

CYCLE DECOMPOSITIONS AND TRAIN TRACKS

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ABSTRACT. We prove that the disjoint cycle decomposition of the permutation $(1\ 2\ \cdots\ n_1)^{k_1}(1\ 2\ \cdots\ n_2)^{k_2}\ \cdots\ (1\ 2\ \cdots\ n_r)^{k_r}$ consists of cycles of at most r distinct lengths. The proof relies on the geometry and topology of simple closed curves and train tracks on a closed surface of genus r .

It is easy to see that the disjoint cycle decomposition of the permutation $(1\ 2\ \cdots\ n)^k$ consists of cycles that all have the same length; the number of disjoint cycles is the greatest common divisor of n and k .

We prove the following generalization of this result:

Theorem. *Let n_i and k_i be positive integers for $1 \leq i \leq r$. Then the disjoint cycle decomposition of the permutation $\rho = (1\ 2\ \cdots\ n_1)^{k_1}\ \cdots\ (1\ 2\ \cdots\ n_r)^{k_r}$ consists of cycles of at most r distinct lengths.*

Although the theorem seems like a very basic and elementary result about permutations and is purely algebraic in nature, we know of no purely algebraic or elementary proof. We supply a proof that uses topology, making the theorem a good example of how topology can be used to prove a new result in algebra. Our proof consists of producing a multiple curve on a smooth surface of genus r which represents the permutation ρ in such a way that disjoint cycles in the disjoint cycle decomposition of ρ correspond to connected components of the multiple curve. Cycles of distinct lengths will correspond to nonhomotopic components. We then show that the multiple curve can have at most r nonhomotopic components by producing a train track on the surface that carries the multiple curve and analyzing the train track. It follows that the cycles of ρ can have at most r distinct lengths.

Train tracks were developed by Thurston in [5] and are discussed in detail in [4]. Our results arose in the context of the problem of determining the number of components of a multiple curve specified by integral weights on a train track. (This problem is discussed for the closed surface of genus 2 in [3].)

Start by conjugating ρ so that n_1 is the largest of the integers n_i , $1 \leq i \leq r$. (It is a well-known fact that two permutations are conjugate if and only if they have the same number of cycles of each length; see [2] or [1], for example.) Next, since $(1\ 2\ \cdots\ n_i)^{k_i} = (1\ 2\ \cdots\ n_i)^{k_i \bmod n_i}$, assume $0 < k_i < n_i$ for each i . In the product of permutations, we use the convention of applying the permutation on the right first. (Some authors prefer to apply the permutation on the left first.) For example,

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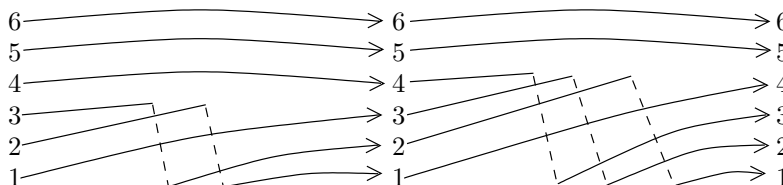


FIGURE 1. The first two factors $(1\ 2\ 3)^2$ and $(1\ 2\ 3\ 4)^3$ in the permutation $(1\ 2\ 3\ 4\ 5\ 6)^4(1\ 2\ 3\ 4)^3(1\ 2\ 3)^2$ acting on the integers from 1 to 6.

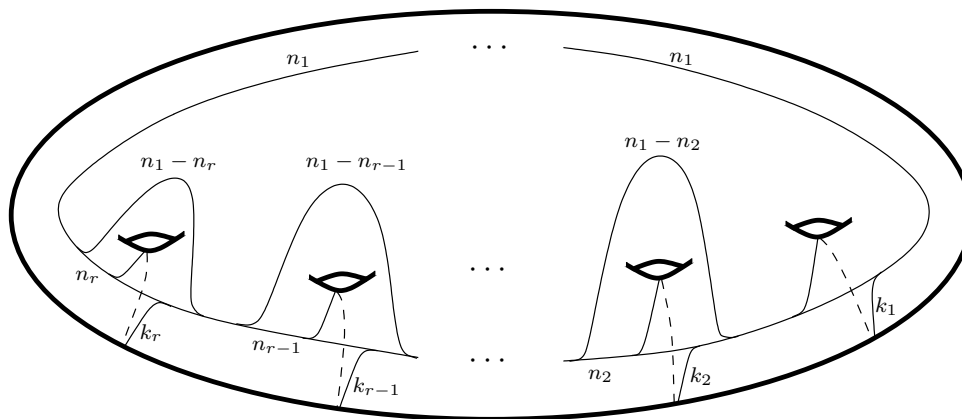


FIGURE 2. A train track τ on a closed surface of genus r . The surface appears in boldface; τ does not. The multiple curve C that embodies the permutation $\rho = (1\ 2 \cdots n_1)^{k_1} \cdots (1\ 2 \cdots n_r)^{k_r}$ is specified by the integral weights shown on the branches of τ .

