THE COMPLEXITY OF THE WORD-PROBLEM
FOR FINITE MATRIX RINGS

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Abstract. We analyze the so-called word-problem for $M_2(Z_2)$, the ring of $2 \times 2$ matrices over $Z_2$. We prove that the term-equivalence problem for the semigroup (and so for the ring) $M_2(Z_2)$ is coNP-complete.

1. INTRODUCTION

In this paper we study the computational complexity of the word-problem for $M_2(Z_2)$. We shall use the standard notation for computational complexity, as P, NP, coNP, etc.

The word-problem for an algebra $\mathcal{A}$ has two different versions for terms and for polynomials. We call an expression term if it contains only variables and we call it polynomial if it may contain elements of $\mathcal{A}$. The term-equivalence problem over $\mathcal{A}$ (TERM-EQ $\mathcal{A}$) asks whether two given terms agree for every substitution. For example $x^6$ and $id$ are terms over the group $S_3$, and they are equivalent because the exponent of $S_3$ is 6. The polynomial-equivalence problem (POL-EQ $\mathcal{A}$) asks the same for polynomials, for example $x(1, 2)yx^2(1, 2, 3)$ and $x^2y$ are polynomials over the group $S_3$, but they are not equivalent, e.g. by substituting $x = y = id$ the two values are not equal. Here $(1, 2)$ denotes the transposition flipping 1 and 2 and $(1, 2, 3)$ denotes the 3-cycle mapping 1 to 2, 2 to 3 and 3 to 1.

Let TERM-SAT $\mathcal{A}$ and POL-SAT $\mathcal{A}$ denote the term- and polynomial-satisfiability problems, respectively. The instance of TERM-SAT (POL-SAT) is a term (polynomial) $t$ and an element $a \in \mathcal{A}$. The question is whether there is an evaluation of $t$ such that $t = a$. Observe that for any finite algebra, TERM-EQ and POL-EQ are both in coNP and TERM-SAT (POL-SAT) is in NP.

2. PRELIMINARIES

We present a few recent results for some algebraic structures.

It is already known [3] that for a commutative ring $\mathcal{R}$ the TERM-EQ problem is in P if $\mathcal{R}$ is nilpotent and coNP-complete otherwise. Burris and Lawrence proved in [2] that the same holds for rings in general. Following their proof it is easy to see that for a nilpotent ring $\mathcal{R}$ the problem POL-EQ $\mathcal{R}$ is in P. A straightforward
consequence of their result is that if the ring is not nilpotent, then \( \text{POL-EQ } R \) is \text{coNP-complete}.

So, for example, for the ring \( \mathbb{Z}_2 \), the \text{coNP-completeness} of \( \text{TERM-EQ } \mathbb{Z}_2 \) is an easy consequence of the \text{NP-completeness} of 3-SAT. But the proof uses high powers of sums. This is the reason why Willard and Lawrence introduced the \( \mathbb{Z}_2 \) version of the problems, where every polynomial is a sum of monomials, e.g., an expression of the form \((x+y)^n\) is too long when expanded. The \( \text{TERM}_{\mathbb{Z}_2} \)-EQ (\( \text{POL}_{\mathbb{Z}_2} \)-EQ) problem asks whether two terms (polynomials), \( p \) and \( q \) — that are sums of monomials — are equal at every substitution. They proved in [5] that

**Theorem 1.** Let \( R \) be a ring and \( J(R) \) denote its Jacobson radical.

If \( R/J(R) \) is commutative, then \( \text{TERM}_{\mathbb{Z}_2} \)-EQ \( R \) is in \( P \).

If \( R = M_n(F) \) is a finite matrix ring whose invertible elements form a non-solvable group, then \( \text{TERM}_{\mathbb{Z}_2} \)-EQ \( R \) is \text{coNP-complete}. That is, if \( n \geq 3 \) or \( |F| \geq 4 \), then \( \text{TERM}_{\mathbb{Z}_2} \)-EQ \( M_n(F) \) is \text{coNP-complete}.

They ask what happens for \( n = 2 \) and \( |F| = 2,3 \) (see Problem 2 in [5]). We examine this question in Section 4.

The group case is only partially solved. An unpublished result of Lawrence and Burris is the following:

**Theorem 2.** Let \( G \) be a group. If \( G \) is nilpotent, then \( \text{TERM-EQ } G \) is in \( P \). If \( G \) is non-solvable, then \( \text{TERM-EQ } G \) is \text{coNP-complete}.

