PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 12, Pages 3599–3606
S 0002-9939(04)07463-5
Article electronically published on July 12, 2004

SUBSPACES OF $L^1(\mathbb{R}^d)$

CAROLINE SWEEZY

(Communicated by Andreas Seeger)

Abstract. The relationship of the Hardy space $H^1(R^d)$ and the space of integrable functions $L^1(R^d)$ is examined in terms of intermediate spaces of functions that are described as sums of atoms. It is proved that these spaces have dual spaces that lie between the space of functions of bounded mean oscillation, BMO, and L^∞. Furthermore, the spaces intermediate to H^1 and L^1 are shown to be dual to spaces similar to the space of functions of vanishing mean oscillation. The proofs are extensions of classical proofs.

One of the most celebrated results on duality in the twentieth century was the discovery of C. Fefferman and E. Stein that the space of functions of bounded mean oscillation (BMO – see Def. 0 below) is the dual space for the Hardy space H^1. BMO was introduced by F. John and L. Nirenberg in 1961, [JN]. The John-Nirenberg Theorem shows that BMO can be identified with the functions that are exponentially integrable (see Def. 4 with $r = 1$). The Hardy spaces, H^p, were originally defined by Thomas Hardy on the unit disk: an analytic function f on $D = \{z = re^{i\theta} : |z| < 1\}$ was in $H^p(D)$ if $\sup_{0<r<1} \int_{|z|=r} |f(z)|^p \, d\theta < \infty$. There is a similar definition for Hardy space functions on the upper half plane. Since Hardy's time the H^p spaces have taken on a real variable characterization that does not depend on analyticity; this was due to work of Stein and Weiss. The real H^p spaces on $\mathbb{R}^d$ can be identified with the $L^p(\mathbb{R}^d)$ spaces if $p > 1$. However for $p = 1$, $H^1 \subsetneqq L^1$. The Hardy space $H^1(\mathbb{R}^d)$ can be defined as sums of atoms (the approach used here) or by using maximal functions (see §2.1 of this paper). See Stein's book [S], Chapters 3 and 4, for detailed descriptions of Hardy spaces and BMO and for further references.

As a result of the duality theorem of Fefferman and Stein the following simple diagram is well known:

$$H^1(\mathbb{R}^d) \subset L^1(\mathbb{R}^d),$$
$$BMO(\mathbb{R}^d) = H^1(\mathbb{R}^d)^* \supset L^1(\mathbb{R}^d)^* = L^\infty(\mathbb{R}^d).$$

This paper adds the spaces $X_{(r)}$ and their duals $BMO(r) = X^*_{(r)}$ for $1 \le r < \infty$ (Defs. 2 & 3) to the diagram as follows:

$$X_{(1)}(\mathbb{R}^d) = H^1(\mathbb{R}^d) \subset X_{(r)}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d),$$
$$BMO = BMO(1) = H^{1*}(\mathbb{R}^d) \supset X^*_{(r)}(\mathbb{R}^d) = BMO(r) \supset L^{1*}(\mathbb{R}^d) = L^\infty.$$

Received by the editors June 23, 2003.
2000 *Mathematics Subject Classification.* Primary 42B25.

Determining the exact relationship of H^1 to L^1 and of L^∞ to BMO is a question of considerable interest. The present paper shows there are function spaces that form a continuum from H^1 to L^1 and proves that there are dual spaces that stretch from BMO to L^∞. There are many unanswered questions about these spaces, especially about the spaces that lie between $H^1$and L^1. Some of the open problems are discussed at the end of this paper. The spaces that lie between $H^1(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$ are defined as sums of atoms (Def. 2 below). The spaces between L^∞ and BMO are already known, although perhaps not in the exact form given in Definition 3 below (see the discussion in Section 2 of S. Janson's function classes).

The main result presented here is the duality relation between the two sets of spaces that is stated in Theorem 1. To introduce some notation that will be used throughout the paper, the following description is given.

