CLOSED SIMILARITY LORENTZIAN AFFINE MANIFOLDS

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Abstract. A Sim$(n-1,1)$ affine manifold is an $n$-dimensional affine manifold whose linear holonomy lies in the similarity Lorentzian group but not in the Lorentzian group. In this paper, we show that a compact Sim$(n-1,1)$ affine manifold is incomplete. Let $h, i$ be the Lorentz form, and $q$ the map on $\mathbb{R}^n$ defined by $q(x) = (x, x)_L$. We show that for a compact radiant Sim$(n-1,1)$ affine manifold $M$, if a connected component $C$ of $\mathbb{R}^n - q^{-1}(0)$ intersects the image of the universal cover of $M$ by the developing map, then either $C$ or a connected component of $C - H$, where $H$ is a hyperplane, is contained in this image.

Introduction

An $n$-dimensional affine manifold $M$ is an $n$-dimensional differentiable manifold endowed with an atlas whose coordinate changes are locally affine maps. The affine structure of $M$ pulls back to its universal cover $\tilde{M}$ and defines on it an affine structure determined by a local diffeomorphism $D : \tilde{M} \to \mathbb{R}^n$, called the developing map. The developing map gives rise to a representation $h : \pi_1(M) \to Aff(\mathbb{R}^n)$, called the holonomy of the affine manifold. Its linear part $L(h)$, is called the linear holonomy of the affine manifold. We will say that the affine manifold is complete if and only if the developing map is a diffeomorphism. An $n$-affine manifold is said to be radiant if its holonomy fixes an element of $\mathbb{R}^n$.

We denote by $O(p,q)$ the subgroup of linear automorphisms of $\mathbb{R}^n$ that preserve a bilinear symmetric form of type $p,q$, and by $Sim(p,q)$ the group generated by $O(p,q)$ and the homotheties. An $O(p,q)$ affine manifold $M$ is an affine manifold $M$ such that the image of its linear holonomy $L(h)$ is a subgroup of $O(p,q)$. A $Sim(p,q)$ affine manifold $M$ is an affine manifold $M$ such that the image of its linear holonomy $L(h)$ is a subgroup of $Sim(p,q)$ and contains an element that is not in $O(p,q)$.

Let us consider the flat riemannian torus $T^n$. Bieberbach has shown that closed $O(n,0)$ affine manifolds are finitely covered by $T^n$. Using the notion of discompacity, Yves Carrière has shown that closed $O(n-1,1)$ affine manifolds are complete. It is obvious that a $Sim(n,0)$ affine manifold is incomplete, since an element of its holonomy that does not lie in $O(n,0)$ fixes an element of $\mathbb{R}^n$. There exist examples of complete $Sim(n-1,1)$ affine manifolds. Let us give one: Endow $\mathbb{R}^n$ with its...
basis \((e_1, \ldots, e_n)\) and with the Lorentzian product defined by
\[
\langle e_i, e_j \rangle_L = \begin{cases} 1; & 0 < i < n; \\
0; & i = j; \\
-1; & i = n.
\end{cases}
\]

We restrict this product to \(\mathbb{R}^2\). The affine map whose linear part is
\[
\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}
\]
in the basis \((e_1 + e_2, e_1 - e_2)\), and whose translation part is \(e_1 - e_2\) generates a group that acts properly and freely on \(\mathbb{R}^2\).

The goal of this paper is to study closed \(\text{Sim}(n-1,1)\) affine manifolds. First we show:

**Theorem 1.** A compact \(\text{Sim}(n-1,1)\) affine manifold is incomplete.

After, using the notion of discompacity, we show

**Theorem 2.** Let \(M\) be a closed radiant \(\text{Sim}(n-1,1)\) affine manifold. If a connected component \(C\) of \(\mathbb{R}^n - q^{-1}(0)\) intersects \(D(M)\), then either \(C\) is contained in \(D(M)\) or a connected component of \(C - H\), where \(H\) is a hyperplane.

Interesting structures of \(\text{Sim}(n-1,1)\) affine manifolds can be constructed using the work of Goldman on projective structures on surfaces; see [Gol]. For instance, a \(\text{Sim}(2,1)\) structure whose linear holonomy is Zariski dense in \(\text{Gl}(3,\mathbb{R})\) is given in this paper.

1. **Closed \(\text{Sim}(n-1,1)\) affine manifolds are incomplete**

The main goal of this part is to show that a closed \(\text{Sim}(n-1,1)\) affine manifold cannot be complete.

Let us suppose that there exists a complete closed \(\text{Sim}(n-1,1)\) affine manifold \(M\); \(M\) is the quotient of \(\mathbb{R}^n\) by a subgroup of affine transformations \(\Gamma\), whose linear part is contained in \(\text{Sim}(n-1,1)\).

