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A CRITERION FOR SATELLITE 1-GENUS 1-BRIDGE KNOTS

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Abstract. Let K be a knot in a closed orientable irreducible 3-manifold M. Suppose M admits a genus 1 Heegaard splitting and we denote by H the splitting torus. We say H is a 1-genus 1-bridge splitting of (M, K) if H intersects K transversely in two points, and divides (M, K) into two pairs of a solid torus and a boundary parallel arc in it. It is known that a 1-genus 1-bridge splitting of a satellite knot admits a satellite diagram disjoint from an essential loop on the splitting torus. If $M = S^3$ and the slope of the loop is longitudinal in one of the solid tori, then K is obtained by twisting a component of a 2-bridge link along the other component. We give a criterion for determining whether a given 1-genus 1-bridge splitting of a knot admits a satellite diagram of a given slope or not. As an application, we show there exist counter examples for a conjecture of Ait Nouh and Yasuhara.

1. Introduction

Let M be a closed orientable irreducible 3-manifold, and K a knot in M. We say that K is a 1-*genus* 1-*bridge knot* if (M, K) has a 1-*genus* 1-*bridge splitting* H, that is, there is a Heegaard splitting torus H of M such that H intersects K transversely in two points and K intersects each of the solid tori bounded by H in a trivial arc. (Here, an arc t embedded in a solid torus V with $t \cap \partial V = \partial t$ is called *trivial* if it is boundary parallel, that is, there is a disc C in V such that $t \subset \partial C$ and $C \cap \partial V = \mathrm{cl}\,(\partial C - t)$. We call such a disc C a *cancelling disc* of t.) The class of 1-genus 1-bridge knots contains all torus knots and 2-bridge knots.

Let $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a 1-genus 1-bridge splitting, and C_i a cancelling disc of t_i in V_i for $i = 1$ and 2. Set $s_i = \partial C_i \cap H$. Then the overstrand s_1 and the understrand s_2 together give *a* 1-*genus* 1-*bridge diagram* of the splitting. It is a *satellite diagram* if there is an essential loop ℓ in H with $\ell \cap (s_1 \cup s_2) = \emptyset$. We call the isotopy class of such a loop ℓ in H (rather than $H - K$) a *slope* of the satellite diagram. A 1-genus 1-bridge splitting *admits a satellite diagram* if there is such a pair of cancelling discs. See [5], and also [8]. If the slope of a 1-genus

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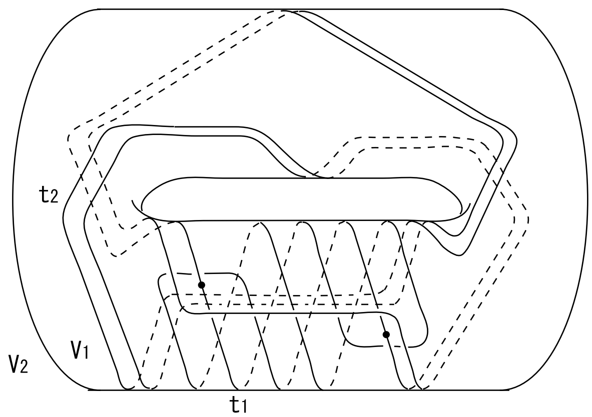

FIGURE 1.

1-bridge satellite diagram is meridional in one of the solid tori V_1 and V_2, then the knot K is trivial. When the slope is longitudinal on ∂V_i, K can be obtained from a component of a 2-bridge link by Dehn surgery on the other component, as is essentially shown in [8]. (In fact, K has a 1-bridge diagram on the annulus $A = \mathrm{cl}\,(\partial V_i - N(\ell))$; that is, shrinking C_1 and C_2, we can isotope K to be the union of an overstrand very near to s_1 and an understrand very close to s_2. We can take a core of the other solid torus V_j to be disjoint from C_1 and C_2. We perform a Dehn surgery on the core so that ∂A bounds two meridian discs Q of the filled solid torus. Thus K is deformed to be in a 1-bridge position with respect to the 2-sphere $A \cup Q$.) When $M = S^3$, the Dehn surgery is the same operation as a twisting.