There are function spaces $\{\underline{\overline{X}}_{(r)}\}_{1\leq r<\infty}$ that lie between $H^1(\mathbb{R}^d) = \underline{\overline{X}}_{(1)}$ and $L^1(\mathbb{R}^d)$. If $s \geq r$, then $\underline{\overline{X}}_{(s)} \supseteq \underline{\overline{X}}_{(r)}$, $1 \leq r \leq s < \infty$. The main result concerning these spaces is that they have dual spaces, $BMO(r) = \underline{\overline{X}}^*_{(r)}$, that lie between $BMO(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$. If $s \geq r$, then $BMO(s) \subseteq BMO(r)$. The $BMO(r)$ spaces are identified with certain exponential spaces (Proposition 1). It is also proved (Theorem 2) that if $VMO(r)$ is defined as the closure of C_0 (the continuous functions that vanish at infinity) in $BMO(r)$, then $\underline{\overline{X}}_{(r)} = VMO(r)^*$ (see [CW] for the case $r = 1$). The proofs of these results are generalizations of classical proofs. The dual of a space $\underline{\overline{X}}$ is denoted by $\underline{\overline{X}}^*$.

The Hardy space $H^1(\mathbb{R}^d)$ is considered in terms of its definition as atomic sums. Recall that the simplest kind of atom that is used to describe $H^1(\mathbb{R}^d)$ is an L^∞ atom; this is a measurable function α, such that α is supported on a ball Q in $\mathbb{R}^d$, $\int_Q \alpha(x)dx = 0$, and $\|\alpha\|_\infty \leq |Q|^{-1}$. Any function $f(x) = \sum \lambda_j \alpha_j(x)$, with the α_j being L^∞ atoms and $\sum |\lambda_j| < \infty$, is an H^1 function. To define the $\underline{\overline{X}}_{(r)}$ spaces for $r > 1$, however, we need to use L^q atoms. Garcia-Cuerva and Rubio de Francia proved that any L^2 atom, a measurable function β, also supported on a ball Q, with mean value zero, such that $(\frac{1}{|Q|}\int_Q (\beta(x))^2 dx)^{1/2} \leq |Q|^{-1}$, can be written as a sum of L^∞ atoms [GCRF]. Their proof shows that any H^1 function can be written as a sum of L^2 atoms; in fact it is easy to see that one can write an H^1 function as a sum of L^q atoms (supp$\beta \subset Q$, $\int_Q \beta(x)dx = 0$, $(\frac{1}{|Q|}\int_Q (\beta(x))^q dx)^{1/q}$ $\leq |Q|^{-1}$) for any $1 < q \leq 2$, where q is fixed. The $\underline{\overline{X}}_{(r)}$ spaces are defined below as sums of L^q atoms, where q can become arbitrarily close to 1. This gives the faster rate of growth in $\underline{\overline{X}}_{(r)}$, $r > 1$, that takes us out of H^1 into larger subspaces of L^1.

The increased rate of growth for the $\underline{\overline{X}}_{(r)}$ spaces is, of course, matched by a slightly more controlled rate of growth for the dual space functions, the $BMO(r)$ functions. The condition is somewhat delicate, and one must look at the p-means of these functions (see Def. 3 below) to obtain the right spaces for duality.

The first section of this paper gives the necessary definitions of the spaces, and presents the results described above and their proofs. The second section sketches a direct proof that $\underline{\overline{X}}_{(1)} = H^1$ using the grand maximal function, shows that the spaces $BMO(r)$ are identical to certain exponential function spaces, and discusses other facts and open problems concerning the spaces $\underline{\overline{X}}_{(r)}$ and $BMO(r)$.

1. Section

First to define the function spaces: let r be a real number greater than or equal to 1. Definition 0 is well known.

Definition 0. $f \in BMO(R^d)$ if $\sup_Q \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx\right) \leq C_0$ for C_0 independent of Q. The Q are balls in R^d, and $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$.

Definition 1. α is an $L^{(q,r)}$ atom if supp $\alpha \subseteq B$ for some ball B in R^d,

$$\int_{\mathbb{R}^d} \alpha(x)dx = 0,$$

and

$$\left(\frac{1}{|B|}\int\limits_B |\alpha(x)|^q\,dx\right)^{\frac{1}{q}} \leq \frac{1}{p^{\frac{1}{r}}\,|B|},\ \frac{1}{q} + \frac{1}{p} = 1.$$

In this paper we will always assume that $1 < q \leq 2$. So, except for the factor $\frac{1}{p^{\frac{1}{r}}}$, these are the usual L^q, H^1 atoms.

Definition 2. $f \in \underline{X}_{(r)}(R^d)$ if $f(x) = \sum\limits_{j=0}^{\infty}\lambda_j\alpha_j(x)$ where $\sum\limits_{j=0}^{\infty}|\lambda_j| < \infty$ and α_j is an $L^{(q_j,r)}$ atom for some $1 < q_j \leq 2$.

Convergence can be taken to be pointwise almost everywhere. The results proved below show that in fact one can have weak convergence instead of point-wise convergence.

