

**BOUNDARY REGULARITY IN THE DIRICHLET PROBLEM
 FOR THE INVARIANT LAPLACIANS Δ_γ
 ON THE UNIT REAL BALL**

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(Communicated by Mei-Chi Shaw)

ABSTRACT. We study the boundary regularity in the Dirichlet problem of the differential operators

$$\Delta_\gamma = (1 - |x|^2) \left\{ \frac{1 - |x|^2}{4} \sum_j \frac{\partial^2}{\partial x_j^2} + \gamma \sum_j x_j \frac{\partial}{\partial x_j} + \gamma \left(\frac{n}{2} - 1 - \gamma \right) \right\}.$$

Our main result is: if $\gamma > -1/2$ is neither an integer nor a half-integer not less than $n/2 - 1$, one cannot expect global smooth solutions of $\Delta_\gamma u = 0$; if $u \in C^\infty(\overline{B}_n)$ satisfies $\Delta_\gamma u = 0$, then u must be either a polynomial of degree at most $2\gamma + 2 - n$ or a polyharmonic function of degree $\gamma + 1$.

1. INTRODUCTION

Let B_n be the open unit ball in \mathbb{R}^n ($n \geq 2$) and S^{n-1} the unit sphere. We consider the differential operators

$$(1.1) \quad \Delta_\gamma = (1 - |x|^2) \left\{ \frac{1 - |x|^2}{4} \sum_j \frac{\partial^2}{\partial x_j^2} + \gamma \sum_j x_j \frac{\partial}{\partial x_j} + \gamma \left(\frac{n}{2} - 1 - \gamma \right) \right\},$$

where $\gamma \in \mathbb{R}$. If $\gamma = n/2 - 1$, $\Delta_{n/2-1}$ is the Laplace-Beltrami operator with respect to the Poincaré metric on B_n . For general γ , it can be shown that

$$(1.2) \quad \Delta_\gamma \left\{ (\det \psi'(x))^{\frac{n-2-2\gamma}{2n}} u(\psi(x)) \right\} = (\det \psi'(x))^{\frac{n-2-2\gamma}{2n}} (\Delta_\gamma u)(\psi(x))$$

for every $u \in C^2(B_n)$ and for every $\psi \in \mathcal{M}(B_n)$. Here $\mathcal{M}(B_n)$ is the group of those Möbius transformations that leave the unit ball B_n invariant, and $\psi'(x)$ is the Jacobian matrix of $\psi \in \mathcal{M}(B_n)$. This is the reason why we call Δ_γ the invariant Laplacians. Actually, from a harmonic analysis perspective, the operators Δ_γ can be considered as the Casimir operators with respect to the following unitary representation of the Lorentz group $SO(n, 1)$ on $L^2(B_n, \mu_\gamma)$:

$$(1.3) \quad (T^\gamma(g)f)(x) = \left\{ \det(g^{-1})'(x) \right\}^{\frac{n-2-2\gamma}{2n}} f(g^{-1} \cdot x),$$

Received by the editors July 4, 2003.

2000 *Mathematics Subject Classification*. Primary 35J25, 32W50; Secondary 35C10, 35C15.

Key words and phrases. Invariant Laplacians, Laplace-Beltrami operator, Weinstein equation, boundary regularity, polyharmonicity.

This research was supported by 973 project of China grant G1999075105.

where μ_γ is the weighted measure on B_n given by $d\mu_\gamma(x) = (1 - |x|^2)^{-2\gamma-2}dx$. The details of these matters will be included in a forthcoming paper by the present authors [14].