In this paper, we give a criterion for determining whether a given 1-genus 1-bridge splitting of a knot has a satellite diagram of a given slope or not. Note that every 1-genus 1-bridge splitting has infinitely many diagrams, since a trivial arc in a solid torus has infinitely many isotopy classes of cancelling discs. (In fact, a trivial arc t is isotopic in the solid torus V to every arc α in ∂V such that $\partial\alpha = \partial t$ and such that α is disjoint from a meridian disc D of V with $D \cap t = \emptyset$. See Lemma 2.5 in [6].) A 1-genus 1-bridge diagram of a satellite knot is not always satellite even if the overstrand and the understrand intersect each other in a minimal number of points up to isotopy in H fixing their endpoints. See Figure 1, where a cable knot of the trefoil knot is described. In fact, the projection of the arc t_1 is isotopic in V_1 to the straight line connecting the two points $H \cap K$.

However, the Heegaard diagram of a 1-genus 1-bridge splitting is unique up to homeomorphism for the homeomorphism class of the splitting. Here the *Heegaard diagram* is a pair of isotopy classes of meridian loops m_1 and m_2 of V_1 and V_2 in the twice punctured torus $H - K$ such that m_i bounds a meridian disc disjoint from t_i in V_i. (Note that such a meridian disc is unique up to isotopy of the pair (V_i, t_i). See Lemma 2.4 in [6].) We can easily obtain the Heegaard diagram from a 1-genus 1-bridge diagram. We use Heegaard diagrams in our criterion instead of 1-genus 1-bridge diagrams.

Theorem 1.1. *Let* $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ *be a* 1*-genus* 1*-bridge splitting, and* D_i *a meridian disc of* V_i *with* $D_i \cap t_i = \emptyset$ *for* $i = 1$ *and* 2*. Suppose that* $\partial D_1 \cap \partial D_2$

is minimized up to isotopy in $H - K$. This splitting has a satellite 1*-genus* 1*-bridge diagram of slope ℓ_0 if and only if there is a simple closed curve ℓ isotopic to ℓ_0 in H such that $\ell \cup \partial D_i$ does not separate the two points $H \cap K$ for $i = 1$ and* 2*.*

Addendum 1.2. (1) *In the "only if part", we can take ℓ so that $|\ell \cap \partial D_i| = |\ell \cdot \partial D_i|$ for $i = 1$ and* 2*, where $\ell \cdot \partial D_i$ denotes the algebraic intersection number.*

(2) *The "if part" holds even if ∂D_1 and ∂D_2 do not intersect each other in a minimal number of points up to isotopy in $H - K$.*

The proof is given in Section 2. We apply this result to torus knots in Section 3, and obtain the next result.

Corollary 1.3. *A* 1*-genus* 1*-bridge splitting of a torus knot $T(p, q)$ with $q = p + 2$ has a satellite diagram of slope* $(1, 1)$*. In particular, $T(p, p + 2)$ is obtained from a component of a* 2*-bridge link in S^3 by twisting along the other component.*

This corollary gives counterexamples for a conjecture by Ait Nouh and Yasuhara [1], which says if a torus knot $T(p, q)$ is obtained by twisting a trivial knot, then $q = kp \pm 1$ for some integer k. The type of the 2-bridge link will be given in [3]. Such a 2-bridge link admits infinitely many exceptional Dehn surgeries, since an n-twisting is realized also by a $(-1/n)$-Dehn surgery, and no Dehn surgery on a torus knot yields a hyperbolic 3-manifold.

In Section 4, applying Theorem 1.1, we show that any 1-genus 1-bridge splitting of the torus knot $T(5, 12)$ does not admit a satellite diagram of longitudinal slope of one of the solid tori bounded by the splitting torus. A similar argument works for $T(p, q)$ with $p = 4k + 1,\ q = np + 2$ for $k \geq 1$ and $n \geq 2$.

2. Proof of Theorem 1.1

We prove Theorem 1.1 and its addendum in this section.