The norm of $\underline{X}_{(r)}$ can be defined as $\inf\{\sum\limits_{j=0}^{\infty}|\lambda_j|$: all possible representations of f as a sum of such atoms $\} \equiv \|f\|_{\underline{X}_{(r)}}$. For $r = 1$ it is well known that

$$\inf\left\{\sum_{i=1}^{\infty}|\lambda_i| : f(x) = \sum_{i=1}^{\infty}\lambda_i\alpha_i(x)\right\}$$

is a norm that is equivalent to the H^1norm of f (see [GCRF], [S]). For $r \neq 1$, by definition the $\inf\sum|\lambda_i|$ forms a semi-norm on the space of functions described in Definition 2. To show that in fact this is a norm, we will see that part of the proof of Theorem 1 below shows that f, as in Definition 2, lies in the dual space of $BMO(r)$; the proof is independent of any norm definition for f. It implies that f is an element of a Banach space and that the norm of f in that Banach space is $\lesssim \sum\limits_{i=1}^{\infty}|\lambda_i|$ for any representation $f(x) = \sum\lambda_i\alpha_i(x)$. If $\inf\sum\limits_{i=1}^{\infty}|\lambda_i| = 0$, then $\|f\|$ must $= 0$, i.e. f is the zero element. So $\inf\sum\limits_{i=1}^{\infty}|\lambda_i|$ is a norm on the linear space of functions described in Definition 2.

It is immediate that if f is given by a finite sum, or if all $\alpha_j \in L^{q_j}$ where $q_j \geq q > 1$ for all j, then $f \in H^1(\mathbb{R}^d)$. The difference between $\underline{X}_{(r)}$ and H^1 is that in infinite sums the atoms can be L^{q_j} atoms where the q_j's become arbitrarily close to 1. For $r = 1$ these sums are still in H^1, but if $r > 1$, in fact $\underline{X}_{(r)} \supsetneq H^1$. A careful examination of the decomposition of an L^q atom into a sum of L^∞ atoms given by Garcia-Cuerva and Rubio de Francia [GCRF] shows that there is a factor of $\frac{1}{q-1}$.

So if one were to write an infinite sum of L^{q_j} atoms, with $q_j \longrightarrow 1$, as a sum of L^∞ atoms, the sum of their coefficients could easily fail to converge. Theorem 1 and an example given in Section 2 actually prove $\underline{\overline{X}}_{(r)} \neq H^1$ if $r > 1$.

The next family of spaces is implicit in the work of John Garnett ([G] , [GJ]). They also appear in some form in unpublished work of S. Janson [CWW].

Definition 3. For $f \in L^1_{loc}(\mathbb{R}^d)$, $f \in BMO(r)$ if

$$\|f\|_{BMO(r)} \equiv \sup_{1 \le p < \infty} \left[\frac{1}{p^{\frac{1}{r}}} \sup_B \left(\frac{1}{|B|} \int\limits_B |f(x) - f_B|^p \, dx \right)^{\frac{1}{p}} \right] \le C < \infty,$$

$$f_B = \frac{1}{|B|} \int\limits_B f(x) dx, \text{ and } 0 < r < \infty.$$

It is well known that for $f \in BMO$, then $(\frac{1}{|B|} \int_B |f(x) - f_B|^p \, dx)^{\frac{1}{p}} \le C_0 p$, where C_0 is independent of p and B (this fact follows from the John-Nirenberg theorem; see [G] and [S] and Proposition 1 in Section 2 of this paper). So $BMO(1) = BMO$ and for $r > 1$ the p-means of a $BMO(r)$ function grow a little more slowly than the averaged function in BMO, as $p \longrightarrow \infty$.

The main results of this paper are:

Theorem 1. *$BMO(r) = \underline{\overline{X}}^*_{(r)}$, $1 \le r < \infty$.*

Theorem 2. *$\underline{\overline{X}}_{(r)} = VMO(r)^*$, $1 \le r < \infty$.*

Remark 1. The spaces of interest here are for $1 \le r < \infty$. There is no reason, however, why most results cannot be extended to $0 < r < 1$.

The rest of this section will present proofs of the two results given above.

The following proof of Theorem 1 is a generalization of Stein's proof of H^1 and BMO duality given in Chapter 4 of **Harmonic Analysis** [S].

