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MÖBIUS FUNCTION OF COORDINATE HYPERPLANES IN COMPLEX ELLIPSOIDS

WITOLD JARNICKI

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ABSTRACT. For $p_1, \dots, p_n > 0$, let $\mathbb{E} = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\}$ be a complex ellipsoid. We present effective formulas for the generalized Möbius and Green functions $m_{\mathbb{E}}(A, \cdot)$, $g_{\mathbb{E}}(A, \cdot)$ in the case where $A := \{z \in \mathbb{E} : z_1 \cdots z_k = 0\}$ $(1 \le k \le n)$.

1. INTRODUCTION AND MAIN RESULTS

Let $G \subset \mathbb{C}^n$ be a domain, and let $A \subset G$. Define the *generalized Green function*

$$g_G(A, z) := \sup\{u(z) : \ u : G \longrightarrow [0, 1), \ \log u \in \mathcal{PSH}(G), \\ \forall_{a \in A} \ \exists_{C = C(u,a) > 0} \ \forall_{w \in G} : \ u(w) \le C\|w - a\|\}, \quad z \in G,$$

and the *generalized Möbius function*

$$m_G(A, z) := \sup\{|f(z)| : f \in \mathcal{O}(G, \mathbb{D}), \ f|_A \equiv 0\}, \quad z \in G,$$

where $\mathbb{D}$ denotes the unit disc (cf. [Jar-Jar-Pfl 2003]).

The generalized Möbius and Green functions were recently studied by many authors (e.g. [Lár-Sig 1998]). Despite various proven properties, effective formulas for $m_G(A, \cdot)$ and $g_G(A, \cdot)$ are known only in a few special cases. An effective formula for the generalized Green function in the case of the Euclidean ball $G = \mathbb{B}_2 \subset \mathbb{C}^2$ and the union of coordinate lines $A = \{(z, w) \in \mathbb{B}_2 : zw = 0\}$ was given in [Ngu 2003].

Let $n \ge 2$. Let $p_1, \dots, p_n > 0$. Define the *complex ellipsoid with exponents* $p_1, \dots, p_n$:

$$\mathbb{E} = \mathbb{E}_{p_1,\dots,p_n} := \Big\{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\Big\}.$$

Observe that $\mathbb{E}_{p_1,\dots,p_n}$ is convex if and only if $p_1, \dots, p_n \ge 1/2$ (cf. [Jar-Pfl 1993], §8.4).

For $G \subset \mathbb{C}^n$, $k \in \{1, \dots, n\}$, we consider the sets

$$A = A_{G,k} := \{z \in G : z_1 \cdots z_k = 0\}.$$

We give an effective formula for $g_{\mathbb{E}}(A_{\mathbb{E},k}, \cdot)$ for any complex ellipsoid $\mathbb{E} \subset \mathbb{C}^n$ and any $k \in \{1, \dots, n\}$ (Theorem 1.2(a)). This is a generalization of a result in [Ngu 2003] (with a much simpler proof). We prove that the $m_{\mathbb{E}}(A_{\mathbb{E},k}, \cdot) \equiv$

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$g_{\mathbb{E}}(A_{\mathbb{E},k},\cdot)$ for the cases where $\mathbb{E}$ is convex (Theorem 1.2(b)) or where $k=1$, $n=2$, $p_2\geq 1/2$ (Theorem 1.2(c)). This is not true in the general nonconvex case; we prove that $m_{\mathbb{E}}(A_{\mathbb{E},k},\cdot)\not\equiv g_{\mathbb{E}}(A_{\mathbb{E},k},\cdot)$ if there exists a $j\in\{k+1,\dots,n\}$ such that $p_j<1/2$ (Theorem 1.2(d)). Partial results for the case $k=n=2$ are given (Theorem 1.2(e)).