First we prove the "if part". Since $\ell \cup \partial D_i$ does not separate the two points ∂t_i, there is an arc α_i (in H) such that $\partial \alpha_i = \partial t_i$ and $\alpha_i \cap (\ell \cup \partial D_i) = \emptyset$. Lemma 2.5 in [6] allows us to take a cancelling disc C_i of t_i in V_i so that $C_i \cap H = \alpha_i$. This is because we obtain a ball by cutting V_i along D_i. Thus the discs C_1 and C_2 give a satellite 1-genus 1-bridge diagram disjoint from ℓ.

Now we prove the "only if part". Suppose that the 1-genus 1-bridge splitting $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ has a satellite diagram with slope ℓ. Then there is a cancelling disc C_i of t_i in V_i such that the arc $s_i = C_i \cap H$ is disjoint from ℓ for $i = 1$ and 2. We can assume that C_1 and C_2 are isotoped so that s_1 and s_2 intersect each other in a minimal number of points. Let A be the annulus obtained by cutting H along ℓ. For $i = 1$ and 2, we will find a meridian disc D_i of V_i with $D_i \cap t_i = \emptyset$ and with the following properties:

(1) ∂D_i either intersects A in essential arcs, or is entirely contained in $\operatorname{int} A$.
(2) $\ell \cup \partial D_i$ is disjoint from the arc s_i for $i = 1$ and 2.
(3) $\partial D_i \cup s_j$ has no bigon in $H - K$ for $(i, j) = (1, 2)$ and $(2, 1)$.
(4) $\partial D_1 \cap \partial D_2$ is minimized up to isotopy of D_1 and D_2 in (V_i, t_i).

Condition (2) implies that $\ell \cup \partial D_i$ does not separate the two points $H \cap K$, since s_i connects them and is disjoint from $\ell \cup \partial D_i$. If such discs are found, then Theorem 1.1 follows from uniqueness of the isotopy class of meridian discs disjoint from the trivial arc (see Lemma 2.4 in [6]). Note that the union $\partial D_1 \cup \partial D_2$ is also unique up to isotopy in $(H, K \cap H)$ if ∂D_1 and ∂D_2 intersect each other minimally. Hence, it is sufficient to find such a pair of discs D_1 and D_2, although Theorem 1.1 is stated for every pair of discs D_1 and D_2 such that their boundary circles intersect each other minimally. Condition (1) implies Addendum 1.2 (1).

First, we take a meridian disc D_i' of V_i so that it satisfies condition (1). D_i' may intersect t_i and s_i. Then we isotope D_i' near $\partial D_i'$ along subarcs of s_i so that it satisfies condition (2). Condition (1) is kept during this operation, because s_i is disjoint from ℓ. Next we will isotope D_i' so that it satisfies condition (3). Suppose that $\partial D_i' \cup s_j$ has a bigon in $H - K$. Note that the bigon face Q is disjoint from ℓ by the conditions $s_j \cap \ell = \emptyset$ and (1). The disc Q is disjoint also from s_i, because of condition (2) and the condition that $s_i \cap s_j$ is minimal. Hence we can isotope D_i' near its boundary in (V_i, t_i) along the disc Q slightly beyond the arc $Q \cap s_j$. This reduces the number of intersection points $\partial D_i' \cap s_j$ by two. By repeating such operations, we can deform D_i' so that it satisfies condition (3). Finally, we isotope D_1' and D_2' so that they satisfy condition (4). If their boundary circles do not intersect each other minimally, then $\partial D_1' \cup \partial D_2'$ has a bigon R in $H - K$. See [2]. For $i = 1$ and 2, R is disjoint from s_i by conditions (2) and (3). The circle ℓ intersects R in subarcs. Each such subarc connects the arcs $\partial D_1' \cap R$ and $\partial D_2' \cap R$ because of condition (1). We isotope D_1' near its boundary along the disc R slightly beyond the arc $\partial D_2' \cap R$. During the isotopy, conditions (1), (2) and (3) are kept. Repeating this process, we can deform D_1' and D_2' so that it satisfies condition (4). We isotope t_i along C_i to be very close to s_i and disjoint from D_i'. Thus we have obtained the desired pair of discs D_1 and D_2. □

3. Proof of Corollary 1.3

We prove Corollary 1.3 in this section.

