INFINITE SYSTEMS OF LINEAR EQUATIONS FOR REAL ANALYTIC FUNCTIONS

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Abstract. We study the problem when an infinite system of linear functional equations
\[ \mu_n(f) = b_n \quad \text{for } n \in \mathbb{N} \]
has a real analytic solution \( f \) on \( \mathbb{R}^d \) for every right-hand side \( (b_n)_{n \in \mathbb{N}} \subseteq \mathbb{C} \) and give a complete characterization of such sequences of analytic functionals \( (\mu_n) \). We also show that every open set \( \mathbb{R}^d \) has a complex neighbourhood \( C^d \) such that the positive answer is equivalent to the positive answer for the analogous question with solutions holomorphic on \( \mathbb{R}^d \).

INTRODUCTION

One of the first classical examples of an infinite system of linear equations is related to the so-called moment problem solved by Hausdorff. It is the question of finding a Borel measure \( \mu \) on \([0, 1]\) such that a given sequence of reals \( (b_n)_{n \in \mathbb{N}} \) is the sequence of moments of \( \mu \) (i.e., \( \int_0^1 t^n \, d\mu(t) = b_n \) for \( n \in \mathbb{N} \)). More generally, we can look for a functional \( f \in X' \) such that for the given sequence of scalars \( (b_n) \) and of vectors \( (x_n) \) belonging to a locally convex (Banach) space \( X \) the following holds:

\[ f(x_n) = b_n \quad \text{for } n \in \mathbb{N}. \]

First solutions for spaces \( X = C[0, 1] \) or \( L^p[0, 1] \) are due to F. Riesz (1909) and the functional analytic approach to the problem is contained in the famous book of Banach [1 Ch. IV §7, §8]; see also [23] pp. 106-107.

Later on, Eidelheit, a colleague of Banach, characterized in all Fréchet spaces what are now called Eidelheit sequences [9] (see [17 Th. 26.27], [12 II.38.6]). Let \( X \) be a locally convex space. We call a sequence \( (\mu_n)_{n \in \mathbb{N}} \) of continuous linear functionals on \( X \) an Eidelheit sequence on \( X \) if for every sequence of scalars \( (b_n)_{n \in \mathbb{N}} \) there is \( f \in X \) satisfying

\[ \mu_n(f) = b_n \quad \text{for } n \in \mathbb{N}. \]

The notion of Eidelheit sequences has been extensively studied in Fréchet spaces later on; see [9, 18, 20, 21] and [22], compare also [17 §26].

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We will prove a completely analogous result for the highly non-metrizable space of real analytic functions $\mathcal{A}(\omega)$ on an arbitrary open set $\omega \subseteq \mathbb{R}^d$ (Theorem 2.2). We show that for every open set $\omega \subseteq \mathbb{R}^d$ there is a domain of holomorphy $\Omega \subseteq \mathbb{C}^d$, $\Omega \cap \mathbb{R}^d = \omega$, such that every Eidelheit sequence on $\mathcal{A}(\omega)$ is automatically Eidelheit on the space $H(\Omega)$ of holomorphic functions on $\Omega$ (Theorem 2.2). The main tool in the proof is given by the so-called distinguished sets, which might be of independent interest (see Lemma 1.1 and Lemma 1.3).

Let $(f_{1,n})_n \in \mathbb{N}$, $(f_{2,n})_n \in \mathbb{N}$ be two Eidelheit sequences on $\mathcal{A}(\omega_1)$ and $\mathcal{A}(\omega_2)$, respectively. We consider the question if there is a continuous linear map (= operator) $T : \mathcal{A}(\omega_1) \to \mathcal{A}(\omega_2)$ such that $f_{1,n} = f_{2,n} \circ T$ for every $n \in \mathbb{N}$ (compare an analogous problem on Fréchet spaces due to Mityagin [18] and its solution in [21]). It turns out that for every $(f_{1,n})_n \in \mathbb{N}$ there is $(f_{2,n})_n \in \mathbb{N}$ without such factorization (Proposition 3.1). On the other hand, we show a positive result for sequences related to the interpolation problem for analytic functions (Theorem 3.2).