These operators also appear in a natural way when we transplant the Weinstein equation

$$(1.4) \quad L_k[u] \equiv \sum_j \frac{\partial^2 u}{\partial x_j^2} + \frac{k}{x_n} \frac{\partial u}{\partial x_n} = 0$$

(considered in the upper half-space $\mathbb{R}_+^n = \{(x_1, \dots, x_n) : x_n > 0\}$) to the unit ball, via a standard Möbius transformation (see [12, 3]). More precisely, we have

$$(1.5) \quad (\Delta_\gamma u)(y) = |x + e_n|^{n-2-2\gamma} \tilde{L}_{-2\gamma} (|x + e_n|^{2\gamma-n+2} u(\psi(x)))$$

for all $u \in C^2(B_n)$, where e_n is the last coordinate vector, $\tilde{L}_k = x_n^2 L_k$, and where the Möbius transformation $\psi : \mathbb{R}_+^n \rightarrow B_n$, $x \mapsto y$ is defined by

$$y_i = \frac{2x_i}{|x + e_n|^2} \quad (i = 1, \dots, n-1), \quad y_n = \frac{|x|^2 - 1}{|x + e_n|^2}.$$

The complex-ball counterparts of these operators, the Laplacians

$$\Delta_{\alpha,\beta} = 4(1 - |z|^2) \left\{ \sum_{i,j} (\delta_{ij} - z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j} + \alpha \sum_j z_j \frac{\partial}{\partial z_j} + \beta \sum_j \bar{z}_j \frac{\partial}{\partial \bar{z}_j} - \alpha\beta \right\},$$

have been considered by many authors; see, e.g., [8, 1, 5]. Among other important results, it was proved by Graham in [9] that, despite the fact that the Dirichlet problem for $\Delta_{0,0}$ is solvable for arbitrary continuous boundary data, in order that the solution is infinitely differentiable up to the boundary it is necessary and sufficient that the C^∞ data be the boundary value of a pluriharmonic function (the real part of a holomorphic function). In particular, his proof in fact showed a striking phenomenon (Corollary 1.5 in [9]) that if an M -harmonic function u (i.e., $\Delta_{0,0}u = 0$) is n -times continuously differentiable up to the boundary, then u must be pluriharmonic. The statement and the proof of this theorem can also be found in Theorem 6.8.12 in [11]. Recently this phenomenon in the polydisc setting was investigated by S-Y. Li and E. Simon, and it was proved that any harmonic functions in Bergman-type metrics in the polydisc that are continuous up to the boundary must be harmonic in each complex variable.

In this paper we will deal with the analogous question in the context of the unit ball of \mathbb{R}^n and for the Laplacians Δ_γ . Interestingly, the boundary regularity in the Dirichlet problem for Δ_γ depends on the value of the parameter γ , and we characterize the global smooth solutions of $\Delta_\gamma u = 0$ as either polynomials of degree at most $2\gamma + 2 - n$ or polyharmonic functions of degree $\gamma + 1$. Here we recall that a function $u \in C^{2k}(\Omega)$ is called polyharmonic of finite degree k in the open set $\Omega \subset \mathbb{R}^n$ if $\Delta^k u(x) = 0$ for all $x \in \Omega$. Here Δ is the Euclidean Laplacian in \mathbb{R}^n given by $\sum_j \partial^2 / \partial x_j^2$, and Δ^k is defined inductively for integers $k \geq 0$ by $\Delta^0 = \text{id}$, $\Delta^1 = \Delta$, $\Delta^k = \Delta \Delta^{k-1}$.

More precisely, our main result states:

Theorem 1.1. *Let $f \in C^\infty(S^{n-1})$, $f \neq 0$. Then the solution u to the Dirichlet problem*

$$(1.6) \quad \begin{cases} \Delta_\gamma u = 0 & \text{in } B_n, \\ u = f & \text{on } S^{n-1} \end{cases}$$

is in $C^\infty(\overline{B}_n)$ if and only if one of the following occurs:

- (H1) γ is a nonnegative integer;
- (H2) the data f has a finite spherical harmonic expansion

$$f = \sum_{k=0}^N Y_k, \quad Y_k \in \mathcal{H}_k,$$

and $\gamma + 1 - n/2 - N$ is a nonnegative integer. Here N is the greatest index k such that Y_k is not identically zero on S^{n-1} , and \mathcal{H}_k is the space of spherical harmonics of degree k .