$(1\ 2\ 3\ 4)(2\ 4) = (1\ 2)(3\ 4)$ but $(2\ 4)(1\ 2\ 3\ 4) = (1\ 4)(2\ 3)$. So the first permutation in the product $\rho = (1\ 2 \cdots n_1)^{k_1} \cdots (1\ 2 \cdots n_r)^{k_r}$ fixes the integers from $n_r + 1$ to n_1 , and shifts the integers 1 through n_r by adding k_r (modulo n_r) as in Figure 1.

Figure 1 suggests that we might study the combinatorics of multiple curves on surfaces to understand the cycle decomposition of ρ . Let S_r denote a closed, smooth surface of genus r . A *multiple curve* on S_r is a collection of disjoint, smooth, simple closed curves on S_r , none of which is homotopically trivial. The multiple curve C that we construct, which embodies the permutation ρ , is carried by the train track τ shown in Figure 2. (For a definition and discussions of train tracks, see [4]. Essentially, a train track carries a multiple curve if the curves can be squashed together smoothly so that they are contained in the train track. The integers labeling the branches of the train track in Figure 2 represent the numbers of curve arcs from C squashed together on that branch.)

Label the curve arcs carried in the branch of τ near the top of Figure 2 with the integers from 1 to n_1 . Figure 3 shows this enumeration for the permutation $\rho = (1\ 2\ 3\ 4\ 5)^3(1\ 2\ 3)^2$ on a surface of genus 2. For $1 \leq j \leq n_1$, the permutation ρ takes j to the label of the next arc traversed as one would travel in a counter-clockwise direction along the multiple curve C carried by τ . Thus, disjoint cycles in the disjoint cycle decomposition of ρ correspond to connected components of

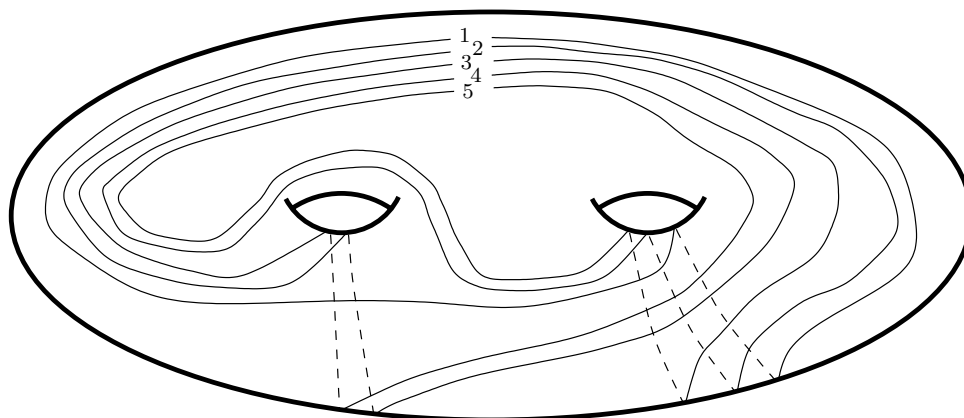


FIGURE 3. The multiple curve C that represents the permutation $\rho = (12345)^3(123)^2$.

C , and disjoint cycles of distinct lengths correspond to nonhomotopic components. (Disjoint cycles of the same length correspond to components of C that may or may not be homotopic.) If we show that C has at most r nonhomotopic components, this will finish the proof of our theorem.

It is not difficult to see that one can smoothly deform the curves in C (or deform τ) so that all the switches of τ lie in a single component X of $S_r - C$ (see Figure 4).

