of \( t_1, t_2 \) in \( V_1, V_2 \) such that \( C_1 \cap C_2 = H \cap K \). Set \( C_i \cap H = s_i \), the arc for \( i = 1 \) and \( 2 \). Then \( L = s_1 \cup s_2 \) forms a simple closed curve isotopic to \( K \). On one of the solid tori, say \( V_1 \), \( L \) goes around 5 times longitudinally, and on the other solid tori \( V_2 \), 12 times longitudinally. We will show that this splitting does not admit a satellite diagram of longitudinal slope of \( V_2 \).

Let \( D'_i \) be a meridian disc of \( V_i \) such that \( \partial D'_i \) intersects \( s_i \) transversely in a single point \( x_0 \) and \( s_1 \) in 4 points \( x_1, x_2, x_3, x_4 \) which appear on \( \partial D'_i \) in this order. We take a meridian disc \( D_2 \) of \( V_2 \) so that:

1. \( \partial D_2 \) is disjoint from \( s_2 \);
2. \( \partial D_2 \) intersects \( s_1 \) transversely in 12 points \( y_1, \ldots, y_{12} \) appearing on \( \partial D_2 \) in this order;
3. \( \partial D_2 \) intersects \( \partial D'_1 \) in a single point \( y_0 \) between the points \( y_{12} \) and \( y_1 \) and between the points \( x_2 \) and \( x_3 \);
4. the subarc of \( L \) between \( x_0 \) and \( y_3 \) contains a point \( x_+ \) of \( H \cap K \); and
5. the subarc of \( L \) between \( x_0 \) and \( y_{10} \) contains a point \( x_- \) of \( H \cap K \).

See Figure \( \text{III} \) where the torus \( H \) cut along \( \partial D_2 \) is described.

We form a Heegaard diagram of this splitting. We isotope \( D'_1 \) near its boundary along subarcs of \( s_1 \) between \( x_2 \) and \( x_+ \) and between \( x_4 \) and \( x_+ \). See Figure \( \text{IV} \) where subarcs of \( \partial D'_1 \) in (a) are deformed to those in (b). Further, we isotope \( D'_1 \) along subarcs of \( s_1 \) between \( x_3 \) and \( x_- \) and between \( x_1 \) and \( x_- \) similarly.

After these isotopies, \( D'_1 \) is transformed into a disc \( D_1 \) whose boundary is disjoint from the arc \( s_1 \). We schematically describe \( \partial D_1 \) as in Figure \( \text{IV} \, (c) \), which implies \( \partial D_1 \) contains 2 subarcs parallel to the segment from \( x_4 \) to \( x_2 \) and 4 subarcs parallel to the segment from \( x_2 \) to \( x_+ \). We call these subarcs multiplied subarcs of \( s_1 \) in the following.
The circles $\partial D_1$ and $\partial D_2$ together give a Heegaard diagram of the 1-genus 1-bridge splitting. This diagram is minimal; that is, $\partial D_1 \cup \partial D_2$ has no bigon in $H - K$. See Figure 5.

By Theorem 1.1 and Addendum 1.2 (1), it is sufficient to confirm that there is no circle $\ell$ such that $\ell$ intersects $\partial D_2$ in a single point, say $z$, and $|\ell \cap \partial D_1| = |\ell \cdot \partial D_1|$. We orient the circle $\partial D_1$ arbitrarily. Then every multiplied subarc of $s_1$ contains a pair of anti-parallel subarcs of $\partial D_1$.

In Figure 6 the multiplied subarc of $s_1$ between $y_7$ and $y_{12}$ and that between $y_{12}$ and $y_3$ together separate the 2 copies of the corner of $\partial D_2$ between $y_7$ and $y_{12}$ (via $y_8$). Hence the point $z = \ell \cap \partial D_2$ cannot be between $y_7$ and $y_{12}$ (via $y_8$). (Otherwise, $\ell$ must intersect a multiplied subarc of $s_1$, and then intersects $\partial D_1$ in more than $|\ell \cdot \partial D_1|$ points.) Similarly, the point $z$ cannot be between $y_{12}$ and $y_3$ (via $y_8$).
Considering the multiplied subarc of $s_1$ between $y_1$ and $y_8$ and that between $y_3$ and $y_8$, we can see that $z$ cannot be between $y_3$ and $y_8$ (via $y_4$) nor between $y_8$ and $y_1$ (via $y_9$). Hence $z$ can be nowhere, and there is no such $\ell$.

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