Definition 1.1. Let $p_1,\dots,p_n>0$, $1\leq k\leq n$. Put $\mathbb{E}:=\mathbb{E}_{p_1,\dots,p_n}$, $A:=A_{\mathbb{E},k}$. Let $z\in\mathbb{E}$ be such that the sequence $(p_j|z_j|^{2p_j})_{j=1}^k$ is monotonically increasing. For $s\in\{1,\dots,k\}$ define

$$q_s:=\sum_{j=1}^{s}(2p_j)^{-1},\quad r_s(z):=1-\sum_{j=s+1}^{n}|z_j|^{2p_j},\quad c_s(z):=r_s(z)/q_s.$$

Let $d:=\max\{s\in\{1,\dots,k\}:2p_s|z_s|^{2p_s}\leq c_s(z)\}$. Define

$$(*)\qquad R_{\mathbb{E}}(A,z)=\prod_{j=1}^{d}|z_j|\Big(\frac{2p_j}{c_d(z)}\Big)^{\frac{1}{2p_j}}.$$

The main results of the paper are the following.

Theorem 1.2. *Under the above assumptions we have:*

(a) $g_{\mathbb{E}}(A,\cdot)\equiv R_{\mathbb{E}}(A,\cdot)$,
(b) $m_{\mathbb{E}}(A,\cdot)\equiv g_{\mathbb{E}}(A,\cdot)\equiv R_{\mathbb{E}}(A,\cdot)$, *for* $p_j\geq 1/2$, $j=1,\dots,n$,
(c) $m_{\mathbb{E}}(A,\cdot)\equiv g_{\mathbb{E}}(A,\cdot)\equiv R_{\mathbb{E}}(A,\cdot)$, *for* $k=1$, $n=2$, $p_2\geq 1/2$,
(d) $m_{\mathbb{E}}(A,\cdot)\not\equiv g_{\mathbb{E}}(A,\cdot)$ *if there exists a* $j\in\{k+1,\dots,n\}$ *with* $p_j<1/2$,
(e) $m_{\mathbb{E}}(A,\cdot)\equiv g_{\mathbb{E}}(A,\cdot)\equiv R_{\mathbb{E}}(A,\cdot)$, *for* $k=n=2$, $p_1\leq p_2$, *and either* $p_2\geq 1/2$ *or* $8p_1+4p_2(1-p_2)>1$.

Observe that one can obtain the monotonicity of $(p_j|z_j|^{2p_j})_{j=1}^k$ for arbitrary $z\in\mathbb{E}$ by permutation of coordinates. The final result is a subdivision of $\mathbb{E}$ into 2^k-1 subsets, each with $R_{\mathbb{E}}(A,\cdot)$ given by a formula of type $(*)$.

Remark 1.3. In the case where $p_1=\dots=p_n=1$, the domain $\mathbb{E}$ is the Euclidean ball $\mathbb{B}_n$. Theorem 1.2 may then be formulated in the following simpler way.

Corollary 1.4. *Assume that* $|z_1|\leq\dots\leq|z_k|$, *and let*

$$d:=\max\Big\{s\in\{1,\dots,k\}:s|z_s|^2+\sum_{j=s+1}^{n}|z_j|^2\leq 1\Big\}.$$

Then

$$m_{\mathbb{B}_n}(A,z)=g_{\mathbb{B}_n}(A,z)=\Big(\frac{d}{1-\sum_{j=d+1}^{n}|z_j|^2}\Big)^{\frac{d}{2}}\prod_{j=1}^{d}|z_j|.$$

Remark 1.5. It is unknown to the author whether $m_{\mathbb{E}}(A,\cdot)\equiv g_{\mathbb{E}}(A,\cdot)$ for any $p_j>0$, $j=1,\dots,k$, $p_j\geq 1/2$, $j=k+1,\dots,n$. However, Theorem 1.2(e) and the author's research (cf. Remark 4.4) indicate that this is true at least in the case $k=n=2$.

Remark 1.6. Take $p_{j,\ell}\nearrow+\infty$, $j=1,\dots,n$. Then

$$G_\ell:=\mathbb{E}_{p_{1,\ell},\dots,p_{n,\ell}}\nearrow\mathbb{D}^n.$$

Consequently,

$$m_{G_\ell}(A_{G_\ell,k},\cdot)=g_{G_\ell}(A_{G_\ell,k},\cdot)\searrow m_{\mathbb{D}^n}(A_{\mathbb{D}^n,k},\cdot)=g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k},\cdot)$$

(cf. [Jar-Jar-Pfl 2003], Property 2.7) and hence

$$m_{\mathbb{D}^n}(A_{\mathbb{D}^n,k},z)=|z_1|\cdots|z_k|,\quad z\in\mathbb{D}^n.$$

However, we need the formula for $g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k},\cdot)$ before we prove Theorem 1.2 (cf. Lemma 2.1).

