PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 11, Pages 3387–3395
S 0002-9939(04)07524-0
Article electronically published on June 21, 2004

PRECISE ASYMPTOTICS FOR A SERIES OF T. L. LAI

AUREL SPĂTARU

(Communicated by Richard C. Bradley)

ABSTRACT. Let $X,\ X_1,\ X_2,...$ be i.i.d. random variables with $EX=0$, and set $S_n = X_1 + ... + X_n$. We prove that, for $1 < p < 3/2$,

$$\lim_{\varepsilon \searrow \sigma\sqrt{2p-2}} \sqrt{\varepsilon^2 - \sigma^2(2p-2)} \sum_{n\geq 2} n^{p-2} P(|S_n| \geq \varepsilon\sqrt{n\log n}) = \sigma\sqrt{\frac{2}{p-1}},$$

under the assumption that $EX^2 = \sigma^2$ and $E[|X|^{2p}(\log^+|X|)^{-p}] < \infty$. Necessary and sufficient conditions for the convergence of the sum above were established by Lai (1974).

1. Introduction

Let $\{X,\ X_k,\ k \geq 1\}$ be i.i.d. random variables with $EX = 0$ and partial sums $\{S_n,\ n \geq 1\}$, and consider series of the type

$$f(\varepsilon) = \sum_n t_n P(|S_n| \geq \varepsilon c_n), \quad \varepsilon > 0, \tag{1}$$

where $t_n, c_n > 0$ and $\sum_n t_n = \infty$. f is a nonincreasing function and, under appropriate moment conditions, a threshold α may be determined such that $f(\varepsilon) = \infty$ for $\varepsilon < \alpha$, while $f(\varepsilon) < \infty$ for $\varepsilon > \alpha$. Then it is sensible to look for a normalizing function $g(\varepsilon),\ \varepsilon > \alpha$, such that the ratio $f(\varepsilon)/g(\varepsilon)$ has a nondegenerate limit l as $\varepsilon \searrow \alpha$. The literature on this so-called precise asymptotics problem is reasonably rich, almost exhaustive references being given in Spătaru [13]. (We record here three new papers in the field by Scheffler [11] and Rozovsky [9, 10], and recent related work on counting processes, record times, and partial maxima: Gut and Steinebach [5], Gut [2], Wang and Yang [14], and Wang, Yan and Yang [15].)

The starting point for the present research is the following theorem, concerning moderate deviations, due to Lai [7].

Theorem A. *Let $p > 1$, and assume that $EX^2 = \sigma^2$ and $E[|X|^{2p}(\log^+|X|)^{-p}] < \infty$. Then*

$$\sum_{n\geq 2} n^{p-2} P(|S_n| \geq \varepsilon\sqrt{n\log n}) < \infty, \quad \varepsilon > \sigma\sqrt{2p-2}. \tag{2}$$

Conversely, if the sum is finite for some ε, then $E[|X|^{2p}(\log^+|X|)^{-p}] < \infty$.

Received by the editors August 1, 2003.
2000 *Mathematics Subject Classification.* Primary 60G50, 60E15.
Key words and phrases. Tail probabilities of sums of i.i.d. random variables, moderate deviations, Lai law.

We obtain the next precise asymptotics describing the behaviour of the series in (2) near the threshold $\sigma\sqrt{2p-2}$.

Theorem 1. *If* $1 < p < 3/2$, $EX^2 = \sigma^2$ *and* $E[|X|^{2p}(\log^+|X|)^{-p}] < \infty$, *then*

$$\lim_{\varepsilon\searrow\sigma\sqrt{2p-2}} \sqrt{\varepsilon^2 - \sigma^2(2p-2)} \sum_{n\geq 2} n^{p-2} P(|S_n| \geq \varepsilon\sqrt{n\log n}) = \sigma\sqrt{\frac{2}{p-1}}.$$

Since truncations of X may have nonzero expectations, we needed to prove in fact the following more general result. (Notice that the convergence of the sum in (2) implies that

$$\sum_{n\geq 2} n^{p-2} P(|S_n| \geq \varepsilon\sqrt{n\log n} + a_n) < \infty$$

whenever $a_n = o(\sqrt{n\log n})$ and $\varepsilon > \sigma\sqrt{2p-2}$.)