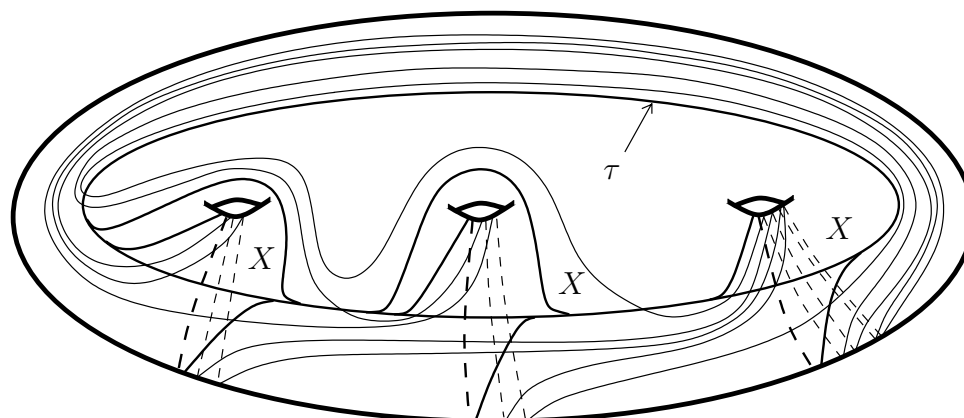


FIGURE 4. The surface S_3 along with the train track τ and a multiple curve C carried by τ . We have arranged the curves in C so that all the switches of τ are in a single component X of $S_3 - C$. The switches are located between the arcs of curves in C that pass through the handles and those that avoid the handles from above.

Cutting the surface S_r along τ produces a connected polygon with $4r - 2$ cuspidal vertices arising from the $4r - 2$ switches on τ . (Each of the r handles of S_r is removed by cutting along τ , and $S_r - \tau$ is connected. We will show momentarily that $S_r - \tau$

is simply connected.) The component X of $S_r - C$ has a smooth boundary. So in order to obtain this component from the polygon $S_r - \tau$, all of the cuspidal vertices must be identified in pairs, as in Figure 5.

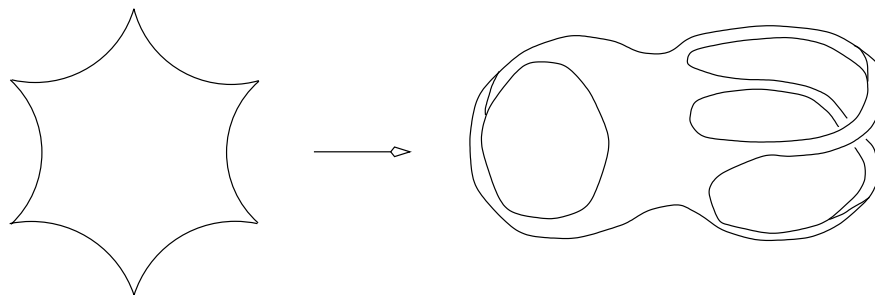


FIGURE 5. One way to identify the cuspidal vertices of the polygon $S_2 - \tau$ in pairs, making the boundary smooth.

The polygon $S_r - \tau$ has Euler characteristic at most 1 (allowing for the possibility that $S_r - \tau$ is not simply connected), and each identification of a pair of cuspidal vertices reduces the Euler characteristic by 1. So the Euler characteristic $\chi(X)$ is at most $1 - (2r - 1) = 2 - 2r = \chi(S_r)$. Since no curve in C is homotopically trivial, no component of $S_r - C$ is homeomorphic to a disc. Thus, no component of $S_r - C$ has positive Euler characteristic. It follows that $\chi(X) = \chi(S_r)$ and that every component of $S_r - C$ other than X has an Euler characteristic of 0 (and that the polygon $S_r - \tau$ is simply connected).

Now let Σ denote any collection of disjoint, smooth, simple closed curves on S_r , none of which is homotopically trivial, and no two of which are homotopic to one another. If A is a subset of Σ and γ is a loop in $\Sigma - A$, then the surface $S_r - (A \cup \{\gamma\})$ either has one more component than $S_r - A$ or the total genus is one less than that of $S_r - A$. Thus, if we let g denote the total genus of $S_r - \Sigma$, $|\Sigma|$ the number of loops in Σ , and c the number of components of $S_r - \Sigma$, then

$$g - c + |\Sigma| = r - 1.$$

Hence, if $|\Sigma| \geq r + 1$, then $g \leq c - 2$, and there must be at least two components of $S_r - \Sigma$ that have genus 0. Since Σ consists of nontrivial, nonhomotopic loops, both of these two components must have negative Euler characteristics.

Therefore, if the multiple curve C had more than r nonhomotopic components, then there would be at least two components of $S_r - C$ which would have negative Euler characteristics.

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