A CRITERION FOR SATELLITE 1-GENUS 1-BRIDGE KNOTS

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Abstract. Let $K$ be a knot in a closed orientable irreducible 3-manifold $M$. Suppose $M$ admits a genus 1 Heegaard splitting and we denote by $H$ the splitting torus. We say $H$ is a 1-genus 1-bridge splitting of $(M, K)$ if $H$ intersects $K$ transversely in two points, and divides $(M, K)$ into two pairs of a solid torus and a boundary parallel arc in it. It is known that a 1-genus 1-bridge splitting of a satellite knot admits a satellite diagram disjoint from an essential loop on the splitting torus. If $M = S^3$ and the slope of the loop is longitudinal in one of the solid tori, then $K$ is obtained by twisting a component of a 2-bridge link along the other component. We give a criterion for determining whether a given 1-genus 1-bridge splitting of a knot admits a satellite diagram of a given slope or not. As an application, we show there exist counter examples for a conjecture of Ait Nouh and Yasuhara.

1. Introduction

Let $M$ be a closed orientable irreducible 3-manifold, and $K$ a knot in $M$. We say that $K$ is a 1-genus 1-bridge knot if $(M, K)$ has a 1-genus 1-bridge splitting $H$, that is, there is a Heegaard splitting torus $H$ of $M$ such that $H$ intersects $K$ transversely in two points and $K$ intersects each of the solid tori bounded by $H$ in a trivial arc. (Here, an arc $t$ embedded in a solid torus $V$ with $t \cap \partial V = \partial t$ is called trivial if it is boundary parallel, that is, there is a disc $C$ in $V$ such that $t \subset \partial C$ and $C \cap \partial V = \partial (\partial C - t)$. We call such a disc $C$ a cancelling disc of $t$.) The class of 1-genus 1-bridge knots contains all torus knots and 2-bridge knots.

Let $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a 1-genus 1-bridge splitting, and $C_i$ a cancelling disc of $t_i$ in $V_i$ for $i = 1$ and 2. Set $s_i = \partial C_i \cap H$. Then the overstrand $s_1$ and the understrand $s_2$ together give a 1-genus 1-bridge diagram of the splitting. It is a satellite diagram if there is an essential loop $\ell$ in $H$ with $\ell \cap (s_1 \cup s_2) = \emptyset$. We call the isotopy class of such a loop $\ell$ in $H$ (rather than $H - K$) a slope of the satellite diagram. A 1-genus 1-bridge splitting admits a satellite diagram if there is such a pair of cancelling discs. See [6], and also [6]. If the slope of a 1-genus

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1-bridge satellite diagram is meridional in one of the solid tori $V_1$ and $V_2$, then
the knot $K$ is trivial. When the slope is longitudinal on $\partial V_i$, $K$ can be obtained
from a component of a 2-bridge link by Dehn surgery on the other component,
as is essentially shown in [3]. (In fact, $K$ has a 1-bridge diagram on the annulus
$A = \text{cl}(\partial V_i - N(\ell))$; that is, shrinking $C_1$ and $C_2$, we can isotope $K$ to be the union
of an overstrand very near to $s_1$ and an understrand very close to $s_2$. We can take
a core of the other solid torus $V_j$ to be disjoint from $C_1$ and $C_2$. We perform a
Dehn surgery on the core so that $\partial A$ bounds two meridian discs $Q$ of the filled solid
torus. Thus $K$ is deformed to be in a 1-bridge position with respect to the 2-sphere
$A \cup Q$.) When $M = S^3$, the Dehn surgery is the same operation as a twisting.