Moreover, in the case when (H1) is fulfilled, the solution u is polyharmonic of degree $\gamma + 1$ in B_n , whereas when (H2) is fulfilled, the solution is a polynomial of degree at most $2\gamma + 2 - n$.

2. PRELIMINARIES

A number of hypergeometric functions will appear throughout. We use the classical notation ${}_2F_1(a, b; c; z)$ to denote

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

with $c \neq 0, -1, -2, \dots$, and where $(a)_0 = 1$, $(a)_k = a(a+1) \cdots (a+k-1)$ for $k \geq 1$. We refer to [6] for the theory of these functions.

Although the following formula is probably well known, we were unable to locate a proof in the literature. Thus we have included a proof.

Lemma 2.1. *For $x \in B_n$ and $\lambda \in \mathbb{C}$, we have*

$$(2.1) \quad \int_{S^{n-1}} \frac{d\sigma(\zeta)}{|x - \zeta|^{2\lambda}} = {}_2F_1\left(\lambda, \lambda - \frac{n}{2} + 1; \frac{n}{2}; |x|^2\right),$$

where σ is the surface measure on S^{n-1} normalized so that $\sigma(S^{n-1}) = 1$.

Proof. Without loss of generality, we assume that $x = -|x|e_1$. In the spherical coordinates,

$$\begin{aligned} \zeta_j &= \sin \theta_1 \cdots \sin \theta_{j-1} \cos \theta_j, \quad 1 \leq j \leq n-1, \\ \zeta_n &= \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1}, \end{aligned}$$

where θ_j are subject to the conditions

$$0 \leq \theta_j \leq \pi, \quad 1 \leq j \leq n-2; \quad 0 \leq \theta_{n-1} < 2\pi,$$

the integral in (2.1) is equal to

$$\begin{aligned} \int_{S^{n-1}} \frac{d\sigma(\zeta)}{(1 - 2x \cdot \zeta + |x|^2)^\lambda} &= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n-1}{2})} \int_0^\pi \frac{\sin^{n-2} \theta_1}{(1 + 2|x| \cos \theta_1 + |x|^2)^\lambda} d\theta_1 \\ &= {}_2F_1\left(\lambda, \lambda - \frac{n}{2} + 1; \frac{n}{2}; |x|^2\right), \end{aligned}$$

where we have used the formula ([15], p. 55)

$$(2.2) \quad {}_2F_1\left(a, a - b + \frac{1}{2}; b + \frac{1}{2}; z^2\right) = \frac{\Gamma(b + \frac{1}{2})}{\sqrt{\pi} \Gamma(b)} \int_0^\pi \frac{(\sin t)^{2b-1}}{(1 + 2z \cos t + z^2)^a} dt, \quad \square$$

(Re b > 0, |z| < 1).

The following simple fact explains why polyharmonicity is involved in our result.

Proposition 2.2. *If $\Delta_\gamma u = 0$, then $\Delta_{\gamma-1}(\Delta u) = 0$.*

Proof. A straightforward computation yields that

$$\Delta_{\gamma-1}(\Delta u) = \Delta(\Delta_\gamma u) + \sum_j x_j \frac{\partial}{\partial x_j} \left(\frac{4}{1 - |x|^2} \Delta_\gamma u \right) + \frac{2n}{1 - |x|^2} \Delta_\gamma u,$$

and the proposition follows immediately. □

Remark 1. This proposition implies, in particular, that if γ is a nonnegative integer, any solution of $\Delta_\gamma u = 0$ is polyharmonic of degree $\gamma + 1$ in B_n . As an immediate consequence, we would like to mention a well-known fact: in even dimension, every harmonic function in the Poincaré metric on B_n (or hyperbolically harmonic function) is polyharmonic of degree $n/2$.