2. Proof of Theorem 1.2

Lemma 2.1. *Let* $n\in\mathbb{N}$, $1\le k\le n$. *Then*

$$m_{\mathbb{D}^n}(A_{\mathbb{D}^n,k},z)=g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k},z)=|z_1|\cdots|z_k|,\quad z\in\mathbb{D}^n.$$

Proof of Lemma 2.1 (due to P. Pflug). Obviously,

$$|z_1|\cdots|z_k|\le m_{\mathbb{D}^n}(A_{\mathbb{D}^n,k},z)\le g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k},z).$$

It remains to prove that $g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k},z)\le|z_1|\cdots|z_k|$.

It is enough to prove that for any $1\le k\le n$ and for any function $u:\mathbb{D}^n\longrightarrow[0,1)$ such that $\log u\in\mathcal{PSH}(\mathbb{D}^n)$ and $u(z)\le C(a)\|z-a\|$, $a\in A_{\mathbb{D}^n,k}$, $z\in\mathbb{D}^n$, we have $u(z)\le|z_1|\cdots|z_k|$. We proceed by induction on k.

For $k=1$ the inequality follows from the Schwarz-type lemma for logarithmically subharmonic functions $u(\cdot,z_2,\dots,z_n)$, $z_2,\dots,z_n\in\mathbb{D}$.

For $k>1$ we first apply the case $k=1$ and get $u(z_1,\dots,z_n)\le|z_1|$, $z\in\mathbb{D}^n$. Applying the inductive assumption to $u(z_1,\cdot)/|z_1|$, $z_1\in\mathbb{D}$, finishes the proof. □

Take a $u:\mathbb{E}\longrightarrow[0,1)$ with $\log u\in\mathcal{PSH}(\mathbb{E})$ and $u(\zeta)\le C(a)\|\zeta-a\|$, $a\in A$, $\zeta\in\mathbb{E}$. Consider the mapping

$$\iota_z:\mathbb{D}^d\ni(\zeta_1,\dots,\zeta_d)\longmapsto\Big(\zeta_1\Big(\frac{c_d(z)}{2p_1}\Big)^{\frac{1}{2p_1}},\dots,\zeta_d\Big(\frac{c_d(z)}{2p_d}\Big)^{\frac{1}{2p_d}},z_{d+1},\dots,z_n\Big)\in\mathbb{E}.$$

Applying the holomorphic contractivity of the generalized Green function and Lemma 2.1 proves that $m_{\mathbb{E}}(A,z)\le g_{\mathbb{E}}(A,z)\le R_{\mathbb{E}}(A,z)$.

It remains to prove that $m_{\mathbb{E}}(A,z)\ge R_{\mathbb{E}}(A,z)$ (resp. $g_{\mathbb{E}}(A,z)\ge R_{\mathbb{E}}(A,z)$). In the case $d=k=n$ it suffices to take

$$f(\zeta):=\prod_{j=1}^{d}\zeta_j\Big(\frac{2p_j}{c_d(z)}\Big)^{\frac{1}{2p_j}}.$$

Then $f\in\mathcal{O}(\mathbb{E},\mathbb{D})$ and $|f(z)|=R_{\mathbb{E}}(A,z)$.

It remains to prove Theorem 1.2 in the remaining cases. We may assume that $z_j\ne0$, $j=1,\dots,d$. Consider the following lemma.

Lemma 2.2. *Define* $\mathbb{E}':=\mathbb{E}_{p_{d+1},\dots,p_n}$.

(a) *Let* $v:\mathbb{E}'\longrightarrow[0,1)$, $v\not\equiv0$ *be such that* $\log v\in\mathcal{PSH}(\mathbb{E}')$ *and* $v(\zeta)\le|\zeta_j|$, $\zeta\in\mathbb{E}'$, $j=d+1,\dots,k$, *and the mapping*

$$\mathbb{E}'\ni(\zeta_{d+1},\dots,\zeta_n)\longmapsto v(\zeta_{d+1},\dots,\zeta_n)r_d^{q_d}(\zeta)$$

attains its maximum value M *at* $(z_{d+1},\dots,z_n)$. *Then* $g_{\mathbb{E}}(A,z)=R_{\mathbb{E}}(A,z)$.