In this paper, we give a criterion for determining whether a given 1-genus 1-
bridge splitting of a knot has a satellite diagram of a given slope or not. Note that
every 1-genus 1-bridge splitting has infinitely many diagrams, since a trivial arc in a
solid torus has infinitely many isotopy classes of cancelling discs. (In fact, a trivial arc $t$ is isotopic in the solid torus $V$ to every arc $\alpha$ in $\partial V$ such that $\partial \alpha = \partial t$ and
such that $\alpha$ is disjoint from a meridian disc $D$ of $V$ with $D \cap t = \emptyset$. See Lemma 2.5
in [6].) A 1-genus 1-bridge diagram of a satellite knot is not always satellite even
if the overstrand and the understrand intersect each other in a minimal number of
points up to isotopy in $H$ fixing their endpoints. See Figure 1 where a cable knot of the trefoil knot is described. In fact, the projection of the arc $t_1$ is isotopic in $V_1$
to the straight line connecting the two points $H \cap K$.

However, the Heegaard diagram of a 1-genus 1-bridge splitting is unique up to
homeomorphism for the homeomorphism class of the splitting. Here the Heegaard
diagram is a pair of isotopy classes of meridian loops $m_1$ and $m_2$ of $V_1$ and $V_2$ in the
twice punctured torus $H - K$ such that $m_i$ bounds a meridian disc disjoint from $t_i$
in $V_i$. (Note that such a meridian disc is unique up to isotopy of the pair $(V_i, t_i)$.
See Lemma 2.4 in [6].) We can easily obtain the Heegaard diagram from a 1-genus
1-bridge diagram. We use Heegaard diagrams in our criterion instead of 1-genus 1-
bridge diagrams.

**Theorem 1.1.** Let $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a 1-genus 1-bridge splitting, and
$D_i$ a meridian disc of $V_i$ with $D_i \cap t_i = \emptyset$ for $i = 1$ and 2. Suppose that $\partial D_1 \cap \partial D_2$
is minimized up to isotopy in $H - K$. This splitting has a satellite 1-genus 1-bridge diagram of slope $\ell_0$ if and only if there is a simple closed curve $\ell$ isotopic to $\ell_0$ in $H$ such that $\ell \cup \partial D_i$ does not separate the two points $H \cap K$ for $i = 1$ and 2.

**Addendum 1.2.**

1. In the “only if part”, we can take $\ell$ so that $|\ell \cap \partial D_i| = |\ell \cdot \partial D_i|$ for $i = 1$ and 2, where $\ell \cdot \partial D_i$ denotes the algebraic intersection number.

2. The “if part” holds even if $\partial D_1$ and $\partial D_2$ do not intersect each other in a minimal number of points up to isotopy in $H - K$.

The proof is given in Section 2. We apply this result to torus knots in Section 3, and obtain the next result.

**Corollary 1.3.** A 1-genus 1-bridge splitting of a torus knot $T(p, q)$ with $q = p + 2$ has a satellite diagram of slope $(1, 1)$. In particular, $T(p, p + 2)$ is obtained from a component of a 2-bridge link in $S^3$ by twisting along the other component.

This corollary gives counterexamples for a conjecture by Ait Nouh and Yasuhara [1], which says if a torus knot $T(p, q)$ is obtained by twisting a trivial knot, then $q = kp \pm 1$ for some integer $k$. The type of the 2-bridge link will be given in [2]. Such a 2-bridge link admits infinitely many exceptional Dehn surgeries, since an $n$-twisting is realized also by a $(-1/n)$-Dehn surgery, and no Dehn surgery on a torus knot yields a hyperbolic 3-manifold.

In Section 4, applying Theorem 1.1, we show that any 1-genus 1-bridge splitting of the torus knot $T(5, 12)$ does not admit a satellite diagram of longitudinal slope of one of the solid tori bounded by the splitting torus. A similar argument works for $T(p, q)$ with $p = 4k + 1$, $q = np + 2$ for $k \geq 1$ and $n \geq 2$.

**2. Proof of Theorem 111**

We prove Theorem 1.1 and its addendum in this section.