From now on, for $x \in \mathbb{R}^n$ we shall always write $x = rx'$ with $r = |x|$ and $|x'| = 1$.

Theorem 2.3. *Let u be a C^2 function in B_n satisfying $\Delta_\gamma u = 0$. Then*

$$(2.3) \quad u(x) = \sum_{k=0}^\infty \Phi_k^\gamma(|x|^2) |x|^k u_k(x'),$$

where Φ_k^γ is the hypergeometric function given by

$$\Phi_k^\gamma(z) = {}_2F_1\left(-\gamma, k + \frac{n}{2} - 1 - \gamma; k + \frac{n}{2}; z\right),$$

$u_k \in \mathcal{H}_k$, and where the series converges uniformly and absolutely on compact sets in B_n .

Proof. The proof of the theorem is much like that of Theorem 2.1 in [1], and we will only sketch it.

For each $0 < r < 1$ the L^2 -decomposition in spherical harmonics of $u_r(\zeta) = u(r\zeta)$ gives that

$$u(r\zeta) = \sum_{k=0}^\infty \int_{S^{n-1}} u(r\eta) Z_k(\zeta, \eta) d\sigma(\eta),$$

where $Z_k(\cdot, \eta)$ is the zonal harmonic of degree k with pole at η . Let $\zeta \in S^{n-1}$ and $k \in \mathbb{Z}^+$ be fixed, and for $0 < r < 1$, let

$$f_k^\zeta(r) = \int_{S^{n-1}} u(r\eta) Z_k(\zeta, \eta) d\sigma(\eta).$$

In polar coordinates, the operator Δ_γ has the form

$$\Delta_\gamma = \frac{(1-r^2)^{2+2\gamma}}{4r^{n-1}} \frac{\partial}{\partial r} \left(\frac{r^{n-1}}{(1-r^2)^{2\gamma}} \frac{\partial}{\partial r} \right) + \gamma \left(\frac{n}{2} - 1 - \gamma \right) (1-r^2) + \frac{(1-r^2)^2}{4r^2} \mathcal{L}_S,$$

where \mathcal{L}_S is the Laplacian on S^{n-1} . Since $\Delta_\gamma u = 0$, the above expression gives

$$\begin{aligned} & \frac{(1-r^2)^{2\gamma}}{r^{n-3}} \frac{\partial}{\partial r} \left(\frac{r^{n-1}}{(1-r^2)^{2\gamma}} \frac{\partial f_k^\zeta}{\partial r} \right) + \frac{2\gamma(n-2-2\gamma)r^2}{1-r^2} f_k^\zeta \\ &= - \int_{S^{n-1}} \mathcal{L}_S u(r\eta) Z_k(\zeta, \eta) d\sigma(\eta) = - \int_{S^{n-1}} u(r\eta) \mathcal{L}_S Z_k(\zeta, \eta) d\sigma(\eta) \\ &= k(k+n-2) f_k^\zeta, \end{aligned}$$

where in the second equality we have used that \mathcal{L}_S is a selfadjoint operator and in the last that $\mathcal{L}_S Y_k = -k(k+n-2)Y_k$ for $Y_k \in \mathcal{H}_k$ (see [16, p. 70]).

Let $f_k^\zeta(r) = r^k \varphi(r^2)$. Then φ satisfies

$$z(1-z)\varphi''(z) + \left\{ k + \frac{n}{2} - \left(k + \frac{n}{2} - 2\gamma \right) z \right\} \varphi'(z) + \gamma \left(k + \frac{n}{2} - 1 - \gamma \right) \varphi(z) = 0.$$