(b) *The following conditions are equivalent:*

- *There exists an* $h\in\mathcal{O}(\mathbb{E}')$, $h\not\equiv0$ *such that* $h(\zeta)=0$ *for* $\zeta_{d+1}\cdots\zeta_k=0$ *and the mapping*

$$\mathbb{E}'\ni(\zeta_{d+1},\dots,\zeta_n)\longmapsto|h(\zeta_{d+1},\dots,\zeta_n)|r_d^{q_d}(\zeta)$$

attains its maximum value M *at the point* $(z_{d+1},\dots,z_n)$,
- $m_{\mathbb{E}}(A,z)=R_{\mathbb{E}}(A,z)$.

We present a proof in Section 3. The result above reduces the proof of Theorem 1.2 to the following propositions.

Proposition 2.3. *There exists a* v *as required in Lemma 2.2(a).*

Proposition 2.4. *Using the notation of Theorem 1.2, assume that* $p_j \geq 1/2$, $j = d+1, \dots, n$. *Then there exists an* $h \in \mathcal{O}(\mathbb{E}')$ *as required in Lemma 2.2(b).*

Proposition 2.5. *Using the notation of Theorem 1.2, assume that* $p_{k+1} < 1/2$. *Then one cannot find an* h *as required in Lemma 2.2(b) for* $|z_\ell| \neq 0$ *small enough,* $\ell = 1, \dots, k+1$ *and* $z_\ell = 0$, $\ell = k+2, \dots, n$.

Proposition 2.6. *Using the notation of Theorem 1.2, assume that* $n = k = 2$. *Additionally, assume that* $p_2 \geq 1/2$ *or* $8p_1 + 4p_2(1-p_2) > 1$. *Then there exists an* $h \in \mathcal{O}(\mathbb{D})$ *as required in Lemma 2.2(b).*

3. Proof of Lemma 2.2

(a) Put

$$u(\zeta_1, \dots, \zeta_n) := M^{-1}\Big(\prod_{j=1}^{d}(2p_j)^{\frac{1}{2p_j}}|\zeta_j|\Big)q_d^{q_d}v(\zeta_{d+1}, \dots, \zeta_n).$$

Obviously $\log u \in \mathcal{PSH}(\mathbb{E})$ and $u(\zeta) \leq C|\zeta_j| \leq C\|\zeta - a\|$, $\zeta \in \mathbb{E}$, whenever $a_j = 0$ for some $j \in \{1, \dots, k\}$. For $\zeta \in \mathbb{E}$ we have

$$u(\zeta) \leq M^{-1}\Big(\frac{\sum_{j=1}^{d}|\zeta_j|^{2p_j}}{q_d}\Big)^{q_d} q_d^{q_d}v(\zeta_{d+1}, \dots, \zeta_n) < 1.$$ [1]

Consequently $u : \mathbb{E} \longrightarrow [0,1)$. On the other hand,

$$u(z) = M^{-1}R_{\mathbb{E}}(A,z)r_d^{q_d}(z)v(z_{d+1}, \dots, z_n) = R_{\mathbb{E}}(A,z).$$

(b) Assume that such an h exists. Put

$$f(\zeta_1, \dots, \zeta_n) := M^{-1}\Big(\prod_{j=1}^{d}(2p_j)^{\frac{1}{2p_j}}\zeta_j\Big)q_d^{q_d}h(\zeta_{d+1}, \dots, \zeta_n).$$

Observe that $f(\zeta) = 0$ for $\zeta \in A$. Similarly as in (a) we prove that $|f(\zeta)| < 1$, $\zeta \in \mathbb{E}$ and $f(z) = R_{\mathbb{E}}(A,z)$.