The answer for semigroups is less complete. In [11] the authors show for a special class of aperiodic monoids that the \( \text{POL-EQ} \) problem is tractable. In [7] the authors prove that

**Theorem 3.** \( \text{POL-EQ } M_n(F) \) and \( \text{POL-SAT } M_n(F) \) are \text{coNP-complete}.

It is also shown that

**Theorem 4.** Let \( S \) be a combinatorial 0-simple semigroup. Then \( \text{POL-EQ } S \), \( \text{POL-SAT } S \), \( \text{TERM-EQ } S \), and \( \text{TERM-SAT } S \) are in \( P \).

In [6] V. Yu. Popov and M. V. Volkov exhibit a semigroup of size \( 2^{1700} \) with a \text{coNP-complete} \( \text{TERM-EQ} \) problem. Later Kisieliewicz in [14] presented an example of size a few hundred. In this paper we investigate the word-problem for the matrix semigroup \( M_2(Z_2) \). We prove in Section 4 that

**Theorem 5.** \( \text{TERM-EQ} \) is \text{coNP-complete} for the semigroup \( M_2(Z_2) \).

This result not only provides a 16 element example of a semigroup with \text{coNP-complete} word problem that is significantly smaller than the previously known examples, but also, as an easy corollary we get that the \( \text{TERM}_{\mathbb{Z}_2} \)-EQ is \text{coNP-complete} for the matrix ring as well. Moreover, following the idea of the proof we exhibit an example of a semigroup of size 13 with a \text{coNP-complete} word problem.

### 3. Combinatorial completely 0-simple semigroups

We give a description of combinatorial completely 0-simple semigroups. Let \( M \) be a 0-1 matrix such that each row and column contains at least one 1 entry. We define \( S = S_M \), the completely 0-simple semigroup belonging to the regular matrix \( M \). Let \( A \) and \( I \) denote the index set for the rows and columns of \( M \). The
underlying set of $S$ consists of all pairs of the form $\langle i, \lambda \rangle$ where $i \in I$, and $\lambda \in \Lambda$, along with 0. An associative multiplication is given by the following rule:

$$\langle i, \lambda \rangle \langle j, \mu \rangle = \begin{cases} \langle i, \mu \rangle & \text{if } M(\lambda, j) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note, that for $a \in S_M$, $a^2 = a$ or 0 according to the matrix $M$, and for every case $a^2 = a^3 = a^4 = \ldots$.

Example 6. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

where $\Lambda = I = \{1, 2, 3\}$. Then $\langle 1, 2 \rangle \langle 3, 1 \rangle = \langle 1, 1 \rangle$ as $A(2, 3) = 1$ and $\langle 1, 2 \rangle \langle 2, 1 \rangle = 0$ as $A(2, 2) = 0$. In $S_A$ the product $\langle i, \lambda \rangle \langle j, \mu \rangle = 0$ if and only if $\lambda = j$.

Note that the completely 0-simple semigroups are also called Rees-matrix semigroups. We continue with an observation:

Lemma 7. Let $S = S_M$ be a combinatorial Rees-matrix semigroup with elements $a_1, \ldots, a_n \in S$, where $a_j = \langle i_j, \lambda_j \rangle$ for $1 \leq j \leq n$.

1. $a_1 \cdots a_n = 0$ if and only if there exists $k$, $1 < k \leq n$, such that $a_{k-1}a_k = 0$.
2. If $a_1 \cdots a_n \neq 0$, then $a_1 \cdots a_n = \langle i_1, \lambda_1 \rangle \cdots \langle i_n, \lambda_n \rangle = \langle i_1, \lambda_n \rangle$.

4. The semigroup $M_2(Z_2)$

We split the semigroup $M_2(Z_2)$ into two parts: the group of invertible matrices and the multiplicative semigroup of the 10 singular matrices.

The group of invertible matrices is isomorphic to the symmetric group $S_3$. An isomorphism is given by simply the action of the matrix on the 3 nonzero vectors of the vectorspace $Z_2^2 = \langle \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \text{id} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \langle 1, 3 \rangle \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \to \langle 2, 3 \rangle \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \to \langle 1, 2, 3 \rangle \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \langle 1, 3, 2 \rangle \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \langle 1, 2 \rangle.$$

The semigroup of singular matrices ($L$) is isomorphic to the combinatorial 0-simple semigroup $S_A$, where $A$ was defined in Example 6. Indeed, let $v_1 = u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_3 = u_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in Z_2^2$.