Theorem 1′. *If* $1 < p < 3/2$, $EX^2 = \sigma^2$ *and* $E[|X|^{2p}(\log^+|X|)^{-p}] < \infty$, *then*

$$\lim_{\varepsilon\searrow\sigma\sqrt{2p-2}} \sqrt{\varepsilon^2 - \sigma^2(2p-2)} \sum_{n\geq 2} n^{p-2} P(|S_n| \geq \varepsilon\sqrt{n\log n} + a_n) = \sigma\sqrt{\frac{2}{p-1}}.$$

There are two rather distinct methods to derive exact asymptotics for the series in (1) as $\varepsilon \searrow \alpha$. The first step of both methods consists in finding l under the special assumption that X is stable. The classical method, as illustrated in Heyde [6], Spătaru [12], Gut and Spătaru [3], etc., proceeds with the assumption that X is in the domain of attraction of a stable law, and uses a version of the powerful Fuk-Nagaev inequality (see Spătaru [12]). This inequality is applicable only if the working condition has the form $E|X|^r < \infty$ for some r. The second approach, introduced and practised by Gut and Spătaru [4], and Spătaru [13], is suitable for cases when Fuk-Nagaev type inequalities prove useless. It assumes existence of finite variance, and after the first step, it goes on via truncation and departure from normality.

The case of Lai's theorem seems to be the most difficult encountered so far, since p may be as large as one wants, while $\sqrt{n\log n}$ is smaller than any $n^{1/s}$, $s < 2$. Because the Fuk-Nagaev inequality is inadequate, we make use of the second method. However, appealing to the Berry-Esseen inequality to estimate the error in the normal approximation, as in Gut and Spătaru [4], and Spătaru [13], does not suffice, so we need the non-uniform estimate of Nagaev (see, e.g., Petrov [8], p. 125). Nevertheless, since the factor $n^{-1/2}$ in the Nagaev inequality (and in the Berry-Esseen inequality) is unimprovable, the case $p \geq 3/2$ remains unsettled in general; cf. also Proposition 1.

Without loss of generality we assume that $\sigma^2 = 1$. Let Φ denote the standard normal distribution function, and put $\Psi(x) = 1 - \Phi(x) + \Phi(-x)$, $x \geq 0$. Since $\varepsilon \searrow \sqrt{2p-2}$, we suppose that $2p-2 < \varepsilon^2 < 2p$ (say). Throughout, C and K will denote positive constants and real constants, respectively, independent of ε, possibly varying from place to place. Also $\gamma > 1/2$, $[x]$ will stand for the largest integer $\leq x$, and $\log^+ x = \log(e \vee x)$, $x \geq 0$. To simplify the exposition, we assume that $\sqrt{(2p-2)\log n} + K(\log n)^{-\gamma} \geq 0$, $n \geq 2$.

2. Proof of Theorem 1

The next proposition shows that Theorem 1 holds without the restriction $p < 3/2$, if Φ is the distribution function of X.

Proposition 1. *If X has a standard normal distribution, then*

$$\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}\sum_{n\geq 2} n^{p-2}P(|S_n|\geq \varepsilon\sqrt{n\log n})$$
$$=\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}\sum_{n\geq 2} n^{p-2}\Psi(\varepsilon\sqrt{\log n}) = \sqrt{\frac{2}{p-1}}.$$

Proof. On account of the Euler-MacLaurin sum formula (see Cramér [1], p. 124), we have

$$\sum_{n\geq 2} n^{p-2}\Psi(\varepsilon\sqrt{\log n})$$
(3)
$$= \int_2^\infty x^{p-2}\Psi(\varepsilon\sqrt{\log x})dx + \frac{1}{4}\Psi(\varepsilon\sqrt{\log 2}) - \int_2^\infty P(x)d[x^{p-2}\Psi(\varepsilon\sqrt{\log x})],$$

where $P(x)=[x]-x+1/2$. Since $|P(x)|\leq 1/2$ and $\Psi(x), |\Psi'(x)| \leq Ce^{-x^2/2},\ x\geq 0$, we obtain, for $\varepsilon>\sqrt{2p-2}$,

$$\left|\int_2^\infty P(x)d[x^{p-2}\Psi(\varepsilon\sqrt{\log x})]\right|$$
$$\leq \frac{|p-2|}{2}\int_2^\infty x^{p-3}\Psi(\varepsilon\sqrt{\log x})dx + \frac{\varepsilon}{4}\int_2^\infty \frac{x^{p-3}}{\sqrt{\log x}}\left|\Psi'(\varepsilon\sqrt{\log x})\right|dx$$
$$\leq C\int_2^\infty x^{p-3-\varepsilon^2/2}dx \leq C\int_2^\infty x^{-2}dx,$$

and so

$$\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}\int_2^\infty P(x)d[x^{p-2}\Psi(\varepsilon\sqrt{\log x})] = 0. \tag{4}$$