By Theorem 3 in [7], all the 1-genus 1-bridge splitting tori of a torus knot are isotopic. Hence it is enough to show that, for a certain 1-genus 1-bridge splitting, the $(p, p+2)$-torus knot has a satellite diagram of slope $(1, 1)$.

Let K be the $(p, p+2)$-torus knot in S^3. It is entirely contained in a standard torus H which divides S^3 into two solid tori V_1 and V_2 such that K goes around p times longitudinally in V_1 and $p+2$ times in V_2. There is a circle ℓ of slope $(1, 1)$ in H such that ℓ intersects K in precisely two points x and y. Let D_i be a meridian disc of V_i for $i = 1$ and 2. We can take D_1 so that its boundary intersects ℓ only in the point x and K in p points, one of which is x. We can take D_2 so that its boundary is away from x and y and intersects ∂D_1 in one point, ℓ in one point z, and K in $p+2$ points. x is the only triple intersection point of ∂D_1, ∂D_2, K, and ℓ. See Figure 2. Let s_2 be a very short subarc of K near x, and s_1 the complementary arc $\mathrm{cl}(K - s_2)$. Because ∂D_2 is away from x, it is disjoint from the arc s_2. Among the $p+1$ subarcs of s_1 obtained by cutting s_1 at the p points $(s_1 \cap \partial D_1) \cup y$, there is an arc α connecting a point of $s_1 \cap \partial D_1$ and a point of ∂s_1. Note that α is disjoint from y. We isotope D_1 near its boundary along the arc α on the torus H. Repeating this operation, we obtain D_1' from D_1 such that $\partial D_1'$ is disjoint from s_1. Since s_1 is disjoint from x, $\partial D_1'$ intersects ℓ only in the single point x. Hence $\ell \cup \partial D_1'$ does not separate the two points $H \cap K$ (though s_1 intersects ℓ in the point y). We call this D_1' simply D_1 again. Moreover, ∂D_2 intersects ℓ only in the single point z, so $\ell \cup \partial D_2$ does not separate the two points $H \cap K$ (though s_2 intersects

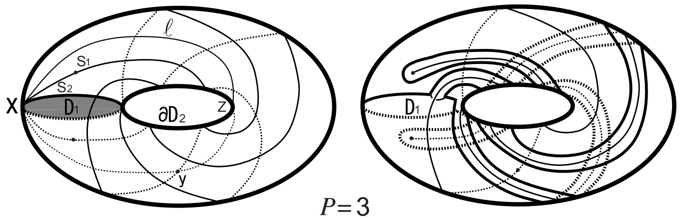

FIGURE 2.

ℓ in the point x). We push the interior of the arc s_i into the interior of V_i, to form a trivial arc t_i in V_i for $i = 1$ and 2. Note that K is isotopic to $t_1 \cup t_2$, and that t_i is disjoint from D_i. Thus H gives a 1-genus 1-bridge splitting of $K = t_1 \cup t_2$, and D_1 and D_2 together give a Heegaard diagram of this splitting such that $\ell \cup \partial D_i$ does not separate the two points $H \cap K = \partial t_i$ for $i = 1$ and 2. Theorem 1.1 with Addendum 1.2 (2) implies that K has a satellite diagram of slope ℓ. □

4. Proof of having no satellite diagram of longitudinal slope

In this section, we show that the torus knot $K = T(5, 12)$ does not have a 1-genus 1-bridge splitting which admits a satellite diagram of longitudinal slope of one of the solid tori.

Let $(S^3, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a 1-genus 1-bridge splitting. By Theorem 3 in [7], there are cancelling discs C_1, C_2 of t_1, t_2 in V_1, V_2 such that $C_1 \cap C_2 = H \cap K$. Set $C_i \cap H = s_i$, the arc for $i = 1$ and 2. Then $L = s_1 \cup s_2$ forms a simple closed curve isotopic to K. On one of the solid tori, say V_1, L goes around 5 times longitudinally, and on the other solid tours V_2, 12 times longitudinally. We will show that this splitting does not admit a satellite diagram of longitudinal slope of V_2.