Let us recall that the space $\mathcal{A}(\omega)$, $\omega \subseteq \mathbb{R}^d$ an open subset, of real analytic functions $f : \omega \to \mathbb{C}$ is equipped with the topology of the projective limit $\text{proj}_N H(K_N)$, where $(K_N)$ is an exhaustion of $\omega$ by a sequence of compact sets

$$K_1 \subseteq K_2 \subseteq \ldots \subseteq K_N \subseteq \ldots \subseteq \omega, \quad \bigcup_{N \in \mathbb{N}} K_N = \omega,$$

and $H(K)$ denotes the space of germs of analytic functions over $K$ with its natural LB-space topology. It is known (see [16] Prop. 1.7, 1.2) that this topology is equal to the inductive limit topology $\text{ind} H(U)$, where $U$ runs over all open neighborhoods of $\omega$ in $\mathbb{C}^d$ and $H(U)$ denotes the Fréchet space of holomorphic functions on $U$ with the compact open topology. Thus $\mathcal{A}(\omega)$ is a complete, separable, ultrabornological, reflexive, webbed nuclear space with the approximation property (but non-metrizable and without a basis [7]) with the dual being a complete LF-space $\mathcal{A}(\omega)' = \text{ind} H(K_N)'$. The space $\mathcal{A}(\omega)$ is the projective limit of a sequence of LB-spaces. Spaces of this type are called PLB-spaces. For more information on the space of real analytic functions, see [2], [4], [5], [7], [8] and [16].

For smooth functions on $\omega \subseteq \mathbb{R}^d$ we use the standard multi-index notation. So if $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, then

$$\frac{\partial^{\alpha}}{\partial x^\alpha} f = \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} f, \quad \delta_2(f) := f(z), \quad \delta_2^{\alpha}(f) := (-1)^{|\alpha|} \frac{\partial^{\alpha}}{\partial x^{\alpha}} f(z),$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_d$.

The elements $\mu \in H(\mathbb{C}^d)'$ are called analytic functionals. A compact subset $K \subseteq \mathbb{C}^d$ is called a carrier for $\mu$ if for every neighborhood $U \subset \subset \mathbb{C}^d$ of $K$ there is a continuity estimate

$$|\mu(f)| \leq C \sup_{z \in U} |f(z)|.$$

We may identify $\mathcal{A}(\mathbb{R}^d)'$ with the analytic functionals that are carried by some $K \subseteq \mathbb{R}^d$. In this case there is a smallest real carrier which is called the support of $\mu$ and denoted by $\text{supp} \mu$ (see [19] p. 44 ff.). For open $\omega \subseteq \mathbb{R}^d$ we may identify $\mathcal{A}(\omega)'$ with the analytic functionals $\mu$ with supp $\mu \subseteq \omega$.

The other notation is standard. We refer for functional analysis to [17] and for complex analysis to [13]. For analytic functionals see [19] or [15].
1. Distinguished open sets

We will need some auxiliary notions. We call a nonnegative $C^1$-function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ distinguished whenever it vanishes at infinity and

$$\sup_{x \in \mathbb{R}^d} \|H_\psi(x)\| < 1,$$

where $H_\psi$ denotes the Hessian of $\psi$ and $\| \cdot \|$ is the norm in the space of quadratic forms on the $d$-dimensional euclidean space. For every distinguished function $\psi$ we define

$$\Omega^\psi := \{ z = x + iy \in \mathbb{C}^d : |y|^2 < \psi(x) \}$$

and call it a distinguished open set with basis $\omega := \Omega^\psi \cap \mathbb{R}^d$. These sets and the construction below are variations of the proof of the Cartan-Grauert Theorem as given in [3, Prop. 1] and [10, Prop. 6 and Prop. 7].

**Lemma 1.1.** (a) Every distinguished open set $\Omega^\psi$ is a domain of holomorphy and for every $\alpha > 0$, $\Omega^\psi_\alpha := \{ z : |y|^2 < \psi(x) - \alpha \} \subset \subset \Omega^\psi$. Moreover, polynomials are dense in $H(\Omega^\psi)$ equipped with the compact-open topology.