But this is just a hypergeometric equation, and the only solutions smooth at 0 are multiples of ${}_2F_1(-\gamma, k+n/2-1-\gamma; k+n/2; z)$. Hence,

$$f_k^\zeta(r) = C_k(\zeta) r^k {}_2F_1 \left(-\gamma, k + \frac{n}{2} - 1 - \gamma; k + \frac{n}{2}; r^2 \right),$$

for some constant $C_k(\zeta)$. This expression, together with the definition of f_k^ζ , gives that for each fixed $0 < r < 1$, the function $\zeta \mapsto f_k^\zeta$ is a spherical harmonic of degree k , and consequently, that there exists $u_k \in \mathcal{H}_k$ so that $C_k(\zeta) = u_k(\zeta)$. Thus,

$$u(x) = \sum_{k=0}^{\infty} \Phi_k^\gamma(|x|^2) |x|^k u_k(x').$$

Since u is regular, each term in the above expansion satisfies an adequate estimate on compact sets of B_n that assures the absolute and uniform convergence of the series (2.3). □

Let us look in detail at the solvability of the Dirichlet problem (1.6) for Δ_γ .

Theorem 2.4. *The Dirichlet problem (1.6) has a solution for all $f \in C(S^{n-1})$ if and only if $\gamma > -1/2$. In this case the solution is unique and is given by*

$$(2.4) \quad u(x) = \int_{S^{n-1}} \mathcal{P}_\gamma(x, \zeta) f(\zeta) d\sigma(\zeta) = \mathcal{P}_\gamma[f](x),$$

with

$$(2.5) \quad \mathcal{P}_\gamma(x, \zeta) = c_{n,\gamma} \frac{(1-|x|^2)^{1+2\gamma}}{|x-\zeta|^{n+2\gamma}}, \quad c_{n,\gamma} = \frac{\Gamma(\frac{n}{2} + \gamma)\Gamma(1 + \gamma)}{\Gamma(\frac{n}{2})\Gamma(1 + 2\gamma)}$$

or, alternatively, by

$$(2.6) \quad u(x) = \sum_{k=0}^{\infty} \frac{\Phi_k^\gamma(|x|^2)}{\Phi_k^\gamma(1)} |x|^k Y_k(x')$$

if $f = \sum_k Y_k$ is the spherical harmonic expansion of f .

Proof. Suppose that the Dirichlet problem (1.6) has a solution for all $f \in C(S^{n-1})$. Take $Y_k \in \mathcal{H}_k, Y_k \neq 0$, and let u be a solution to the Dirichlet problem. By Theorem 2.3,

$$u(r\zeta) = \sum_j \Phi_j^\gamma(r^2)r^j u_j(\zeta),$$

and hence

$$\int_{S^{n-1}} u(r\zeta)\overline{Y_k(\zeta)}d\sigma(\zeta) = {}_2F_1\left(-\gamma, k + \frac{n}{2} - 1 - \gamma; k + \frac{n}{2}; r^2\right) r^k \langle u_k, Y_k \rangle.$$

Since the left-hand side has limit $\|Y_k\|_2^2$, it follows that

$$\lim_{r \rightarrow 1} {}_2F_1\left(-\gamma, k + \frac{n}{2} - 1 - \gamma; k + \frac{n}{2}; r^2\right)$$

exists and is not zero. From [6] we know that if $\text{Re}(c-a-b) \leq 0$, the hypergeometric function ${}_2F_1(a, b; c; z)$ has a limit at 1 only if a or b is a nonpositive integer. Taking k large enough it follows that we must have $\gamma > -1/2$.

Again by Theorem 2.3,

$$\Phi_k^\gamma(r^2)r^k u_k(\zeta) = \int_{S^{n-1}} u(r\eta)Z_k(\zeta, \eta)d\sigma(\eta),$$

where $Z_k(\cdot, \eta)$ is the zonal harmonic of degree k with pole at η . Letting $r \rightarrow 1$ we see that $\Phi_k^\gamma(1)u_k = Y_k$, which shows unicity and establishes formula (2.6).