Let D'_1 be a meridian disc of V_1 such that $\partial D'_1$ intersects s_2 transversely in a single point x_0 and s_1 in 4 points x_1, x_2, x_3, x_4 which appear on $\partial D'_1$ in this order. We take a meridian disc D_2 of V_2 so that:

(1) ∂D_2 is disjoint from s_2;
(2) ∂D_2 intersects s_1 transversely in 12 points $y_1, \dots, y_{12}$ appearing on ∂D_2 in this order;
(3) ∂D_2 intersects $\partial D'_1$ in a single point y_0 between the points y_{12} and y_1 and between the points x_2 and x_3;
(4) the subarc of L between x_0 and y_3 contains a point x_+ of $H \cap K$; and
(5) the subarc of L between x_0 and y_{10} contains a point x_- of $H \cap K$.

See Figure 3, where the torus H cut along ∂D_2 is described.

We form a Heegaard diagram of this splitting. We isotope D'_1 near its boundary along subarcs of s_1 between x_2 and x_+ and between x_4 and x_+. See Figure 4, where subarcs of $\partial D'_1$ in (a) are deformed to those in (b). Further, we isotope D'_1 along subarcs of s_1 between x_3 and x_- and between x_1 and x_- similarly.

After these isotopies, D'_1 is transformed into a disc D_1 whose boundary is disjoint from the arc s_1. We schematically describe ∂D_1 as in Figure 4(c), which implies ∂D_1 contains 2 subarcs parallel to the segment from x_4 to x_2 and 4 subarcs parallel to the segment from x_2 to x_+. We call these subarcs *multiplied subarcs of* s_1 in the following.

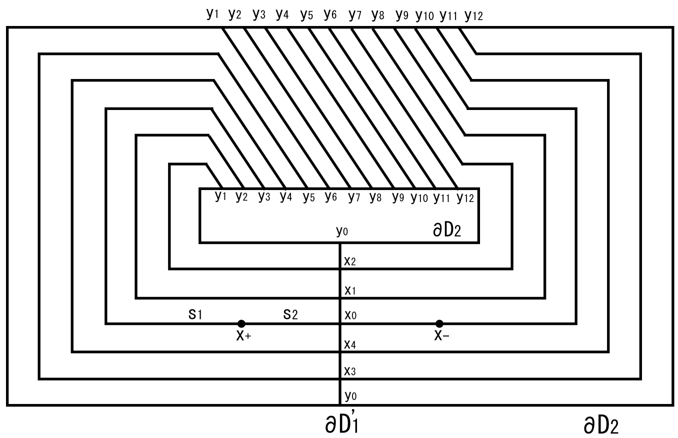

FIGURE 3.

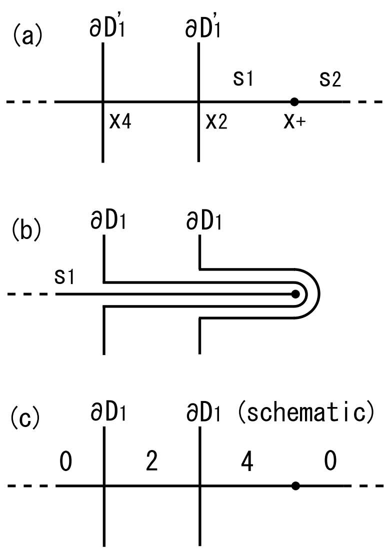

FIGURE 4.

The circles ∂D_1 and ∂D_2 together give a Heegaard diagram of the 1-genus 1-bridge splitting. This diagram is minimal; that is, $\partial D_1 \cup \partial D_2$ has no bigon in $H - K$. See Figure 5.

By Theorem 1.1 and Addendum 1.2 (1), it is sufficient to confirm that there is no circle ℓ such that ℓ intersects ∂D_2 in a single point, say z, and $|\ell \cap \partial D_1| = |\ell \cdot \partial D_1|$. We orient the circle ∂D_1 arbitrarily. Then every multiplied subarc of s_1 contains a pair of anti-parallel subarcs of ∂D_1.