By putting $y=\varepsilon\sqrt{\log x}$ and partial integration, as $\Psi(x)\leq e^{-x^2/2},\ x\geq 0$, we get, for $\varepsilon>\sqrt{2p-2}$,

$$\int_2^\infty x^{p-2}\Psi(\varepsilon\sqrt{\log x})dx = \int_{\varepsilon\sqrt{\log 2}}^\infty \Psi(y)e^{\frac{y^2}{\varepsilon^2}(p-1)}\frac{2y}{\varepsilon^2}dy$$
$$=\frac{1}{p-1}\left[-2^{p-1}\Psi(\varepsilon\sqrt{\log 2}) - \int_{\varepsilon\sqrt{\log 2}}^\infty \Psi'(y)e^{\frac{y^2}{\varepsilon^2}(p-1)}dy\right].$$

Therefore, the substitution $y\sqrt{\varepsilon^2 - 2p + 2} = \varepsilon z$ yields

$$\begin{aligned}
&\lim_{\varepsilon \searrow \sqrt{2p-2}} \sqrt{\varepsilon^2 - 2p + 2} \int_2^\infty x^{p-2} \Psi(\varepsilon\sqrt{\log x}) dx \\
(5) \qquad &= \lim_{\varepsilon \searrow \sqrt{2p-2}} \frac{\sqrt{\varepsilon^2 - 2p + 2}}{p-1} \frac{2}{\sqrt{2\pi}} \int_{\varepsilon\sqrt{\log 2}}^\infty e^{-y^2(\varepsilon^2-2p+2)/2\varepsilon^2} dy \\
&= \lim_{\varepsilon \searrow \sqrt{2p-2}} \frac{\varepsilon}{p-1} \Psi(\sqrt{(\varepsilon^2 - 2p + 2)\log 2}) = \sqrt{\frac{2}{p-1}}.
\end{aligned}$$

The conclusion now follows from (3)-(5). □

The following two propositions also hold without the restriction $p < 3/2$.

Proposition 2. *For any $K \in \mathbb{R}$, we have*

$$\lim_{\varepsilon \searrow \sqrt{2p-2}} \sqrt{\varepsilon^2 - 2p + 2} \sum_{n \geq 2} n^{p-2} \left| \Psi(\varepsilon\sqrt{\log n}) - \Psi(\varepsilon\sqrt{\log n} + K(\log n)^{-\gamma}) \right| = 0.$$

Proof. By Lagrange's theorem,

$$\begin{aligned}
&\left| \Psi(\varepsilon\sqrt{\log n}) - \Psi(\varepsilon\sqrt{\log n} + K(\log n)^{-\gamma}) \right| \\
&= -|K| (\log n)^{-\gamma} \Psi'(\theta_n) = C(\log n)^{-\gamma} e^{-\theta_n^2/2}, \ n \geq 2,
\end{aligned}$$

where $\varepsilon\sqrt{\log n} \leq \theta_n$ or $\varepsilon\sqrt{\log n} + K(\log n)^{-\gamma} \leq \theta_n$, i.e., $\theta_n^2 \geq \varepsilon^2 \log n$ or $\theta_n^2 \geq \varepsilon^2 \log n + 2\varepsilon K(\log n)^{1/2-\gamma} \geq \varepsilon^2 \log n - C$, according as $K \geq 0$ or $K < 0$. Thus $e^{-\theta_n^2/2} \leq Cn^{-\varepsilon^2/2}$, $n \geq 2$, and so

$$\sum_{n \geq 2} n^{p-2} \left| \Psi(\varepsilon\sqrt{\log n}) - \Psi(\varepsilon\sqrt{\log n} + K(\log n)^{-\gamma}) \right| \leq C \sum_{n \geq 2} n^{p-2-\varepsilon^2/2} (\log n)^{-\gamma}.$$