Proof of Theorem 1. To show that $BMO(r) \subseteq \underline{\overline{X}}^*_{(r)}$ take $f \in BMO(r)$ and let $h(x) = \sum \lambda_j \alpha_j(x)$ be any finite sum in $\underline{\overline{X}}_{(r)}$. Choose the representation for $h(x)$ to be such that $\sum |\lambda_j| \le (1+\varepsilon) \|h\|_{\underline{\overline{X}}_{(r)}}$. Then

$$\begin{aligned}
\left| \int\limits_{\mathbb{R}^d} h(x) f(x) dx \right| &= \left| \sum \int\limits_{\mathbb{R}^d} \lambda_j \alpha_j(x) f(x) dx \right| \\
&\le \sum |\lambda_j| \int\limits_{B_j} |\alpha_j(x)| \left| f(x) - f_{B_j} \right| dx \\
&\le \sum |\lambda_j| \left(\int\limits_{B_j} |\alpha_j(x)|^{q_j} \, dx \right)^{\frac{1}{q_j}} \left(\int\limits_{B_j} \left| f(x) - f_{B_j} \right|^{p_j} dx \right)^{\frac{1}{p_j}} \\
&\le \sum |\lambda_j| \frac{1}{p_j^{\frac{1}{r}} |B_j|^{\frac{1}{p_j}}} \left(\int\limits_{B_j} \left| f(x) - f_{B_j} \right|^{p_j} dx \right)^{\frac{1}{p_j}}
\end{aligned}$$

$$\leq \sum |\lambda_j| \, \|f\|_{BMO(r)}$$
$$\leq (1+\varepsilon) \, \|h\|_{\overline{X}_{(r)}} \, \|f\|_{BMO(r)} \, .$$

Taking $\int_{\mathbb{R}^d} h(x)f(x)dx = \lim_{N\to\infty} \int_{\mathbb{R}^d} \sum_{j=0}^{N} \lambda_j \alpha_j(x) f(x)dx$ where $h(x) = \sum_{j=0}^{\infty} \lambda_j \alpha_j(x)$, one has the same upper bound for infinite sums.

Conversely, suppose $\Lambda \in \overline{X}^*_{(r)}$ with $\|\Lambda\| = 1$. Then Λ defines a linear functional on all the spaces $L^q_0(B)$ for any ball $B \subseteq \mathbb{R}^d$, $1 < q \leq 2$. Here $L^q_0\,(B) = \{f \in L^q(B) : \int_B f(x)dx = 0\}$. This is true because any $\beta \in L^q_0(B)$ can be written as a multiple of an $L^q(B)$ atom α, ($p^{\frac{1}{r}} \, |B|^{\frac{1}{p}} \, \|\beta\|_q \, \alpha(x) = \beta(x)$ will work), and atoms lie in $X_{(r)}$.

So the norm of Λ in $L^q_0(B)^*$ is $\leq p^{\frac{1}{r}} \; |B|^{\frac{1}{p}}$ since $|\Lambda(\beta)| \leq p^{\frac{1}{r}} \, |B|^{\frac{1}{p}} \, \|\beta\|_q$ by the fact that $\|\alpha\|_{X_{(r)}} \leq 1$ from the definition of the norm on $\overline{X}_{(r)}$, and that $\|\Lambda\|_{\overline{X}^*_{(r)}} = 1$. Putting Λ equal to zero on constants gives that $\Lambda \in L^q(B)^*$. So the Riesz representation theorem guarantees there is a function $f \in L^p(B)$ so that $\Lambda(\beta) = \int f\,(x)\,\beta\,(x)\;dx$. In fact $f \in L^p_0(B)$.

Now holding q fixed, notice that on $L^q_0(B)$ f is only uniquely defined up to additive constants. Call the original representative f^B, so that $(\int_B \left|f^B\,(x)\right|^p dx)^{\frac{1}{p}} = \|\Lambda\|_{L^q(B)^*} \leq p^{\frac{1}{r}} \; |B|^{\frac{1}{p}}$. To obtain a single representative f for Λ on all $B \subseteq \mathbb{R}^d$ one can require, as Stein does, that $\int_{B_1(0)} f(x)dx = 0$. Then $f(x) = f^B(x) + C_B$ where C_B is constant, and

$$\left(\frac{1}{|B|}\int\limits_B |f\,(x) - f_B|^p \,dx\right)^{\frac{1}{p}} \leq 2\left(\frac{1}{|B|}\int\limits_B |f\,(x) - C_B|^p \,dx\right)^{\frac{1}{p}}$$
$$= 2\left(\frac{1}{|B|}\int\limits_B \left|f^B\,(x)\right|^p dx\right)^{\frac{1}{p}} \leq 2p^{\frac{1}{r}}.$$

If f can be shown to be the representative for Λ on all the spaces $L^q_0(B), 1 < q \leq 2$, for all $B \subseteq \mathbb{R}^d$, then $f \in BMO(r)$.