To show formula (2.4), one can argue as follows. By direct differentiation, one first shows that $\Delta_\gamma \mathcal{P}_\gamma[f] = 0$. It is also clear that for any $\eta \in S^{n-1}$ and any neighborhood V of η in S^{n-1} ,

$$\lim_{r \rightarrow 1} \int_{S^{n-1} \setminus V} \mathcal{P}_\gamma(r\eta, \zeta)d\sigma(\zeta) = 0.$$

By Lemma 2.1 and the formula

$$(2.7) \quad {}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z),$$

one can show that

$$(2.8) \quad \int_{S^{n-1}} \mathcal{P}_\gamma(x, \zeta)d\sigma(\zeta) = c_{n,\gamma} {}_2F_1\left(-\gamma, \frac{n}{2} - 1 - \gamma; \frac{n}{2}; |x|^2\right) = \frac{\Phi_0^\gamma(|x|^2)}{\Phi_0^\gamma(1)}$$

for all $x \in B$. It then follows that $\lim_{x \rightarrow \zeta} \mathcal{P}_\gamma(x, \cdot) = \delta_\zeta$ weakly as measures, where δ_ζ is the Dirac mass at ζ . Therefore, for $f \in C(S^{n-1})$ the integral $\mathcal{P}_\gamma[f]$ is continuous on $\overline{B_n}$ and solves the Dirichlet problem (1.6). \square

3. PROOF OF THEOREM 1.1

We start with two simple examples to illustrate the flavor of our result.

Example 1. Let $n = 3$ and $f \equiv 1$. We have

$$(3.1) \quad \begin{aligned} \mathcal{P}_\gamma[f](x) &= \frac{\Gamma(\frac{3}{2} + \gamma)\Gamma(1 + \gamma)}{\Gamma(\frac{3}{2})\Gamma(1 + 2\gamma)} {}_2F_1\left(-\gamma, \frac{1}{2} - \gamma; \frac{3}{2}; |x|^2\right) \\ &= \frac{2^{-2\gamma-1}(1 + |x|)^{1+2\gamma} - (1 - |x|)^{1+2\gamma}}{|x|}, \end{aligned}$$

where in the last equality we have used the formula ([15], p. 39)

$$(1 + z)^{1-2a} - (1 - z)^{1-2a} = 2z(1 - 2a) {}_2F_1\left(a, a + \frac{1}{2}; \frac{3}{2}; z^2\right).$$

Thus, it is easy to see that $\mathcal{P}_\gamma[f]$ is $C^{2+[2\gamma]}$ up to the boundary if and only if $1 + 2\gamma$ is a nonnegative integer, where $[a]$ denotes the greatest integer that does not exceed a . If this is the case, it is clear that the solution is a polynomial of degree $2\gamma - 1$.

Example 2. Let $n = 2$. Define $f(\zeta_1, \zeta_2) = \zeta_2^2$. Of course $f \in C^\infty(S^{n-1})$. Note that

$$\begin{aligned} \mathcal{P}_\gamma[f](x_1, 0) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - x_1^2)^{1+2\gamma}}{(1 - 2x_1 \cos \theta + x_1^2)^{1+\gamma}} \sin^2 \theta d\theta \\ &= \frac{(1 - x_1^2)^2}{2} {}_2F_1(1 - \gamma, 2 - \gamma; 2; x_1^2), \end{aligned}$$

where we have used the formulas (2.2) and (2.7). For any nonnegative integer m ,

$$\begin{aligned} \frac{d^m}{dr^m} {}_2F_1(1 - \gamma, 2 - \gamma; 2; r) \\ = \frac{(1 - \gamma)_m(2 - \gamma)_m}{(m + 1)!} {}_2F_1(m + 1 - \gamma, m + 2 - \gamma; m + 2; r), \end{aligned}$$

and the hypergeometric function in the right-hand side behaves like $(1 - r)^{2\gamma-1-m}$ near $r = 1$, whenever γ is not a nonnegative integer and $m > 2\gamma - 1$. Hence we cannot expect $\mathcal{P}_\gamma[f]$ to be $C^{2+[2\gamma]}$ up to the boundary if γ is not a nonnegative integer.