Hence, by the change of variable $y = (\varepsilon^2 - 2p + 2)\log x$, we see that

$$\begin{aligned}
&\sum_{n \geq 2} n^{p-2} \left| \Psi(\varepsilon\sqrt{\log n}) - \Psi(\varepsilon\sqrt{\log n} + K(\log n)^{-\gamma}) \right| \\
&\leq C + C \int_2^\infty x^{p-2-\varepsilon^2/2} (\log x)^{-\gamma} dx \\
&= C + C(\varepsilon^2 - 2p + 2)^{\gamma-1} \int_{(\varepsilon^2-2p+2)\log 2}^\infty y^{-\gamma} e^{-y/2} dy.
\end{aligned}$$

Therefore, since $\gamma > 1/2$,

$$\begin{aligned}
&\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}\sum_{n\geq 2} n^{p-2}\left|\Psi(\varepsilon\sqrt{\log n})-\Psi(\varepsilon\sqrt{\log n}+K(\log n)^{-\gamma})\right| \\
&\leq C\lim_{\varepsilon\searrow\sqrt{2p-2}}(\varepsilon^2-2p+2)^{\gamma-1/2}\left(\int_{(\varepsilon^2-2p+2)\log 2}^{1} y^{-\gamma}e^{-y/2}dy+\int_1^{\infty}e^{-y/2}dy\right) \\
&\qquad\leq C\lim_{\varepsilon\searrow\sqrt{2p-2}}(\varepsilon^2-2p+2)^{\gamma-1/2}\int_{(\varepsilon^2-2p+2)C}^{1} y^{-\gamma}dy=0. \qquad \square
\end{aligned}$$

Now, let $\psi(x)=\sqrt{x\log x},\ x\geq 1$, and let $\varphi(x),\ x\geq 0$, denote the inverse function of ψ. Then

$$\varphi(x)\sim\frac{x^2}{2\log x}\text{ as }x\to\infty. \tag{6}$$

Also, for $\varepsilon>0$ and $n\geq 2$, put $Y_{n,k}=X_kI\{|X_k|<\varepsilon\psi(n)\},\ 1\leq k\leq n,\ U_n=Y_{n,1}+...+Y_{n,n}$, $\sigma_n^2=\sigma_n^2(\varepsilon)=\operatorname{Var}Y_{n,1}$ and $\rho_n=\sigma_n^{-3}E\left|Y_{n,1}-EY_{n,1}\right|^3$. The moment assumption in the next proposition is weaker than that required for Theorem 1.

Proposition 3. *Assume that $E[X^2\log^+|X|]<\infty$. Then, for any $K\in\mathbb{R}$,*

$$\lim_{\varepsilon\searrow\sqrt{2p-2}}\sqrt{\varepsilon^2-2p+2}\sum_{n\geq 2}n^{p-2}\left(\begin{array}{c}\Psi(\varepsilon\sqrt{\log n}+K(\log n)^{-\gamma})\\ -\Psi(\varepsilon\sigma_n^{-1}\sqrt{\log n}+K\sigma_n^{-1}(\log n)^{-\gamma})\end{array}\right)=0.$$

Proof. We begin by noticing that $\lim_{n\to\infty}\sigma_n(\varepsilon)=1$ uniformly with respect to $\varepsilon>\sqrt{2p-2}$. Then choose n_0 (independent of ε) such that $\sigma_n(1+\sigma_n)\geq 1$ for $n\geq n_0$. Hence, by Lagrange's theorem, arguing as in the proof of Proposition 2, we see that

$$\begin{aligned}
&\Psi(\varepsilon\sqrt{\log n}+K(\log n)^{-\gamma})-\Psi(\varepsilon\sigma_n^{-1}\sqrt{\log n}+K\sigma_n^{-1}(\log n)^{-\gamma}) \\
&\leq(\varepsilon+|K|)(1-\sigma_n^{-1})\sqrt{\log n}\Psi'(\theta_n)\leq C(1-\sigma_n^2)\sqrt{\log n}n^{-\varepsilon^2/2},\ n\geq n_0,
\end{aligned}\tag{7}$$