By Hölder's inequality, $L^p(B) \subseteq L^2(B)$ and $L^2_0(B) \subseteq L^q_0(B)$ (remember that $\frac{1}{p} + \frac{1}{q} = 1$). So any atom in $L^2_0(B)$ is also a function in $L^q_0(B)$. Now Λg, $g \in L^2_0(B)$, does not change if g is considered to be in $L^q_0(B)$, $q < 2$, although the upper bound for the norm of Λ changes from L^2 to L^q. Any representative for Λ in $L^p(B)$ is also an L^2 function and f is uniquely defined in L^2 , the normalization for f being the same for all $L^p(B)$. It follows that the function $f \in L^2(B)$ in fact represents Λ on all the $L^q_0(B)$ spaces. □

Theorem 2 follows from a generalization of Coifman and Weiss' proof that H^1 is the dual of VMO, when VMO is defined to be the closure of C_0 in BMO [CW]. The proof for $\overline{X}_{(r)}$ in place of H^1 and $VMO(r)$ in place of VMO requires only technical modifications of their proof. The proof is long, and therefore it is not included here. The main changes come from the fact that one can now have infinitely many different kinds of atoms in the decomposition of an $\overline{X}_{(r)}$ function, and the BMO norm must be replaced by the $BMO(r)$ norm.

2. Section

Some relevant facts about the new spaces are detailed in this section for the interested reader.

2.0. The function $f(x) = \ln|x| - \ln|x-1|$ is shown to have

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p \, dx\right)^{\frac{1}{p}} \geq cp$$

in [G] (Chapter 6, Corollary 2.3). Since $f = H(\chi_{(0,1)})$, where $H(g)$ is the Hilbert transform of g, this means that $f \in BMO$ ([G], Chapter 6, Theorem 1.5). This example shows that BMO is strictly larger than $BMO(r)$ if $r > 1$. So by Theorem 1, and §2.1 below, $\underline{\overline{X}}_{(r)} \supsetneqq \underline{\overline{X}}_{(1)}$, for $r > 1$.

2.1. One can prove directly that $\underline{\overline{X}}_{(1)} = H^1$ by using the grand maximal function (see Stein's book, Chapter 3 [S]). Let $M_\Phi f(x) = \sup_{t>0} |\Phi_t * f(x)|$ be the canonical representative for the grand maximal function. Briefly, $f \in H^1(\mathbb{R}^d)$ if $M_\Phi f \in L^1(\mathbb{R}^d)$. Φ is a smooth, rapidly decreasing function so that $\int \Phi \neq 0$. For $h(x) = \sum \lambda_j \alpha_j(x)$ a finite sum of L^{q_j} atoms, such that $\sum |\lambda_j| < \infty$,

$$\begin{aligned}\int_{\mathbb{R}^d} |M_\Phi h(x)| dx &\leq \sum_j \int\limits_{\mathbb{R}^d \setminus 2B_j} |M_\Phi(\lambda_j \alpha_j)(x)| dx + \int\limits_{2B_j} |M_\Phi(\lambda_j\alpha_j)(x)| dx \\ &\leq \sum_j |\lambda_j| \left(\int\limits_{\mathbb{R}^d\setminus 2B_j} |M_\Phi \alpha_j(x)| dx + \int\limits_{2B_j} |M_\Phi \alpha_j(x)| dx \right).\end{aligned}$$

The terms $\int_{\mathbb{R}^d \setminus 2B_j} |M_\Phi \alpha_j(x)| dx$ can be shown to be bounded by a universal constant for any $L^{(q_j,1)}(B)$ atom by almost the same argument Stein uses for L^∞ atoms (Chapter 3 [S]).