(b) For every pair of open sets $U \subset W \subset \mathbb{C}^d$, $U$ distinguished, there is a distinguished set $\Omega^\psi$ with the basis $W \cap \mathbb{R}^d$ such that $U \subset \subset \Omega^\psi \cap W$. If $U \subset \subset W$, then we can choose $\psi$ such that $U \subset \subset \Omega^\psi$.

(c) Every open set $\omega \subset \mathbb{R}^d$ has a basis of open neighbourhoods in $\mathbb{C}^d$ consisting of distinguished sets.

(d) Every compact set in $\mathbb{C}^d$ is contained in some bounded distinguished set.

Parts (a) and (c) imply a corollary going back to Cartan (see [11, Cor. II. 3.15]):

**Corollary 1.2.** Every open set $\omega$ in $\mathbb{R}^d$ has a basis of neighbourhoods in $\mathbb{C}^d$ consisting of domains of holomorphy.

**Proof of Lemma 1.1** (a): Let $f(z) := |y|^2 - \psi(x)$ for $z = x + iy \in \mathbb{C}^d$. Since $\|H_\psi(x)\| < 1$, the Levi form of $f$ is positive definite. Thus, $f$ is plurisubharmonic on $\mathbb{C}^d$. Clearly $\Omega^\psi_\alpha = \{ z : f(z) < 0 \}$. Since $\psi$ vanishes at infinity, $\Omega^\psi_\alpha \subset \subset \Omega^\psi$.

By [13, Cor. 5.4.3], every function holomorphic on a neighbourhood of $\Omega^\psi_\alpha$ can be approximated uniformly on this set by entire functions.

(b): Let $U = \Omega^\psi_\alpha$ and

$$s(x) := \inf\{|w| : x + i(y + w) \not\in W \text{ for some } y, |y|^2 \leq \psi_0(x)\}.$$

Observe that

(i) since $\psi_0$ is continuous and $W$ is open, $s$ is lower semi-continuous;

(ii) $V := \{ x \in W \cap \mathbb{R}^d : s(x) > 0 \}$ is open and contains $(W \setminus U) \cap \mathbb{R}^d$;

(iii) if $U \subset \subset W$, then $V = W \cap \mathbb{R}^d$.

We choose a covering of $V$ by sets

$$U_{r_j}(x_j) = \{ x \in \mathbb{R}^d : |x - x_j| < r_j \} \subset \subset W \cap \mathbb{R}^d.$$

We set

$$\varphi(x) = \begin{cases} e^{1/|x|^2 - 1} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases} \quad \psi_1(x) = \sum_{j=1}^{\infty} \varepsilon_j \varphi \left( \frac{1}{r_j} (x - x_j) \right).$$

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By (i) and a suitable choice of $\varepsilon_j$, we get that the series is convergent and

(iv) $\psi_1(x) < s^2(x)$;
(v) $\sup_{x \in \mathbb{R}^d} \|H\psi_1(x)\| < 1 - \sup_{x \in \mathbb{R}^d} \|H\psi_3(x)\|$.

By (v), $\psi := \psi_0 + \psi_1$ is a distinguished function. It is immediate that $U \subseteq \Omega^\psi$ and, by (iv), we get $\Omega^\psi \subseteq W$. By (ii), the basis of $\psi$ is equal to $W \cap \mathbb{R}^d$. Finally, if $U \subseteq W$, then, by (iii), $\psi_1|_{U \cap \mathbb{R}^d} > \delta > 0$. Thus $U \subseteq \Omega^\psi \subseteq \Omega$. (c): Take in (b) $U = \emptyset$ and $\omega = W \cap \mathbb{R}^d$.

(d): It suffices to observe that if $\varphi$ is defined as in (b) and $\varphi_a := a \varphi(\frac{x}{a})$, then $H\varphi_a = a^{-1}H\varphi$. Therefore, for a big enough, $\varphi_a$ is a distinguished function but $\varphi_a \longrightarrow \infty$ uniformly on compact sets as $a \longrightarrow \infty$. \hfill \Box

The proof of the following lemma follows the line of, e.g., the proof of [19, Théorème 111].