where $\varepsilon\sqrt{\log n}+K(\log n)^{-\gamma}\leq\theta_n\leq\varepsilon\sigma_n^{-1}\sqrt{\log n}+K\sigma_n^{-1}(\log n)^{-\gamma}$. Since $EX^2=1$, (7) shows that

$$\begin{aligned}
&\sum_{n\geq 2}n^{p-2}\left(\Psi(\varepsilon\sqrt{\log n}+K(\log n)^{-\gamma})-\Psi(\varepsilon\sigma_n^{-1}\sqrt{\log n}+K\sigma_n^{-1}(\log n)^{-\gamma})\right) \\
&\qquad\leq C+C\sum_{n\geq n_0}n^{p-2-\varepsilon^2/2}\sqrt{\log n}E[X^2I\{|X|\geq\varepsilon\psi(n)\}] \\
&\qquad\quad+C\sum_{n\geq n_0}n^{p-2-\varepsilon^2/2}\sqrt{\log n}(EY_{n,1})^2 \\
&\qquad=C+C\Sigma_1+C\Sigma_2\text{ (say)}.
\end{aligned}\tag{8}$$

Taking into account that the function $x^{-1}\sqrt{\log x}$ is decreasing for $x > \sqrt{e}$, and applying Fubini's theorem, we obtain

$$\Sigma_1 \le E\left[X^2 I\{\varphi(|X|/\varepsilon) \ge n_0\} \sum_{n_0 \le n \le \varphi(|X|/\varepsilon)} n^{p-2-\varepsilon^2/2}\sqrt{\log n}\right]$$
$$\le CE\left[X^2 I\{\varphi(|X|/\sqrt{2p-2}) \ge n_0\} \int_1^{\varphi(|X|/\sqrt{2p-2})} x^{p-2-\varepsilon^2/2}\sqrt{\log x}\,dx\right].$$

Further, the substitution $y = (\varepsilon^2 - 2p + 2)\log x$ yields

(9)
$$\Sigma_1 \le CE\left[\frac{X^2 I\{\varphi(|X|/\sqrt{2p-2}) \ge n_0\}}{(\varepsilon^2 - 2p + 2)^{3/2}} \int_0^{(\varepsilon^2-2p+2)\log\varphi(|X|/\sqrt{2p-2})} \sqrt{y}e^{-y/2}dy\right].$$

Now, by l'Hôpital's rule, for $\varphi(|X|/\sqrt{2p-2}) \ge n_0$,

$$\lim_{\varepsilon \searrow \sqrt{2p-2}} \frac{1}{\varepsilon^2 - 2p + 2} \int_0^{(\varepsilon^2-2p+2)\log\varphi(|X|/\sqrt{2p-2})} \sqrt{y}e^{-y/2}dy$$

(10)
$$= \lim_{\varepsilon \searrow \sqrt{2p-2}} \sqrt{\varepsilon^2 - 2p + 2}(\log\varphi(|X|/\sqrt{2p-2}))^{3/2}\varphi(|X|/\sqrt{2p-2})^{p-1-\varepsilon^2/2} = 0.$$

Notice also that $E[X^2 \log^+|X|] < \infty$ entails

$$E[X^2 \log\varphi(|X|/\sqrt{2p-2}) I\{\varphi(|X|/\sqrt{2p-2}) \ge n_0\}] < \infty,$$

by virtue of (6). Since the function $\sqrt{y}e^{-y/2}$, $y \ge 0$, is bounded, (9) and (10) show that

$$\lim_{\varepsilon \searrow \sqrt{2p-2}} \sqrt{\varepsilon^2 - 2p + 2}\,\Sigma_1 = 0, \tag{11}$$

by the bounded convergence theorem. As for Σ_2, $EX = 0$ and $E[X^2 \log^+|X|] < \infty$ imply that

$$|EY_{n,1}| \le E[|X|\, I\{|X| \ge \sqrt{2p-2}\psi(n)\}] \le Cn^{-1/2}(\log n)^{-3/2},\ n \ge 2, \tag{12}$$

and so

(13)
$$\lim_{\varepsilon \searrow \sqrt{2p-2}} \sqrt{\varepsilon^2 - 2p + 2}\,\Sigma_2 \le C \lim_{\varepsilon \searrow \sqrt{2p-2}} \sqrt{\varepsilon^2 - 2p + 2} \sum_{n \ge n_0} n^{p-3-\varepsilon^2/2}$$
$$\le C \lim_{\varepsilon \searrow \sqrt{2p-2}} \sqrt{\varepsilon^2 - 2p + 2} \sum_{n \ge n_0} n^{-2} = 0.$$