First we prove the “if part”. Since $\ell \cup \partial D_i$ does not separate the two points $\partial t_i$, there is an arc $\alpha_i$ (in $H$) such that $\partial \alpha_i = \partial t_i$ and $\alpha_i \cap (\ell \cup \partial D_i) = \emptyset$. Lemma 2.5 in [6] allows us to take a cancelling disc $C_i$ of $t_i$ in $V_i$ so that $C_i \cap H = \alpha_i$. This is because we obtain a ball by cutting $V_i$ along $D_i$. Thus the discs $C_1$ and $C_2$ give a satellite 1-genus 1-bridge diagram disjoint from $\ell$.

Now we prove the “only if part”. Suppose that the 1-genus 1-bridge splitting $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ has a satellite diagram with slope $\ell$. Then there is a cancelling disc $C_i$ of $t_i$ in $V_i$ such that the arc $s_i = C_i \cap H$ is disjoint from $\ell$ for $i = 1$ and 2. We can assume that $C_1$ and $C_2$ are isotoped so that $s_1$ and $s_2$ intersect each other in a minimal number of points. Let $A$ be the annulus obtained by cutting $H$ along $\ell$. For $i = 1$ and 2, we will find a meridian disc $D_i$ of $V_i$ with $D_i \cap t_i = \emptyset$ and with the following properties:

1. $\partial D_i$ either intersects $A$ in essential arcs, or is entirely contained in $\text{int} A$.
2. $\ell \cup \partial D_i$ is disjoint from the arc $s_j$ for $i = 1$ and 2.
3. $\partial D_i \cup s_j$ has no bigon in $H - K$ for $(i, j) = (1, 2)$ and $(2, 1)$.
4. $\partial D_1 \cap \partial D_2$ is minimized up to isotopy of $D_1$ and $D_2$ in $(V_i, t_i)$.

Condition (2) implies that $\ell \cup \partial D_i$ does not separate the two points $H \cap K$, since $s_i$ connects them and is disjoint from $\ell \cup \partial D_i$. If such discs are found, then Theorem 1.1 follows from uniqueness of the isotopy class of meridian discs disjoint from the
trivial arc (see Lemma 2.4 in [?]). Note that the union $\partial D_1 \cup \partial D_2$ is also unique up to isotopy in $(H, K \cap H)$ if $\partial D_1$ and $\partial D_2$ intersect each other minimally. Hence, it is sufficient to find such a pair of discs $D_1$ and $D_2$, although Theorem 1.1 is stated for every pair of discs $D_1$ and $D_2$ such that their boundary circles intersect each other minimally. Condition (1) implies Addendum 1.2 (1).

First, we take a meridian disc $D'_i$ of $V_i$ so that it satisfies condition (1). $D'_i$ may intersect $t_i$ and $s_i$. Then we isotope $D'_i$ near $\partial D'_i$ along subarcs of $s_i$ so that it satisfies condition (2). Condition (1) is kept during this operation, because $s_i$ is disjoint from $\ell$. Next we will isotope $D'_j$ so that it satisfies condition (3). Suppose that $\partial D'_i \cup s_j$ has a bigon in $H - K$. Note that the bigon face $Q$ is disjoint from $\ell$ by the conditions $s_j \cap \ell = \emptyset$ and (1). The disc $Q$ is disjoint also from $s_i$, because of condition (2) and the condition that $s_i \cap s_j$ is minimal. Hence we can isotope $D'_i$ near its boundary in $(V_i, t_i)$ along the disc $Q$ slightly beyond the arc $Q \cap s_j$. This reduces the number of intersection points $\partial D'_i \cap s_j$ by two. By repeating such operations, we can deform $D'_i$ so that it satisfies condition (3). Finally, we isotope $D'_1$ and $D'_2$ so that they satisfy condition (4). If their boundary circles do not intersect each other minimally, then $\partial D'_1 \cup \partial D'_2$ has a bigon $R$ in $H - K$. See [2]. For $i = 1$ and 2, $R$ is disjoint from $s_i$ by conditions (2) and (3). The circle $\ell$ intersects $R$ in subarcs. Each such subarc connects the arcs $\partial D'_i \cap R$ and $\partial D'_j \cap R$ because of condition (1). We isotope $D'_i$ near its boundary along the disc $R$ slightly beyond the arc $\partial D'_2 \cap R$. During the isotopy, conditions (1), (2) and (3) are kept. Repeating this process, we can deform $D'_1$ and $D'_2$ so that it satisfies condition (4). We isotope $t_i$ along $C_i$ to be very close to $s_i$ and disjoint from $D'_i$. Thus we have obtained the desired pair of discs $D_1$ and $D_2$. $\square$

3. Proof of Corollary 1.3

We prove Corollary 1.3 in this section.