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Supposing that the solution u is even in $C^{2+[2\gamma]}(\overline{B}_n)$, let us show that either (H1) or (H2) occurs.

We write the spherical harmonic expansion

$$(3.2) \quad f = \sum_{k=0}^{\infty} Y_k, \quad Y_k \in \mathcal{H}_k.$$

Theorem 2.4 shows that the solution u is given by

$$(3.3) \quad u(r\zeta) = \sum_{k=0}^{\infty} \frac{\Phi_k^\gamma(r^2)}{\Phi_k^\gamma(1)} r^k Y_k(\zeta).$$

Define for each k the function

$$(3.4) \quad Q_k(r) = \int_{S^{n-1}} u(r\zeta) \overline{Y_k(\zeta)} d\sigma(\zeta) = \frac{\Phi_k^\gamma(r^2)}{\Phi_k^\gamma(1)} r^k \|Y_k\|_2^2.$$

Since u is $C^{2+[2\gamma]}$ up to the boundary, by differentiation under the integral sign, Q_k is $C^{2+[2\gamma]}$ up to $r = 1$.

On the other hand, we claim that: if either

- (a) there are infinitely many Y_k in (3.2) that are not identically zero on S^{n-1} , and γ is not a nonnegative integer;

or

- (b) $f = \sum_{k=0}^N Y_k$, and neither γ nor $\gamma + 1 - n/2 - N$ is a nonnegative integer, then Q_k is not $C^{2+[2\gamma]}$ at $r = 1$. This contradiction proves the “only if” part of the theorem.

We first treat the case (a). By the assumption, we can choose k so large that $k + n/2 - 1 - \gamma > 0$ and Y_k is not identically zero on S^{n-1} . Note that the m^{th} derivative of Q_k is

$$(3.5) \quad Q_k^{(m)}(r) = \frac{\|Y_k\|_2^2}{\Phi_k^\gamma(1)} \sum_{j=\lceil \frac{m+1}{2} \rceil}^\infty \frac{(-\gamma)_j (k + \frac{n}{2} - 1 - \gamma)_j (2j + k - m + 1)_m}{(k + \frac{n}{2})_j j!} r^{2j+k-m},$$

in which $(-\gamma)_j \neq 0$ and $(k + n/2 - 1 - \gamma)_j \neq 0$ for all j . The coefficients in this series are of order $j^{m-2-2\gamma}$, as $j \rightarrow \infty$. Thus the behavior of $Q_k^{(m)}(r)$ at $r = 1$ is like $(1 - r)^{2\gamma+1-m}$ whenever $m > 2\gamma + 1$. In particular, we conclude that Q_k is not $C^{\lfloor 2\gamma \rfloor + 2}$ at $r = 1$.

For the case (b), let us take $k = N$ in (3.5). The assumption guarantees that $(-\gamma)_j \neq 0$ and $(N + n/2 - 1 - \gamma)_j \neq 0$ for all j . The remainder of the proof is just the same as that in the case (a).

For the converse direction, we deal only with the case when (H1) is fulfilled. The other one is even simpler. Recall that if $f \in C^\infty(S^{n-1})$ has the expansion (3.2), then

$$(3.6) \quad \int_{S^{n-1}} |Y_k(\zeta)|^2 d\sigma(\zeta) = O(k^{-N'}) \quad \text{as } k \rightarrow \infty, \text{ for each fixed } N'.$$

(See [16, p. 70, 3.1.5].) When γ is a nonnegative integer, the functions $\Phi_k^\gamma(|x|^2)$ occurring in (2.6) reduce to polynomials of degree 2γ . More precisely,

$$(3.7) \quad \Phi_k^\gamma(|x|^2) = \sum_{j=0}^\gamma \frac{(-\gamma)_j (k + \frac{n}{2} - 1 - \gamma)_j}{(k + \frac{n}{2})_j} \frac{|x|^{2j}}{j!}.$$