Finally, the result follows from (8), (11) and (13). □

Proposition 4. *Assume that* $1 < p < 3/2$ *and* $E[|X|^{2p}(\log^+|X|)^{-p}] < \infty$. *Then*

$$\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}\sum_{n\geq 2} n^{p-2}\left| \begin{array}{c} P(|U_n-EU_n|\geq \varepsilon\sqrt{n\log n}+K\sqrt{n}(\log n)^{-\gamma}) \\ -\Psi(\varepsilon\sigma_n^{-1}\sqrt{\log n}+K\sigma_n^{-1}(\log n)^{-\gamma}) \end{array}\right| = 0. \tag{14}$$

Proof. For sufficiently large $n \geq n_1$ (say), by the Nagaev inequality, we have

$$\begin{aligned}
&\Big|P(|U_n-EU_n)|\geq \varepsilon\sqrt{n\log n}+K\sqrt{n}(\log n)^{-\gamma}) \\
&\qquad -\Psi(\varepsilon\sigma_n^{-1}\sqrt{\log n}+K\sigma_n^{-1}(\log n)^{-\gamma})\Big| \\
&\qquad \leq C\frac{\rho_n}{\sqrt{n}(\varepsilon\sigma_n^{-1}\sqrt{\log n}+K\sigma_n^{-1}(\log n)^{-\gamma})^3} = C\frac{E|Y_{n,1}-EY_{n,1}|^3}{\sqrt{n}(\varepsilon\sqrt{\log n}+K(\log n)^{-\gamma})^3} \\
&\qquad \leq C\frac{E|Y_{n,1}|^3+|EY_{n,1}|^3}{\sqrt{n}(\log n)^{3/2}}.
\end{aligned}$$

Let Σ stand for the sum in (14). Since $p < 3/2$ and $\sum_{n\geq m} n^{p-5/2}(\log n)^{-3/2} \leq Cm^{p-3/2}(\log m)^{-3/2}$, by Fubini's theorem and (12), we get

$$\begin{aligned}
&\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}\,\Sigma \\
&\leq C\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}\sum_{n\geq n_1} n^{p-5/2}(\log n)^{-3/2}E[|X|^3 I\{|X|<\varepsilon\psi(n)\}] \\
&\quad +C\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}\sum_{n\geq n_1} n^{p-4}(\log n)^{-6} \\
&= C\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2} \\
&\quad \times E\left[|X|^3 I\{\varphi(|X|/\varepsilon)\geq 2\}\sum_{n>\varphi(|X|/\varepsilon)} n^{p-5/2}(\log n)^{-3/2}\right] \\
&\leq C\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2} \\
&\quad \times E\left[|X|^3 I\{\varphi(|X|/\varepsilon)\geq 2\}[\varphi(|X|/\varepsilon)]^{p-3/2}(\log[\varphi(|X|/\varepsilon)])^{-3/2}\right] \\
&\leq C\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}E[|X|^{2p}(\log^+|X|)^{-p}] = 0,
\end{aligned}$$

where the last inequality above is obtained via (6). ∎

We are now prepared to prove Theorem 1′.

Proof of Theorem 1′. From Propositions 1-4, we see that

$$\lim_{\varepsilon\searrow\sqrt{2p-2}} \sqrt{\varepsilon^2-2p+2}\sum_{n\geq 2} n^{p-2}P(|U_n-EU_n|\geq \varepsilon\sqrt{n\log n}+K\sqrt{n}(\log n)^{-\gamma}) = \sqrt{\frac{2}{p-1}}. \tag{15}$$