**Lemma 1.3.** Let $\omega$ be open and bounded in $\mathbb{R}^d$, $\Omega^\psi$ a distinguished set with basis $\omega$. Then for every $\mu \in H(\Omega^\psi)' \cap \mathfrak{A}(\mathbb{R}^d)'$ we have $\text{supp}\mu \subset \overline{\omega}$.

**Proof.** Let $\Omega \supset \overline{\omega}$ be open and bounded in $\mathbb{C}^d$. Let $\Omega_1$ be an open neighbourhood of $\mathbb{R}^d$ in $\mathbb{C}^d$ such that $\Omega^\psi \cap \Omega_1 \subseteq \Omega$. By Lemma [11] there is a distinguished set $\Omega^{\psi_1}$ with basis $\mathbb{R}^d$ such that $\Omega^\psi \subseteq \Omega^{\psi_1} \subseteq \Omega^\psi \cup \Omega_1$. Since $\Omega^{\psi_1}$ is a domain of holomorphy with covering $\Omega_1 \cap \Omega^{\psi_1}$ and $\Omega^\psi \cap \Omega^{\psi_1} = \Omega^\psi$, the solution of the first Cousin problem yields the following exact sequence:

$$0 \longrightarrow H(\Omega^{\psi_1}) \longrightarrow H(\Omega_1 \cap \Omega^{\psi_1}) \oplus H(\Omega^\psi) \longrightarrow \delta \longrightarrow H(\Omega_1 \cap \Omega^\psi) \longrightarrow 0$$

where $j(f) = (f, f)$ and $\delta(f, g) = f - g$.

On $H(\Omega_1 \cap \Omega^{\psi_1}) \oplus H(\Omega^\psi)$ we define $u(f, g) = \mu f - \mu g$. Since $u \circ j = 0$ on $H(\mathbb{C}^d)$ and, by density of $H(\mathbb{C}^d)$ in $H(\Omega^{\psi_1})$ (see Lemma [11] (a)), we have $u \circ j = 0$ on $H(\Omega^{\psi_1})$ and $u$ gives rise to an element $\mu \in H(\Omega_1 \cap \Omega^\psi)'$ that clearly extends the given $\mu \in \mathfrak{A}(\mathbb{R}^d)'$.

Since $\Omega^\psi \cap \Omega_1 \subseteq \Omega$ this proves the result. \hfill \Box

2. **Eidelheit sequences**

It turns out that a condition very similar to Eidelheit’s characterization of Eidelheit sequences on Fréchet spaces is also necessary for any PLB-space.

**Lemma 2.1.** If $(f_n)_{n \in \mathbb{N}}$ is an Eidelheit sequence on a PLB-space $X = \text{proj}_{N \in \mathbb{N}} X_N$, then for every $N \in \mathbb{N}$ we have

$$(3) \quad \dim(\text{span}\{f_n : n \in \mathbb{N}\} \cap X_N') < \infty.$$ 

**Proof.** Let us assume that for some $N$ the condition (3) does not hold. Since $X_N$ is an LB-space, there is a bounded set $B \subset (X_N)'$ such that $\text{span}\{f_n : n \in \mathbb{N}\} \cap B$ has infinite dimension. Clearly, for a continuous surjection $T : X \longrightarrow \mathbb{C}^N$, $T(g) := (f_n(g))_{n \in \mathbb{N}}$,

$$T^{-1}(B) \subseteq (T')^{-1}(B^{\circ}) = \{g \in (\mathbb{C}^N)': T'g \in B^{\circ}\}$$

$$= \{g \in (\mathbb{C}^N)': |g(Tx)| \leq 1 \text{ for } x \in B^\circ\} = (T(B^\circ))^\circ.$$ 

Since $T$ is open (as a surjective continuous operator from the webbed space onto a Fréchet space), $T(B^\circ)$ is a 0-neighbourhood and $(T(B^\circ))^\circ$ is bounded in $\varphi = (\mathbb{C}^N)'$. 