Assume now that $m_{\mathbb{E}}(A,z) = R_{\mathbb{E}}(A,z)$. Let $f \in \mathcal{O}(\mathbb{E}, \mathbb{D})$ be such that $f|_A \equiv 0$ and $|f(z)| = R_{\mathbb{E}}(A,z)$ (cf. [Jar-Jar-Pfl 2003], Property 2.5). Put

$$h(\zeta) := \frac{\partial^d f}{\partial z_1 \dots \partial z_d}(0, \zeta).$$

[1] Let $a_1, \dots, a_d \geq 0$, $w_1, \dots, w_d > 0$. Then

$$\prod_{j=1}^{d} a_j^{w_j} \leq \left(\frac{\sum_{j=1}^{d} w_j a_j}{\sum_{j=1}^{d} w_j}\right)^{\sum_{j=1}^{d} w_j}.$$

By definition, we have $h(\zeta) = 0$ for $\zeta_{d+1}\cdots\zeta_k = 0$. Applying the Schwarz lemma to the mapping $f \circ \iota_\zeta$, $\zeta \in \mathbb{E}'$, we get

$$|h(\zeta_{d+1},\dots,\zeta_n)| r_d^{q_d}(\zeta) \le \Big(\prod_{j=1}^{d} (2p_j)^{\frac{1}{2p_j}}\Big) q_d^{q_d}, \quad \zeta \in \mathbb{E}',$$
$$|h(z_{d+1},\dots,z_n)| r_d^{q_d}(z) = \Big(\prod_{j=1}^{d} (2p_j)^{\frac{1}{2p_j}}\Big) q_d^{q_d}.$$

4. Proof of Propositions 2.3, 2.4, 2.5, and 2.6

Proof of Proposition 2.3. We may assume that $z_{d+1},\dots,z_n \ge 0$. Consider functions of the form

$$v_\alpha(\zeta_{d+1},\dots,\zeta_n) = \Big(\prod_{j=d+1}^{k} |\zeta_j|^{1+\alpha_j}\Big)\Big(\prod_{j=k+1}^{n} |\zeta_j|^{\alpha_j}\Big),$$

where $\alpha = (\alpha_{d+1},\dots,\alpha_n)$, $\alpha_{d+1},\dots,\alpha_n \ge 0$. Obviously $v : \mathbb{E}' \longrightarrow [0,1)$, $\log v \in \mathcal{PSH}(\mathbb{E}')$, and $v(\zeta) \le |\zeta_j| \le \|\zeta - a\|$, $\zeta \in \mathbb{E}'$, whenever $a_j = 0$ for some $j \in \{d+1,\dots,k\}$. Since $v_\alpha(\zeta_{d+1},\dots,\zeta_n) = v_\alpha(|\zeta_{d+1}|,\dots,|\zeta_n|)$, it is enough to find an α such that the function

$$\mathbb{E}' \cap \mathbb{R}_+^{n-d} \ni (t_{d+1},\dots,t_n) \longmapsto v_\alpha(t_{d+1},\dots,t_n) r_d^{q_d}(t)$$

attains its maximum at $(z_{d+1},\dots,z_n)$. Considering the partial (logarithmic) derivatives results in the following equations:

$$0 = 1+\alpha_j - 2p_j q_d \frac{t_j^{2p_j}}{r_d(t)}, \quad j = d+1,\dots,k,$$
$$0 = \alpha_j - 2p_j q_d \frac{t_j^{2p_j}}{r_d(t)}, \quad j = k+1,\dots,n.$$

These give formulas for $\alpha_{d+1},\dots,\alpha_n$ such that $(z_{d+1},\dots,z_n)$ is the common zero of the derivatives. To prove that there are no other points like this, consider a reformulation of the above equations:

$$r_d(t) = \frac{2p_j q_d t_j^{2p_j}}{1+\alpha_j}, \quad j = d+1,\dots,k,$$
$$r_d(t) = \frac{2p_j q_d t_j^{2p_j}}{\alpha_j}, \quad j = k+1,\dots,n.$$

The left side is decreasing in any of the variables, while the right sides are increasing. Thus, at most one common zero is allowed.