By Theorem 3 in [7], all the 1-genus 1-bridge splitting tori of a torus knot are isotopic. Hence it is enough to show that, for a certain 1-genus 1-bridge splitting, the $(p, p + 2)$-torus knot has a satellite diagram of slope $(1, 1)$.

Let $K$ be the $(p, p + 2)$-torus knot in $S^3$. It is entirely contained in a standard torus $H$ which divides $S^3$ into two solid tori $V_1$ and $V_2$ such that $K$ goes around $p$ times longitudinally in $V_1$ and $p + 2$ times in $V_2$. There is a circle $\ell$ of slope $(1, 1)$ in $H$ such that $\ell$ intersects $K$ in precisely two points $x$ and $y$. Let $D_i$ be a meridian disc of $V_i$ for $i = 1$ and 2. We can take $D_1$ so that its boundary intersects $\ell$ only in the point $x$ and $K$ in $p$ points, one of which is $x$. We can take $D_2$ so that its boundary is away from $x$ and $y$ and intersects $\partial D_1$ in one point, $\ell$ in one point $z$, and $K$ in $p + 2$ points. $x$ is the only triple intersection point of $\partial D_1$, $\partial D_2$, $K$, and $\ell$. See Figure 2. Let $s_2$ be a very short subarc of $K$ near $x$, and $s_1$ the complementary arc $\text{cl}(K - s_2)$. Because $\partial D_2$ is away from $x$, it is disjoint from the arc $s_2$. Among the $p + 1$ subarcs of $s_1$ obtained by cutting $s_1$ at the $p$ points $(s_1 \cap \partial D_1) \cup y$, there is an arc $\alpha$ connecting a point of $s_1 \cap \partial D_1$ and a point of $\partial s_1$. Note that $\alpha$ is disjoint from $y$. We isotope $D_1$ near its boundary along the arc $\alpha$ on the torus $H$. Repeating this operation, we obtain $D'_1$ from $D_1$ such that $\partial D'_1$ is disjoint from $s_1$. Since $s_1$ is disjoint from $x$, $\partial D'_1$ intersects $\ell$ only in the single point $x$. Hence $\ell \cup \partial D'_1$ does not separate the two points $H \cap K$ (though $s_1$ intersects $\ell$ in the point $y$). We call this $D'_1$ simply $D_1$ again. Moreover, $\partial D_2$ intersects $\ell$ only in the single point $z$, so $\ell \cup \partial D_2$ does not separate the two points $H \cap K$ (though $s_2$ intersects
ℓ in the point x). We push the interior of the arc si into the interior of Vi, to form a trivial arc ti in Vi for i = 1 and 2. Note that K is isotopic to t1 ∪ t2, and that ti is disjoint from Di. Thus H gives a 1-genus 1-bridge splitting of K = t1 ∪ t2, and D1 and D2 together give a Heegaard diagram of this splitting such that ℓ ∪ ∂Di does not separate the two points H ∩ K = ∂ti for i = 1 and 2. Theorem A with Addendum 1.2(2) implies that K has a satellite diagram of slope ℓ. □

4. Proof of Having No Satellite Diagram of Longitudinal Slope

In this section, we show that the torus knot $K = T(5, 12)$ does not have a 1-genus 1-bridge splitting which admits a satellite diagram of longitudinal slope of one of the solid tori.