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Therefore \((T(B^n))^\circ\) and \((T')^{-1}(B)\) are finite dimensional. Since \(T'\) is injective we get
\[
\dim(\text{Im}T' \cap B) = \dim(\text{span}\{f_n : n \in \mathbb{N}\} \cap B) < \infty, \quad \text{a contradiction.}
\]
\[\square\]

**Remark.** The condition (2) does not characterize Eidelheit sequences on an arbitrary PLB-space. In [6, Ex. 2.8] the authors constructed an example of a PLB-space \(X\) with nuclear steps such that its ultrabornological associated topology makes it a nuclear LB-space. Clearly, no sequence on \(X\) is Eidelheit (since it must be an Eidelheit sequence on its ultrabornological associated space, and therefore, on an LB-space). On the other hand, by the construction, \(X\) cannot be represented as a projective limit of LB-spaces with surjective linking maps, and therefore, there exists a sequence \((f_n)_n \in \mathbb{N} \subseteq X'\) satisfying (2).

Surprisingly, for the space \(\mathcal{A}(\omega)\) the above condition is also sufficient.

**Theorem 2.2.** Let \(\omega \subseteq \mathbb{R}^d\) be an arbitrary domain. Then there exists a domain of holomorphy \(\Omega\) in \(\mathbb{C}^d\) (\(\Omega = \mathbb{C}^d\) if \(\omega = \mathbb{R}^d\)), \(\Omega \cap \mathbb{R}^d = \omega\), such that for any sequence \((\mu_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}(\omega)'\) of linearly independent analytic functionals the following assertions are equivalent:

(a) \((\mu_n)_{n \in \mathbb{N}}\) is an Eidelheit sequence on \(\mathcal{A}(\omega)\);
(b) \((\mu_n)_{n \in \mathbb{N}}\) is an Eidelheit sequence on \(H(\Omega)\);
(c) for every compact set \(K \subseteq \omega\) we have
\[
\dim(\text{span}\{\mu_n : n \in \mathbb{N}\} \cap \{\mu \in \mathcal{A}(\mathbb{R}^d)' : \text{supp}\mu \subseteq K\}) < \infty.
\]

**Proof.** Clearly (b) \(\implies\) (a) \(\implies\) (c) follows from Lemma 2.1. Observe that by [19, Lemma 112], \(\mu\) extends to \(\mu \in H(K)'\) if and only if \(\text{supp}\mu \subseteq K\).

(c) \(\implies\) (b): Let \((\omega_n)\) be an exhaustion of \(\omega\), \(\omega_n \subseteq \omega_{n+1}\) for \(n \in \mathbb{N}\). By Lemma 1.1 (b), there exists an increasing family of distinguished sets \(\Omega^\omega_n\) with bases \(\omega_n\), \(\Omega^\omega_n \subseteq \Omega^\omega_{n+1}\) for \(n \in \mathbb{N}\). In the case of \(\omega = \mathbb{R}^d\) we can take \(\Omega^\omega_n\) containing the ball of radius \(n\) in \(\mathbb{C}^d\) (use Lemma 1.1 (a)) and [13, 3.3.7]. By Lemma 1.3 the assumption implies that for every compact set \(L \subseteq \Omega\) we have
\[
\dim(\text{span}\{\mu_n : n \in \mathbb{N}\} \cap \{\mu \in H(\Omega)' : \text{supp}\mu \subseteq L\}) < \infty.
\]
Therefore the classical Eidelheit example (see [9, 17, Th. 26.27]) yields the result.
\[\square\]

3. **Factorization of Eidelheit sequences**

**Proposition 3.1.** For every Eidelheit sequence \((f_n)_n \in \mathbb{N} \subseteq \mathcal{A}(\omega)\) there is an Eidelheit sequence \((g_n)_n \in \mathbb{N} \subseteq \mathcal{A}(\omega)\) such that there is no operator \(T : \mathcal{A}(\omega) \to \mathcal{A}(\omega)\) so that \(g_n \circ T = f_n\) for all \(n \in \mathbb{N}\).