Estimating the right-hand side of $\int\limits_{2B_j} |M_\Phi \alpha_j(x)| dx \leq \left(\int_{\mathbb{R}^d} |M_\Phi \alpha_j(x)|^{q_j} dx\right)^{\frac{1}{q_j}} \cdot |2B_j|^{1-\frac{1}{q_j}}$ also appears as an exercise in Stein, but it is of interest here to see how the proof selects the index $r = 1$. $|M_\Phi \alpha_i(x)| \leq |M\alpha_i(x)|$ where $M\alpha_i(x)$ is the Hardy-Littlewood maximal function of $\alpha_i(x)$. The L^{q_j} maximal inequality for the Hardy-Littlewood maximal function now gives, for α_j an $L^{q_j,r}$ atom,

$$\begin{aligned}\left(\int_{\mathbb{R}^d} |M\alpha_j(x)|^{q_j} dx\right)^{\frac{1}{q_j}} &\leq C\frac{1}{q_j - 1}\left(\int_{\mathbb{R}^d} |\alpha_j(x)|^{q_j} dx\right)^{\frac{1}{q_j}} \\ &= C' \cdot p_j \left(\int_{B_j} |\alpha_j(x)|^{q_j} dx\right)^{\frac{1}{q_j}} \leq C'' \frac{p_j}{p_j^{\frac{1}{r}} |B_j|^{\frac{1}{p_j}}}.\end{aligned}$$

So for $r = 1$,

$$\int_{2B_j} |M_\Phi \alpha_j(x)| dx \leq C|B_j|^{\frac{1}{p_j}} \frac{p_j}{p_j^{\frac{1}{r}} |B_j|^{\frac{1}{p_j}}} = C.$$

Also,

$$\int_{\mathbb{R}^d} |M_\Phi h(x)| dx \leq C \sum |\lambda_j| < \infty \text{ . This implies that } h \in H^1.$$

The argument works for infinite sums as well.

There are several facts about the spaces $BMO(r)$ that are worth noting.

2.2. Proposition 1 is elementary; it identifies the $BMO(r)$ spaces with certain exponential spaces, defined by

Definition 4. $g \in \exp L^r$ if there are constants $C_1, C_2 > 0$ so that for all balls $B \subseteq R^d$, $(\frac{1}{|B|}\int\limits_B \exp\{C_1 |g(x) - g_B|^r \, dx\})^{\frac{1}{r}} \leq C_2 < \infty$. C_1 and C_2 are independent of B, $0 < r < \infty$.

Proposition 1. $BMO(r) = \exp L^r$, $r \geq 1$.

The proof of Proposition 1 appears in Chapter 6, proof of Corollary 2.3, pp. 233-234 of Garnett's book for the case $r = 1$ [G]. To prove the result for $r > 1$ requires only a few obvious modifications of Garnett's proof. The John-Nirenberg theorem shows that $BMO = BMO(1)$.

2.3. One application of the $\underline{\overline{X}}_{(r)}$ duality result proved in Section 1 is a new way to prove that a function lies in the exponential square class (or, indeed, in any of the $\exp L^r$ classes, $r = 2$ would be of particular interest). If $\left|\int f(x) h(x) \, dx\right| \leq C \|h\|_{\underline{\overline{X}}_{(r)}}$ for all $h \in \underline{\overline{X}}_{(r)}$, then $f \in \underline{\overline{X}}^*_{(r)}$ and for $r = 2$, $\underline{\overline{X}}^*_{(2)} = BMO(2) = \exp L^2$, by Proposition 1.

2.4. S. Janson defined a grid of function classes in BMO that give an exact determination of the distance of a function in BMO to L^∞ (mentioned on p. 243 [CWW]). This can be done by defining, as in Garnett, p. 258 (6.6) for the $r = 1$ case,

$$dist(f, L^\infty)_{BMO(r)} = \inf\{\varepsilon : \frac{|\{x \in Q : |f(x) - f_Q| > \lambda\}|}{|Q|} \leq \exp\{\frac{-\lambda^r}{\varepsilon^r}\} \text{ for all } Q, \text{ all } \lambda > \lambda(\varepsilon)\}.$$

($\lambda(\varepsilon)$ can become arbitrarily large as $\varepsilon \downarrow 0$.) So if $f \in BMO(r)$ as defined here, then for all $s > r$, $dist(f, L^\infty)_{BMO(s)} = 0$ (see also [GJ] in connection with this).

Two open problems of interest are:

(1) to find a maximal function characterization for $\underline{\overline{X}}_{(r)}, r > 1$;
(2) to determine exactly how singular integrals act on the spaces $\underline{\overline{X}}_{(r)}$ and $BMO(r)$.

So far the author and J. M. Wilson have found a partial result in a dyadic maximal function characterization for dyadic $\underline{\overline{X}}_{(r)}$. The author would like to thank J. M. Wilson for helpful conversations concerning atomic decompositions and H^p spaces.

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Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003

E-mail address: csweezy@nmsu.edu

URL: http://www.math.nmsu.edu