Let us define the map $\phi$ from the semigroup of singular matrices in $M_2(Z_2)$ to $S_A$ in the following way:

$$\phi(v_i \cdot u_\lambda^T) = \langle i, \lambda \rangle \text{ and } \phi\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 0.$$ 

Here, $u_\lambda^T \cdot v_j = \begin{cases} 0 & \text{if } \lambda = j \\ 1 & \text{if } \lambda \neq j. \end{cases}$ Hence

$$(v_i \cdot u_\lambda^T) \cdot (v_j \cdot u_\mu^T) = v_i \cdot (u_\lambda^T \cdot v_j) \cdot u_\mu^T = \begin{cases} v_i \cdot u_\mu^T & \text{if } u_\lambda^T \cdot v_j \neq 0, \\ 0 & \text{if } u_\lambda^T \cdot v_j = 1. \end{cases}$$
This verifies that \( \phi \) is an isomorphism between the multiplicative semigroup of singular matrices of \( M_2(Z_2) \) and \( S_A \). These two isomorphisms extend a map between \( M_2(Z_2) \) and \( S_3 \cup S_A \) that is an isomorphism where the multiplication between the elements of \( S_3 \) and \( S_A \) is defined as follows. For \( \pi \in S_3 \) and \( \langle i, j \rangle \in S_A \) let

\[
\langle i, \lambda \rangle \pi = \langle i, \pi(\lambda) \rangle \quad \text{and} \quad \pi \langle i, \lambda \rangle = \langle \pi^{-1}(i), \lambda \rangle.
\]

It is easy to see that

Lemma 8. Let \( m_1, \ldots, m_n \in M_2(Z_2) = S_3 \cup S_A, \langle i, \lambda \rangle, \langle j, \mu \rangle \in S_A \) and \( \pi \in S_3 \).

1. \( m_1 \cdots m_n \in S_A \) if and only if there exists a \( k \) for which \( m_k \in S_A \), i.e. a product is in \( S_A \) if and only if at least one of the factors is in \( S_A \).

2. \( \langle i, \lambda \rangle \pi(j, \mu) = 0 \) if and only if \( \pi(\lambda) = j \).

We shall need the analogue of Lemma 7.

Lemma 9. Let \( a_1, \ldots, a_n \in S_A \) where \( a_j = \langle i_j, \lambda_j \rangle \) for \( 1 \leq j \leq n \) and \( \pi_1, \pi_2, \ldots, \pi_{n+1} \in S_3 \). Then

1. \( \pi_1 a_1 a_2 a_3 \cdots a_n a_{n+1} = 0 \) if and only if there exists \( k, 1 < k \leq n \), such that \( a_{k-1} a_k a_k = 0 \) (i.e. \( \pi_k(\lambda_{k-1}) = i_k \)).

2. If \( \pi_1 a_1 a_2 a_3 \cdots a_n a_{n+1} \neq 0 \), then

\[
\pi_1 a_1 \pi_2 a_2 a_3 \cdots a_n a_{n+1} a_{n+1} = \langle \pi_1^{-1}(i_1), \pi_{n+1}(\lambda_n) \rangle.
\]

Now, we prove Theorem 5.

Proof of Theorem 5. For every simple graph we exhibit two terms over \( M_2(Z_2) \), whose lengths are polynomial in the size of the graph, such that the graph is not 6-colorable if and only if the two terms are equivalent. Thus, we reduce the graph 6-coloring problem into TERM-EQ \( M_2(Z_2) \). Let \( \Gamma = \Gamma(V, E) \) be a simple graph (with no loops and double edges). Notice that any possible isolated vertex can be ignored, so we can assume that the graph does not contain isolated vertices. For every vertex \( j \in V \) we introduce a vertex-variable \( v_j \) and for every edge \( i \in E \) we introduce an edge-variable \( e_i \). We define a few terms. Let

\[
P = \prod_{i \in E} (x w_i^4)^6,
\]

\[
Q = \prod_{i \in E} (x w_i^3)^6,
\]

\[
H = \prod_{(i,j) \in E^2} (w_i w_j w_i^2 w_j^2)^6.
\]

Here

\[
w_i = e_i^5 v_j v_k^5 e_j v_k v_j^5,
\]

where \( i \) is the edge connecting the vertices \( j \) and \( k \). The arguments will neither depend on the order of \( v_j \) and \( v_k \) in the definition of \( w_i \) nor on the order in which the product defining \( H \) is arranged. The products \( P \) and \( Q \) are running through the edges of \( \Gamma \) in the same order. Finally, let

\[
p = PPP \times H,
\]

\[
q = PPQ \times H.
\]