It remains to check whether $\alpha_j \ge 0$, $j = d+1,\dots,n$. Obviously, this is true for $j = k+1,\dots,n$ and in the remaining cases we have

$$\alpha_j = \frac{2p_j q_d t_j^{2p_j} - r_d(t)}{t_j r_d(t)} \ge 0,$$

since this is the way we have chosen d. □

Proof of Proposition 2.4. We may assume that $z_{d+1}, \dots, z_n \geq 0$. Consider functions of the form

$$h_\alpha(\zeta_{d+1}, \dots, \zeta_n) = \Big(\prod_{j=d+1}^{k} \zeta_j e^{\alpha_j \zeta_j} \Big) \Big(\prod_{j=k+1}^{n} e^{\alpha_j \zeta_j} \Big),$$

where $\alpha = (\alpha_{d+1}, \dots, \alpha_n)$, $\alpha_{d+1}, \dots, \alpha_n \geq 0$.

Since $|h_\alpha(\zeta_{d+1}, \dots, \zeta_n)| \leq h_\alpha(|\zeta_{d+1}|, \dots, |\zeta_n|)$, it is enough to find an α such that

$$\mathbb{E}' \cap \mathbb{R}_+^{n-d} \ni (t_{d+1}, \dots, t_n) \longmapsto h_\alpha(t_{d+1}, \dots, t_n) r_d^{q_d}(t)$$

attains its maximum at $(z_{d+1}, \dots, z_n)$. Considering the partial (logarithmic) derivatives results in the following equations:

$$0 = \frac{1}{t_j} + \alpha_j - 2p_j q_d \frac{t_j^{2p_j - 1}}{r_d(t)}, \quad j = d+1, \dots, k,$$
$$0 = \alpha_j - 2p_j q_d \frac{t_j^{2p_j - 1}}{r_d(t)}, \quad j = k+1, \dots, n.$$

We continue as in the proof of Proposition 2.3. □

Proof of Proposition 2.5. Consider the following two lemmas.

Lemma 4.1. *Assume that there exist* $0 < c < b < 1$ *such that the function*

$$\varphi : [0, b] \ni t \longmapsto (1 - t^p)^{-q}$$

is strictly concave and

$$\varphi(0) + \frac{b}{c}(\varphi(c) - \varphi(0)) > \varphi(b) + 2.$$

Let $f \in \mathcal{O}(\mathbb{D})$, $f \not\equiv 0$ *such that* $|f(\zeta)|/\varphi(|\zeta|)$ *attains its maximum at* w_0. *Then* $w_0 = 0$ *or* $|w_0| \geq c$.

Proof. Assume that $|w_0| \in (0, c)$. We may assume that $|f(w_0)| = \varphi(|w_0|)$. Consider the function

$$\psi : [0, b] \ni t \longmapsto |f(0)| + \frac{t}{|w_0|}|f(w_0) - f(0)|.$$

From $\psi(0) \leq \varphi(0)$, $\psi(|w_0|) \geq \varphi(|w_0|)$, and the convexity condition we get

$$\psi(b) = |f(0)| + \frac{b}{|w_0|}|f(w_0) - f(0)| \geq \varphi(0) + \frac{b}{|w_0|}|\varphi(|w_0|) - \varphi(0)|$$
$$\geq \varphi(0) + \frac{b}{c}|\varphi(c) - \varphi(0)| > \varphi(b) + 2.$$

The Schwarz lemma and the maximum principle imply that there exists a $w \in \mathbb{D}$ with $|w| = b$ and

$$\frac{|f(w) - f(0)|}{|w|} \geq \frac{|f(w_0) - f(0)|}{|w_0|}.$$

This means that

$$|f(w)| \geq |f(w) - f(0)| - |f(0)| = |f(0)| + |f(w) - f(0)| - 2|f(0)|$$
$$\geq \psi(b) - 2|f(0)| > \varphi(b) + 2 - 2|f(0)| \geq \varphi(b) = \varphi(|w|).$$

This contradicts the maximality of $|f(\zeta)|/\varphi(|\zeta|)$ at w_0. □

Lemma 4.2. *Assume that* $p \in (0,1)$, $q > 0$. *Then there exist* $b \in (0,1)$, $c \in (0,b)$ *as required in Lemma 4.1.*