**Proof.** By Theorem 2.2, there is a domain of holomorphy \(\Omega \subseteq \mathbb{C}^d\) such that \(\Omega \cap \mathbb{R}^d = \omega\) and \((f_n)_n \in \mathbb{N}\) is an Eidelheit sequence on \(H(\Omega)\) as well. By [8] (compare [4]), there is a Fréchet quotient \(F : \mathcal{A}(\omega) \to F\) that is not a quotient. Since \(H(\Omega)\) is quasinormable, by [21, Th. 2.4], there is an Eidelheit sequence \((h_n)_{n \in \mathbb{N}}\) on \(F\) such that for no operator \(S : H(\Omega) \to F\) the factorization \(h_n \circ S = f_n\) holds.

Let \(q : \mathcal{A}(\omega) \to F\) be the quotient map. Observe that \(g_n := h_n \circ q\) is the Eidelheit sequence we are looking for. Indeed, if \(g_n \circ T = f_n\), then \(h_n \circ q \circ T = f_n\), which contradicts the choice of \((h_n)_{n \in \mathbb{N}}\).
\[\square\]
We will consider now Eidelheit sequences of the form $(δ^α_n)_{n ∈ \mathbb{N}, |α| ≤ k_n}$ on $\mathcal{A}(ω)$, $ω ⊆ \mathbb{R}^d$, where $(k_n)_{n ∈ \mathbb{N}}$ is an arbitrary sequence of natural numbers and $(z_n)$ is a discrete sequence. We call such sequences interpolation sequences.

**Theorem 3.2.** Let $(f^{(1)}_n) ∈ \mathcal{A}(ω)$ and $(f^{(2)}_n) ∈ \mathcal{A}(ω)'$ be two arbitrary interpolation sequences. There is always an operator $T: \mathcal{A}(ω_1) → \mathcal{A}(ω_2)$ such that $T'$ maps $(f^{(2)}_n)_{n ∈ \mathbb{N}}$ one-to-one onto $(f^{(1)}_n)_{n ∈ \mathbb{N}}$.

The proof of Theorem 3.2 follows immediately from Corollary 3.4 and Lemmas 3.3 and 3.6 below.

**Lemma 3.3.** Let $ω_1 ⊆ \mathbb{R}^d$ and $ω_2 ⊆ \mathbb{R}^d$ be open domains, and let $(z_n) ⊆ ω_1$ and $(y_n) ⊆ ω_2$ be discrete sequences. Let $(k_n) ∈ \mathbb{N}$ be a sequence of natural numbers. Then there is a real analytic map $φ: ω_1 → ω_2$ such that for every $n ∈ \mathbb{N}$, $φ(z_n) = y_n$, the Jacobian matrix $J φ$ of $φ$ at the points $z_n$ is just the unit matrix and all the partial derivatives of $φ$ at $z_n$ of rank strictly larger than one and less than or equal to $k_n$ are vanishing.

**Corollary 3.4.** Under the notation of the previous lemma, the composition operator $Cφ: \mathcal{A}(ω_2) → \mathcal{A}(ω_1)$, $Cφ(f) := f ∘ φ$, satisfies $Cφ(δ^α_n) = δ^α_{y_n}$ for $|α| ≤ k_n$, $n ∈ \mathbb{N}$.

**Proof of Lemma 3.3.** By [4, Lemma 4.2], there is a surjective real analytic map $ψ: \mathbb{R}^d → ω_2$ of constant rank $d$. It is easily seen that for every $z ∈ \mathbb{R}^d$ the map $C'_ψ$ maps $\text{span}\{δ^α : |α| ≤ k\} ⊆ \mathcal{A}(\mathbb{R}^d)'$ onto $\text{span}\{δ^α_ψ(z) : |α| ≤ k\} ⊆ \mathcal{A}(ω_2)'$. By the classical interpolation problem on domains of holomorphy and Cor. 1.2 we find $η: ω_1 → \mathbb{R}^d$ such that $φ := ψ ∘ η$ is the map we are looking for.