We claim that \( p \equiv q \) if and only if \( \Gamma \) is not 6-colorable.
For this, first we analyze the expression \( w_i = e_i^5 v_j^6 v_k^6 e_i v_j v_k^6 \). If the variables \( e_i, v_j, v_k \) are all from the group \( S_3 \), then \( w_i = e_i^{-1} v_j^6 v_k^6 e_i v_j v_k^{-1} = [e_i, v_j v_k^{-1}] \), that is, the commutator of the group elements \( e_i \) and \( v_j v_k^{-1} \). In \( S_3 \) the commutator subgroup is \( A_3 = \{id, (1,2,3), (1,3,2) \} \). Since the centre of \( S_3 \) is trivial, we have the following.

**Lemma 10.** Fix \( i = \{j,k \} \in E \). Let us assume \( v_j, v_k, e_i, a \in S_3 \) and put \( w_i = [e_i, v_j v_k^{-1}] \). Then

1. \( a^6 = id; \)
2. \( w_i \in A_3, \) the commutator subgroup of \( S_3; \)
3. \( w_i^3 = id; \)
4. \( w_i^3 = w_i; \)
5. \( w_i \) stabilizes \( 1 \) if and only if \( w_i = id. \)
6. For \( u, v \in S_3 \) there is an \( e \in S_3 \) such that \( w = [e, uv^{-1}] \neq id, \) if and only if \( w^{-1} \neq id, \) that is, if and only if \( u \neq v. \)
7. If \( w_i \in S_3 \setminus \{id \} \), then the set \( \{w_i, w_i^2, id \} \) is transitive on \( \{1,2,3 \}. \)
8. If \( w_i \in S_3 \setminus \{id \} \), and \( s \in S_A, \) then \( sw_i sw_i^2 s^2 = 0. \)

Note that there exists at least one edge or vertex variable taking a value from \( S_A \) if and only if there exists at least one word \( w_i \) such that the value of \( w_i \) is in \( S_A. \)

We will distinguish some cases that are described in Table 1.

**Table 1.** The four different cases

<table>
<thead>
<tr>
<th>( x \in S_3 )</th>
<th>( \langle i, \lambda \rangle = x \in S_A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i \neq \lambda )</td>
<td>( i = \lambda )</td>
</tr>
<tr>
<td>( \exists w_j \in S_A )</td>
<td>( \exists w_i \in S_3 \setminus {id } )</td>
</tr>
<tr>
<td>( w_i \notin S_A \Rightarrow w_i = id )</td>
<td>Case 2</td>
</tr>
<tr>
<td>( p = q = 0 )</td>
<td></td>
</tr>
<tr>
<td>Case 3</td>
<td>( P = Q ) hence ( p = q )</td>
</tr>
<tr>
<td>( \forall w_j \in S_3 )</td>
<td>Case 1</td>
</tr>
<tr>
<td>( p = q = x )</td>
<td></td>
</tr>
<tr>
<td>Case 4a</td>
<td>( q = p^2 x = P^1 x = p )</td>
</tr>
<tr>
<td>Case 4b</td>
<td>6-coloring</td>
</tr>
</tbody>
</table>

In the following we will show that except for Case 4b \( p \) is always equivalent to \( q. \)

**Case 1:** When all variables are in \( S_3 \). If all variables are from \( S_3 \), then \( (ux_i^k)^6 = id \) for every edge and \( (w_j w_i w_i^j w_i^2)^6 = id \) for every pair of edges, hence both terms are equal to \( x. \)

**Case 2:** When there exists \( w_j \in S_A \) and there is an \( i \) such that \( w_i \in S_3 \setminus \{id \}. \)

The last item of Lemma 10 says the following: If there is an edge \( j \), such that \( w_j \in S_3 \setminus \{id \} \) and at least one \( w_j \in S_A, \) then \( H = 0, \) and so \( p = q = 0. \)

**Case 3:** When there exists \( w_j \in S_A \) and besides \( w_i \in S_A \cup \{id \} \) for every \( i \in E. \) In both cases \( w_i^4 = w_i^3 = w_i^2, \) hence the two terms are equal.