Let $(S^3, K) = (V_1, t_1) \cup (V_2, t_2)$ be a 1-genus 1-bridge splitting. By Theorem 3 in [7], there are cancelling discs $C_1, C_2$ of $t_1, t_2$ in $V_1, V_2$ such that $C_1 \cap C_2 = H \cap K$. Set $C_i \cap H = s_i$, the arc for $i = 1$ and 2. Then $L = s_1 \cup s_2$ forms a simple closed curve isotopic to K. On one of the solid tori, say $V_1$, L goes around 5 times longitudinally, and on the other solid tori $V_2$, 12 times longitudinally. We will show that this splitting does not admit a satellite diagram of longitudinal slope of $V_2$.

Let $D'_1$ be a meridian disc of $V_1$ such that $\partial D'_1$ intersects $s_1$ transversely in a single point $x_0$ and $s_1$ in 4 points $x_1, x_2, x_3, x_4$ which appear on $\partial D'_1$ in this order. We take a meridian disc $D_2$ of $V_2$ so that:

1. $\partial D_2$ is disjoint from $s_2$;
2. $\partial D_2$ intersects $s_1$ transversely in 12 points $y_1, \ldots, y_{12}$ appearing on $\partial D_2$ in this order;
3. $\partial D_2$ intersects $\partial D'_1$ in a single point $y_0$ between the points $y_{12}$ and $y_1$ and between the points $x_2$ and $x_3$;
4. the subarc of $L$ between $x_0$ and $y_3$ contains a point $x_+$ of $H \cap K$; and
5. the subarc of $L$ between $x_0$ and $y_{10}$ contains a point $x_-$ of $H \cap K$.

See Figure 3 where the torus H cut along $\partial D_2$ is described.

We form a Heegaard diagram of this splitting. We isotope $D'_1$ near its boundary along subarcs of $s_1$ between $x_2$ and $x_+$ and between $x_4$ and $x_+$. See Figure 4 where subarcs of $\partial D'_1$ in (a) are deformed to those in (b). Further, we isotope $D'_1$ along subarcs of $s_1$ between $x_3$ and $x_-$ and between $x_1$ and $x_-$ similarly.

After these isotopies, $D'_1$ is transformed into a disc $D_1$ whose boundary is disjoint from the arc $s_1$. We schematically describe $\partial D_1$ as in Figure 4(c), which implies $\partial D_1$ contains 2 subarcs parallel to the segment from $x_4$ to $x_2$ and 4 subarcs parallel to the segment from $x_2$ to $x_+$. We call these subarcs multiplied subarcs of $s_1$ in the following.
The circles $\partial D_1$ and $\partial D_2$ together give a Heegaard diagram of the 1-genus 1-bridge splitting. This diagram is minimal; that is, $\partial D_1 \cup \partial D_2$ has no bigon in $H - K$. See Figure 5.

By Theorem 1.1 and Addendum 1.2 (1), it is sufficient to confirm that there is no circle $\ell$ such that $\ell$ intersects $\partial D_2$ in a single point, say $z$, and $|\ell \cap \partial D_1| = |\ell \cdot \partial D_1|$. We orient the circle $\partial D_1$ arbitrarily. Then every multiplied subarc of $s_1$ contains a pair of anti-parallel subarcs of $\partial D_1$.

In Figure 6 the multiplied subarc of $s_1$ between $y_7$ and $y_{12}$ and that between $y_{12}$ and $y_5$ together separate the 2 copies of the corner of $\partial D_2$ between $y_7$ and $y_{12}$ (via $y_8$). Hence the point $z = \ell \cap \partial D_2$ cannot be between $y_7$ and $y_{12}$ (via $y_8$). (Otherwise, $\ell$ must intersect a multiplied subarc of $s_1$, and then intersects $\partial D_1$ in more than $|\ell \cdot \partial D_1|$ points.) Similarly, the point $z$ cannot be between $y_{12}$ and $y_5$ (via...
y_0). Considering the multiplied subarc of s_1 between y_1 and y_8 and that between y_3 and y_8, we can see that z cannot be between y_3 and y_8 (via y_4) nor between y_8 and y_1 (via y_6). Hence z can be nowhere, and there is no such ℓ.

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