In Figure 6, the multiplied subarc of s_1 between y_7 and y_{12} and that between y_{12} and y_5 together separate the 2 copies of the corner of ∂D_2 between y_7 and y_{12} (via y_8). Hence the point $z = \ell \cap \partial D_2$ cannot be between y_7 and y_{12} (via y_8). (Otherwise, ℓ must intersect a multiplied subarc of s_1, and then intersects ∂D_1 in more than $|\ell \cdot \partial D_1|$ points.) Similarly, the point z cannot be between y_{12} and y_5 (via

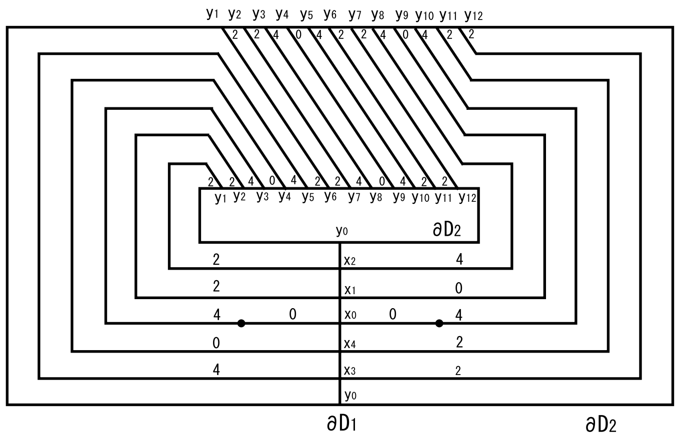

FIGURE 5.

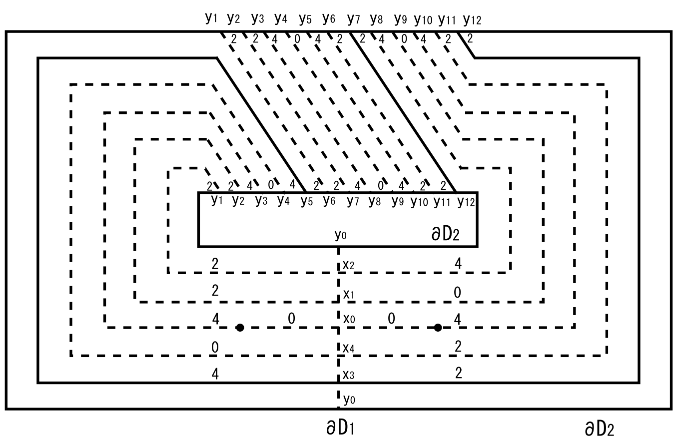

FIGURE 6.

y_0). Considering the multiplied subarc of s_1 between y_1 and y_8 and that between y_3 and y_8, we can see that z cannot be between y_3 and y_8 (via y_4) nor between y_8 and y_1 (via y_9). Hence z can be nowhere, and there is no such ℓ. $\square$

REFERENCES

1. M. Ait Nouh and A. Yasuhara, *Torus knots that cannot be untied by twisting*, Rev. Mat. Complut. **14** (2001), 423–437. MR **2002j:**57005
2. D.B.A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83-107. MR **35:**4938
3. H. Goda, C. Hayashi and H. Song, *Dehn surgeries on* 2*-bridge links which yield reducible* 3*-manifolds*, preprint.
4. C. Hayashi, *Genus one* 1*-bridge positions for the trivial knot and cabled knots*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 1, 53–65. MR **99j:**57005
5. C. Hayashi, *Satellite knots in 1-genus 1-bridge positions*, Osaka J. Math. **36** (1999), 711–729. MR **2001a:**57009
6. C. Hayashi, 1*-genus* 1*-bridge splittings for knots*, to appear in Osaka J. Math. **41** (2004).

7. K. Morimoto, *On minimum genus Heegaard splittings of some orientable closed 3-manifolds*, Tokyo J. Math. **12** (1989), 321–355. MR **91b:**57020
8. K. Morimoto and M. Sakuma, *On unknotting tunnels for knots*, Math. Ann. **289** (1991), 143–167. MR **92e:**57015

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