Put $\gamma' = (3/2) \wedge \gamma$. On account of (12), we have

$$|E\,U_n| + |a_n| \le C\sqrt{n}(\log n)^{-3/2} + C\sqrt{n}(\log n)^{-\gamma} \le C\sqrt{n}(\log n)^{-\gamma'}, \quad n \ge 2,$$

and so

$$\begin{aligned} P(|U_n - EU_n| \ge \varepsilon\sqrt{n\log n} + C\sqrt{n}(\log n)^{-\gamma'}) &\le P(|U_n| \ge \varepsilon\sqrt{n\log n} + a_n) \\ &\le P(|U_n - EU_n| \ge \varepsilon\sqrt{n\log n} - C\sqrt{n}(\log n)^{-\gamma'}), \quad n \ge 2. \end{aligned} \tag{16}$$

From (15) and (16), we obtain

$$\lim_{\varepsilon \searrow \sqrt{2p-2}} \sqrt{\varepsilon^2 - 2p + 2} \sum_{n \ge 2} n^{p-2} P(|U_n| \ge \varepsilon\sqrt{n\log n} + a_n) = \sqrt{\frac{2}{p-1}},$$

which completes the proof, since

$$\begin{aligned} &\sum_{n \ge 2} n^{p-2} P(S_n \ne U_n) \le \sum_{n \ge 2} n^{p-1} P(|X| \ge \varepsilon\psi(n)) \\ &\le CE\varphi(|X|/\varepsilon)^p \le CE[|X|^{2p}(\log^+|X|)^{-p}] < \infty. \end{aligned}$$

□

References

[1] CRAMÉR, H. (1946). *Mathematical Methods in Statistics.* Princeton Univ. Press. MR 8:39f

[2] GUT, A. (2002). Precise asymptotics for record times and the associated counting processes. *Stochast. Process. Appl.* **101,** 233-239. MR 2003i:60091

[3] GUT, A. and SPĂTARU, A. (2000a). Precise asymptotics in the Baum-Katz and Davis law of large numbers. *J. Math. Anal. Appl.* **248,** 233-246. MR 2001g:60064

[4] GUT, A. and SPĂTARU, A. (2000b). Precise asymptotics in the law of the iterated logarithm. *Ann. Probab.* **28,** 1870-1883. MR 2001m:60100

[5] GUT, A. and STEINEBACH, J. (2002). Convergence rates and precise asymptotics for renewal counting processes and some first passage times. Tech. Rept. no. 2002:7, Dept. of Mathematics, Uppsala Univ.

[6] HEYDE, C. C. (1975). A supplement to the strong law of large numbers. *J. Appl. Probab.* **12,** 173-175. MR 51:4358

[7] LAI, T. L. (1974). Limit theorems for delayed sums. *Ann. Probab.* **2,** 432-440. MR 50:8664

[8] PETROV, V. V. (1975). *Sums of Independent Random Variables.* Springer-Verlag. MR 52:9335

[9] ROZOVSKY, L. V. (2002). Some limit theorems for large deviations of sums of independent random variables with a common distribution function from the domain of attraction of a normal law (Russian). *Zapiski Nauch. Semin. POMI* **294,** 165-193; English transl., to appear in *J. Math. Sci. (New York).* MR 2004b:60086

[10] ROZOVSKY, L. V. (2003). On precise asymptotics in the weak law of large numbers for sums of independent random variables with a common distribution function from the domain of attraction of a stable law (Russian). *Teoriya Veroyatn. Primen.* **48,** no. 3.

[11] SCHEFFLER, H.-P. (2003). Precise asymptotics in Spitzer's and Baum-Katz law of large numbers: the semistable case. *J. Math. Anal. Appl.* **288**, 285-298.

[12] SPĂTARU, A. (1999). Precise asymptotics in Spitzer's law of large numbers. *J. Theor. Probab.* **12,** 811-819. MR 2000j:60035

[13] SPĂTARU, A. (2002). Exact asymptotics in loglog laws for random fields. *J. Theor. Probab.*, to appear.

[14] WANG Y. and YANG, Y. (2003). A general law of precise asymptotics for the counting process of record times. *J. Math. Anal. Appl.* **286**, 753-764.

[15] WANG Y., YAN J., and YANG Y. (2003). The precise asymptotics for the order statistics generated by the random samples of maximum domain of attraction of the Fréchet distribution. Preprint, Dept. of Mathematics, Soochow Univ.

Institute of Mathematical Statistics and Applied Mathematics, Romanian Academy, Calea 13 Septembrie.13, 76100 Bucharest, Romania

E-mail address: aspataru@pcnet.ro

URL: http://www.csm.ro/~aspataru