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BOHR'S INEQUALITY FOR UNIFORM ALGEBRAS

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Abstract. We prove a uniform algebra analogue of a classical inequality of Bohr's concerning Fourier coefficients of bounded holomorphic functions. The classical inequality follows trivially.

Bohr's inequality says that if $f(e^{i\theta})$ is a continuous function on the unit circle with analytic Fourier series $\sum_0^\infty a_n e^{in\theta}$ and $\|f\|_\infty \leq 1$, then

$$\sum_0^\infty |a_n| \, r^n \leq 1$$

for $0 \leq r \leq \frac{1}{3}$ ($\frac{1}{3}$ is the best possible bound). See [3], [4], [5] and [7] for details. This inequality was discovered by Bohr [4] and used in a problem in number theory that had connections with Dirchlet series. As mentioned in [5], several mathematicians such as Wiener [4], Riesz [5], Schur [5], Sidon [8], and others discovered their own proofs of the inequality. However, these proofs were closely tied to the unit disk and to holomorphic functions.

Interest in the inequality was recently revived when Dixon [5], used it to settle in the negative the conjecture that if the non-unital von Neumann's inequality held for a Banach algebra, then it was necessarily an operator algebra. Bohr's inequality can also be derived by operator-theoretic arguments; see Paulsen-Popesu-Singh [7], where it is also established, with the help of Bohr's inequality, that every Banach algebra has an equivalent norm that satisfies the non-unital von Neumann inequality. In [7] we (along with Popescu) also established versions of Bohr's inequality in several other contexts including the situation for various higher-dimensional domains, for the non-commutative disc algbera as well as for the reduced and full group C^*-algebras of the free group on n generators. We also used a simple method to actually establish the inequality for a large class of holomorphic functions on the open unit disc. This method also sheds light on several multivariable situations as well. See Aizenberg [1], Boas and Khavinson [2], Boas [3], for further connections and references.

The purpose of this note is to show that a very general version of Bohr's inequality is valid in the context of uniform algebras from which the classical inequality mentioned above follows as a trivial case. The proof is elementary and straightforward with ideas from [7]. Let $\mathbf{X}$ be a compact Hausdorf space and $\mathbf{A}$ a uniform

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algebra on $\mathbf{X}$. Let ϕ be a fixed complex homomorphism on $\mathbf{A}$ and m a representing measure for ϕ. Let $\mathbf{H}^1(dm)$ be the closure of $\mathbf{A}$ in $\mathbf{L}^1(dm)$. Furthermore, let $\mathbf{A}_0 = \ker\phi$ and $\mathbf{H}_0^\infty$ be the weak star closure of $\mathbf{A}_0$ in $\mathbf{L}^\infty(dm)$. It is easy to see and well known that if $f \in \mathbf{H}^1$ and $\psi \in \mathbf{H}_0^\infty(dm)$, then $\int_X f\psi dm = 0$. See Gamelin [6, page 97].

Theorem. *Let f be in $\mathbf{H}^1(dm)$ such that* $\mathrm{Re}\ f \leq 1$ *and* $\int_X f dm \geq 0$. *Let $\{\psi_n\}_0^\infty$ be a sequence in $\mathbf{H}_0^\infty$ such that $\|\psi_n\| \leq 1$ and $\psi_0 = 1$. Let $a_n = \int_X f\overline{\psi_n}dm$. Then*

$$\sum_0^\infty |a_n|\, r^n \leq 1$$

for $0 \leq r \leq \frac{1}{3}$.

Proof. It is obvious that we need to prove the inequality when $r = \frac{1}{3}$. Since for each n, $\int_X \psi_n dm = 0$ (since $\psi_n \in \mathbf{H}_0^\infty$) we see that

$$\begin{aligned} -a_n &= \int_X (1-f)\overline{\psi_n}dm \\ &= \int_X (1-f)\overline{\psi_n}dm + \int_X \overline{(1-f)\overline{\psi_n}}dm \\ &= 2\int_X \mathrm{Re}(1-f)\overline{\psi_n}dm. \end{aligned}$$

Hence

$$\begin{aligned} |a_n| &\leq 2\int_X |\mathrm{Re}(1-f)|\,|\overline{\psi_n}|\,dm \\ &\leq 2\int_X |\mathrm{Re}(1-f)|\,dm \quad (\text{since } |\psi_n| \leq 1) \\ &= 2(1-a_0) \quad (\text{since } \mathrm{Re} f \leq 1). \end{aligned}$$

Clearly then

$$\sum_{n=1}^\infty \frac{|a_n|}{3^n} \leq 1 - a_0,$$

which proves the result. □

To capture the classical Bohr inequality, let $\mathbf{A}$ be the disk algebra, ϕ be evaluation at 0 and let $\psi_n(e^{i\theta}) = e^{in\theta}$.

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