**Lemma 3.5.** Let $(y_n)$, $(x_n)$ be sequences in $\mathbb{R}^d$ tending to infinity. Let $(α(n))$ and $(β(n))$ be sequences of multi-indices such that if $y_n = y_k$, then $α(n) ≠ α(k)$. Then there is an operator $T: \mathcal{A}(\mathbb{R}^d) → \mathcal{A}(\mathbb{R}^d)$ such that $\left( \frac{∂^{α(n)}}{∂x^{α(n)}} Tf \right)(y_n) = \left( \frac{∂^{β(n)}}{∂x^{β(n)}} f \right)(x_n)$ for every $n ∈ \mathbb{N}$.

**Proof.** Let $(k_n)$ be an increasing sequence of natural numbers such that $k_n ≥ \max\{|α(l)| : y_l = y_n\}$. Let us choose $φ_n: \mathbb{C}^d → \mathbb{C}^d$, $φ_n(\mathbb{R}^d) ⊆ \mathbb{R}^d$, holomorphic such that $φ_n(y_k) = \begin{cases} x_n & \text{for } y_k = y_n, \\ 0 & \text{for } y_k ≠ y_n, \end{cases}$ and $\left( \frac{∂^{β}}{∂x^{β}} φ_n \right)(y_k) = 0$ for $1 ≤ |β| ≤ k_n$.

By the solution of the classical interpolation problem for entire functions, such $φ_n$ exist and we may assume that $φ_n → 0$ uniformly on compact sets. Analogously, we define functions $g_n: \mathbb{C}^d → \mathbb{C}$ such that $\left( \frac{∂^{α(n)}}{∂x^{α(n)}} g_n \right)(y_n) = 1$ and $\left( \frac{∂^{β}}{∂x^{β}} g_n \right)(y_k) = 0$ for $|β| ≤ k_n$ and either $y_k ≠ y_n$ or $y_k = y_n, β ≠ α(n)$.

Since the map

$R: H(\mathbb{C}^d) → \mathbb{C}^N, \quad R(f) := \left( \left( \frac{∂^{β}}{∂x^{β}} f \right)(y_n) \right)_{n ∈ \mathbb{N}, |β| ≤ k_n},$
Clearly, the series converges for every $S \in \mathcal{S}$, a discrete sequence in $\mathbb{R}^d$.

**Proof.** One takes $T: \mathcal{A}(\mathbb{R}^d) \to \mathcal{A}(\mathbb{R}^d)$, $(Tf)(z) := \sum_{n \in \mathbb{N}} g_n(z) \cdot \left( \frac{\partial^{\beta(n)}}{\partial x^{\beta(n)}} f \right) (\varphi_n(z))$.

Clearly, the series converges for every $f \in \mathcal{A}(\mathbb{R}^d)$ and, by the Closed Graph Theorem, the map is continuous. Moreover, for each $n$,

$$\left( \frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} T f \right)(y_n) = \sum_{y_k = y_n} \frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} \left[ g_k \cdot \left( \frac{\partial^{\beta(k)}}{\partial x^{\beta(k)}} f \right) \circ \varphi_k \right] (y_k)$$

$$= \sum_{y_k = y_n} \left( \frac{\partial^{\beta(k)}}{\partial x^{\beta(k)}} f \right) (x_n) \cdot \left( \frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} g_k \right) (y_k)$$

$$= \frac{\partial^{\beta(n)}}{\partial x^{\beta(n)}} f(x_n).$$

This completes the proof. \qed

**Lemma 3.6.** Let $\mathbb{R}$ be embedded in a standard way into $\mathbb{R}^d$, and let $(z_n)$ be a discrete sequence in $\mathbb{R}$. Then there are operators $T: \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R}^d)$ and $S: \mathcal{A}(\mathbb{R}^d) \to \mathcal{A}(\mathbb{R})$ such that $T'(\delta_{z_n}) = (\delta_{z_n})$ and $S'(\delta_{z_n}) = (\delta_{z_n})$ for $n \in \mathbb{N}$.

**Proof.** One takes $T(f)(x_1, \ldots, x_d) := f(x_1), S(f)(x) := f(x, 0, \ldots, 0)$. \qed

**References**


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