Proof. We have

$$\varphi'(t) = \frac{pq}{t^{1-p}(1-t^p)^{q+1}},$$

which is decreasing for small t. To prove the existence of c observe that

$$\lim_{d \longrightarrow 0^+} \frac{\varphi(d) - \varphi(0)}{d} = +\infty.$$

□

Observe that $d = k$ for $|z_\ell|$ small enough, $\ell = 1, \dots, k$. Let $h \in \mathcal{O}(\mathbb{E}')$ and consider $f(\zeta) := h(\zeta, 0, \dots, 0)$, $\zeta \in \mathbb{D}$. Put $p := 2p_{k+1}$, $q = q_d$. Let b, c be as in Lemma 4.2. It follows from Lemma 4.1 that $0 \neq |w_0| < c$ cannot be a maximum of $\mathbb{D} \ni \zeta \longmapsto |f(\zeta)|/\varphi(|\zeta|)$. In particular, z cannot be a maximum of $\mathbb{E}' \ni \zeta \longmapsto |h(\zeta)| r_d^{q_d}(\zeta)$ for $0 \neq |z_{k+1}| < c$. □

Proof of Proposition 2.6. Consider the following lemma.

Lemma 4.3. *Let* $a, c > 0$, $t_0 \in (0,1)$ *be such that* $c \geq 1$ *or* $4a + 2c > 1 + c^2$, $t_0^c > \tau := a/(a+c)$. *Then there exist* $b > 0$ *and* $r \geq 1$ *such that*

$$h : [0,1] \ni t \longmapsto \frac{t^a}{(r-t)^b}(1-t^c) \in [0,1]$$

admits its maximum at t_0.

Proof. Comparing $\varphi := th'(t)/h(t)$ to zero we get the equation

$$\varphi(r,t) = a + \frac{bt}{r-t} - \frac{ct^c}{1-t^c} = 0.$$

This gives us a formula for r:

$$r(t) = \frac{t(b - bt^c - a + at^c + t^c c)}{-a + at^c + t^c c}.$$

Observe that

$$\lim_{t \longrightarrow \tau^{1/c}} r(t) = +\infty, \quad \lim_{t \longrightarrow 1} r(t) = 1.$$

In order to prove that $r(t) > 1$ and that $\varphi(r, \cdot)$ has only one zero it suffices to show that $r'(t) < 0$. We have

$$r'(t) = \frac{(-(a+c)t^{2c} + (2a + c - c^2)t^c - a)b + (at^c + t^c c - a)^2}{(at^c + t^c c - a)^2}.$$

It remains to show that the coefficient $\alpha(t^c)$ next to b is negative. We have

$$\alpha(\tau) = \frac{-ac^2}{a+c} < 0,$$
$$\alpha(1) = -c^2 < 0,$$
$$\alpha'(u) = -2(a+c)u + 2a + c - c^2.$$

Let u_0 be the zero of $\alpha'(u)$. For $c \geq 1$ we have $u_0 \leq \tau$ and we are done. Otherwise $u_0 \in (\tau, 1)$ and $4(a+c)\alpha(u_0) = c^2(1 + c^2 - 4a - 2c) < 0$. □

We may assume that $z_0 > 0$. Put $a = 2p_1$, $c = 2p_2$, $t_0 = z_0$. Let r be as in Lemma 4.3. Putting

$$h(\zeta) = \frac{\zeta}{(r-\zeta)^{b/a}}$$

completes the proof. □

Remark 4.4. The calculations in the proof of Proposition 2.6 can be performed using alternative function families, e.g.,

$$\begin{aligned} h_j(\zeta) &:= (\zeta + r)^k, \quad r \geq 0,\ k = 0, 1, 2, \ldots, \\ h_j(\zeta) &:= (\zeta + 1)^\alpha, \quad \alpha > 0, \\ h_j(\zeta) &:= (r - \zeta)^\alpha \zeta^k, \quad \alpha < 0, r \geq 1, k = 0, 1, 2, \ldots, \\ h_j(\zeta) &:= \Big(\frac{\zeta + \delta}{1 + \delta\zeta}\Big)\zeta^k, \quad \delta \in [0, 1],\ k = 0, 1, 2, \ldots. \end{aligned}$$

However, the author was unable to solve the general case using any of them.

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Jagiellonian University, Institute of Mathematics, Reymonta 4, 30-059 Kraków, Poland

E-mail address: wmj@im.uj.edu.pl

Current address: Universität Osnabrück, Fachbereich Mathematik/Informatik, Albrechtstraße 28, 49069 Osnabrück, Germany