**Case 4:** When \( w_i \in S_3 \) for every \( i \in E \) and \( x \in S_A. \)

a) First, let \( x = \langle i, \lambda \rangle, \) where \( i \neq \lambda. \) In this case by item 3 of Lemma 10 \( H^3 = id \) and so either \( P = 0 \) or both sides are equal to \( x, \) hence the equation is obvious.
b) Finally, without loss of generality, we may assume that \( x = \langle 1, 1 \rangle \). Now, \( p = P P P x \) and \( q = P Q P x \) and — because of item 3 of Lemma 10 — \( Q = (1, 1)^k = 0 \), where \( k \geq 2 \), hence \( q = 0 \). Thus \( p \neq q \) if and only if there is a substitution, where \( p = P P P x \neq 0 \), by item 4 of Lemma 10 that holds if and only if \( x w_1 x w_2 x \cdots w_k x \neq 0 \). By Lemmas 3 and 9 we get that it is true if and only if none of the \( w_i \)'s stabilize 1, which is by item 5 of Lemma 10 equivalent to \( w_i \neq id \). Recall that \( w_i = e_i^{-1} v_j v_k^{-1} e_i v_k v_j^{-1} = [e_i, v_k v_j^{-1}] \), where \( e_i \) is the edge variable and \( v_k \) and \( v_j \) are the elements assigned to the endpoints of \( e_i \). According to item 6 of Lemma 10 we can choose an \( e_i \in S_3 \) such that \( w_i \neq id \) if and only if \( v_k \neq v_j \), that is, if and only if the group elements assigned to the neighbor vertices are distinct, that is, if and only if \( \Gamma \) is 6-colorable.

\[
\square
\]

**Corollary 11.** TERM-EQ and TERM_{\Sigma^*}-EQ are \( \text{coNP-complete} \) for the ring \( M_2(Z_2) \).

**Theorem 12.** There exists a 13 element semigroup \( T \) for which the TERM-EQ problem is coNP-complete.

**Proof.** Namely, let \( T = A_3 \cup S_A \) be a subsemigroup of \( M_2(Z_2) \). The proof is based on the same idea as the case of \( M_2(Z_2) \): for an arbitrary simple graph \( \Gamma \) with no isolated vertices we define the same polynomials as we did in the case of \( M_2(Z_2) \) with the difference that here we let \( w_i = v_j v_k^{-1} \). We will prove that \( p \neq q \) if and only if \( \Gamma \) is 3-colorable. We can claim a similar statement to Lemma 10.

**Lemma 13.** Let us assume that \( w_i \in A_3 \).

1. \( w_i^3 = id \);
2. \( w_i^3 = w_i \);
3. \( w_i \) stabilizes 1 if and only if \( w_i = id \), i.e. \( v_j \neq v_k \).
4. If \( w_i \in A_3 \setminus \{id\} \), then the set \( \{w_i, w_i^2, id\} \) is transitive on \( \{1, 2, 3\} \).
5. If \( w_i \in A_3 \setminus \{id\} \), and \( s \in S_A \), then \( sw_i s w_i^2 s^2 = 0 \).

Accordingly, we can distinguish the same cases and in these cases except for Case 4b the proof is word-by-word the same as for \( M_2(Z_2) \). For Case 4b again we may assume without loss of generality that \( x = \langle 1, 1 \rangle \). Here by item 1 of Lemma 13 \( q = 0 \). \( p = P P P x \neq q = 0 \) holds if and only if \( x w_1 x w_2 x \cdots x w_k x \neq 0 \). This is true if and only if none of the \( w_i \)'s stabilize 1, which is by item 3 of Lemma 13 equivalent to \( v_j \neq v_k \), that is, if and only if the group elements assigned to the neighbor vertices are distinct, that is, if and only if \( \Gamma \) is 3-colorable.

\[
\square
\]

5. **Further remarks**

At this point the following two problems arise.

**Problem 1.** Find the smallest semigroup for which the TERM-EQ is coNP-complete.

**Problem 2.** Find the computational complexity of TERM-EQ for the semigroup \( M_n(F) \).

However, it is not clear from the final version of the paper. During the proof we had the following interesting problem, which is the generalization of EQN* in some sense.
**Problem 3.** Given two sets of words \( \{w_1, w_2, \ldots, w_n\} \) and \( \{v_1, v_2, \ldots, v_m\} \) over \( G \leq S_k \), the symmetric group acting on the set \( \Omega = \{1, 2, \ldots, k\} \). For an evaluation of the variables and for \( l \in \Omega \) let \( I_l = \{w_1(l), w_2(l), \ldots, w_n(l)\} \) and \( J_l = \{v_1(l), v_2(l), \ldots, v_m(l)\} \). Find the complexity of the question whether the set-equation system \( I_1 = J_1, I_2 = J_2, \ldots, I_k = J_k \) holds for every evaluation.

If \( n = m = 1 \), then we get the word-problem for the fixed permutation group.

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**References**