Note also that

$$(3.8) \quad 0 < \frac{(k + \frac{n}{2} - 1 - \gamma)_j}{(k + \frac{n}{2})_j} < 1, \quad \text{for all } k > \gamma + 1 - n/2.$$

Thus, for each fixed multi-index α , we have

$$(3.9) \quad \sup_{|x| \leq 1} \left| \frac{\partial^\alpha}{\partial x^\alpha} (\Phi_k^\gamma(|x|^2)) \right| \leq C_{\alpha, \gamma},$$

where $C_{\alpha, \gamma}$ is a positive constant depending only on α and γ . Together with the fact that

$$(3.10) \quad \frac{1}{\Phi_k^\gamma(1)} = \frac{\Gamma(k + \frac{n}{2} + \gamma)\Gamma(1 + \gamma)}{\Gamma(k + \frac{n}{2})\Gamma(1 + 2\gamma)} \leq C_\gamma k^\gamma$$

for some constant $C_\gamma > 0$, this gives

$$(3.11) \quad \sup_{|x| \leq 1} \left| \frac{\partial^\alpha}{\partial x^\alpha} \left(\frac{\Phi_k^\gamma(|x|^2)}{\Phi_k^\gamma(1)} \right) \right| \leq C_{\alpha, \gamma} k^\gamma.$$

If we set $P_k(x) = |x|^k Y_k(x')$, then P_k is a solid harmonic of degree k . It was shown in [16, p. 276, Appendix C] that

$$(3.12) \quad \sup_{|x| \leq 1} \left| \frac{\partial^\alpha}{\partial x^\alpha} P_k(x) \right| \leq A_\alpha k^{n/2 + |\alpha|} \left(\int_{S^{n-1}} |Y_k(x')|^2 d\sigma(x') \right)^{1/2}.$$

Putting all this together, we conclude that each term in the series

$$\sum_{k=0}^{\infty} \frac{\partial^\alpha}{\partial x^\alpha} \left(\frac{\Phi_k^\gamma(|x|^2)}{\Phi_k^\gamma(1)} |x|^k Y_k(x') \right)$$

satisfies an adequate estimate on \overline{B}_n that assures the absolute and uniform convergence of the series. Therefore, u is indefinitely differentiable on \overline{B}_n .

For the second assertion of the theorem, we first observe that in the case when (H1) is fulfilled, this is an immediate consequence of Proposition 2.2. When (H2) is fulfilled, there are only finitely many nonzero terms in the series (2.6) and each hypergeometric function Φ_k^γ reduces to a polynomial of degree $\gamma + 1 - n/2 - k$. Hence, the k^{th} term of the series (2.6) is a polynomial of degree $2\gamma + 2 - n - k$ in x . This completes the proof. \square

Our proof in fact shows

Corollary 3.1. *If $\gamma > -1/2$ is neither an integer nor a half-integer not less than $n/2 - 1$, then one cannot expect $C^{2+[2\gamma]}$ up to the boundary solutions of the Dirichlet problem (1.6). Moreover, if $u \in C^{2+[2\gamma]}(\overline{B}_n)$ satisfies $\Delta_\gamma u = 0$, then u is either a polynomial of degree at most $2\gamma + 2 - n$ or a polyharmonic function of degree $\gamma + 1$.*

Remark 2. A special case of our result is probably known. Let us consider the boundary regularity for the Laplace-Beltrami operator on B_n . This corresponds to the special case of our result when $\gamma = n/2 - 1$, and one can conclude that, in even dimension, the situation is much like the uniformly elliptic case: if the data is C^∞ , then the solution is C^∞ up to the boundary; whereas in odd dimension, if a solution u is C^n up to the closure, then u must be a constant.

ACKNOWLEDGEMENT

Most of the results in this paper are part of the first author's doctoral dissertation under the direction of Professor Jihuai Shi, to whom the first author wishes to express his sincere gratitude. The authors also want to thank Professor Song-Ying Li for valuable comments.

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