**Lemma 1.1.** Let \(\gamma\) be an element of \(\Gamma\) whose linear part has a determinant < 1. Then there exists a basis \((e_1, \ldots, e_n)\) of \(\mathbb{R}^n\) such that the linear part of \(\gamma\) in this basis has the following form:
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \frac{1}{\lambda}B''
\end{pmatrix}
\]
where \(\lambda\) is a real number strictly superior to 1 in absolute value, and \(B''\) is a matrix that preserves the restriction of a euclidean product to the vector subspace generated by \(e_3, \ldots, e_n\).

**Proof.** We have supposed that the determinant of the linear part \(L(\gamma)\) of \(\gamma\) is strictly inferior to 1 in absolute value. This implies that there exists a real number \(\lambda > 1\) such that \(\lambda L(\gamma) = L(\gamma)'\), where \(L(\gamma)'\) is an element of \(O(n-1,1)\). The linear map \(L(\gamma)\) has 1 as an eigenvalue, since \(\gamma\) acts freely. We deduce that \(\lambda\) is an eigenvalue of \(L(\gamma)'\). We remark that \(L(\gamma)'\) has another eigenvalue \(\alpha\) whose module is different from 1 and the module of \(\lambda\) since the absolute value of its determinant is 1. If \(\alpha\) is not a real number, then \(\alpha\) and its complex conjugate \(\bar{\alpha}\) are eigenvalues associated to the complex eigenvectors \(u_1\) and \(u_2\). In this case the restriction of \(L(\gamma)'\) to the plane generated by \(u_1 + u_2\) and \(i(u_1 - u_2)\) is a euclidean similitude whose ratio is different from 1. This is impossible since \(L(\gamma)'\) lies in \(O(n-1,1)\). Let \(v_1\) and \(v_2\)
be the eigenvectors associated to \( \lambda \) and \( \alpha \), and let \( \langle , \rangle_L \) be the Lorentzian product preserved by the linear holonomy. We have:

\[
\langle v_1, v_1 \rangle_L = \langle v_2, v_2 \rangle_L = 0.
\]

We deduce that the restriction of \( \langle , \rangle_L \) to the plane \( P \) generated by \( v_1 \) and \( v_2 \) is nondegenerate and has signature \( (1, 1) \). This implies that the restriction of \( \langle , \rangle_L \) to the orthogonal \( W \) of \( P \) with respect to itself is a scalar product. The restriction \( B'' \) of \( L(\gamma)^f \) to \( W \) is an orthogonal linear map. We can suppose that its determinant is 1. We deduce that \( \alpha = \frac{1}{2} \).  

Up to a change of origin, we can suppose that \( \gamma(0) = (a_1, 0, \ldots, 0) \) where \( a_1 \) is a real number. The restriction \( B \) of \( L(\gamma) \) to the linear subspace generated by \( (e_2, \ldots, e_n) \) is strictly contracting. It is easy to show that the group generated by \( \gamma \) is not cocompact. So \( \Gamma \) contains another element \( \gamma_1 \) different from \( \gamma \).

**Lemma 1.2.** Let \( C \) be the linear part of \( \gamma_1 \). Then \( C(e_1) = e_1 + b \) where \( b \) lies in the linear subspace generated by \( e_2, \ldots, e_n \).

**Proof.** Let \( k \) be an element of \( \mathbb{N} \). Consider the element \( \gamma^k \gamma_1 \). Its linear part has 1 as an eigenvalue. The matrix of this linear part in the basis \( (e_1, \ldots, e_n) \) is \( A^k C \), where \( A \) is the matrix of the linear part of \( \gamma \). Let \( u_k \) be an eigenvector of \( A^k C \) associated to 1. We assume that the norm of \( u_k \) with respect to the euclidean scalar product defined by \( \langle e_i, e_j \rangle = \delta_{ij} \) is 1. Let \( u_k = (u_k^1, u_k^2) \). We have \( C(u_k) = (v_k^1, v_k^2) \), where \( u_k^1 \) and \( v_k^1 \) are elements of \( \mathbb{R} \), and \( u_k^2 \) and \( v_k^2 \) are elements of the vector space \( F \) generated by \( e_2, \ldots, e_n \). We have \( v_k^1 = u_k^1 \) since \( A^k(e_1) = e_1 \), and \( A^k \) preserves \( F \). Since \( B \) is strictly contracting, the norm of \( A^k(0, v_k^2) \) goes to 0 with respect to the euclidean norm. So \( u_k \) goes to \( e_1 \), and \( C(e_1) \), which is the limit of \( C(u_k) = A^{-k}(u_k) \), is \( e_1 + b \) where \( b \) is an element of \( F \).  

**Proof of the Theorem 1.** Let \( c \) be the translational part of \( \gamma_1 \) in the basis \( (e_1, \ldots, e_n) \). Put \( c = (c_1, \ldots, c_n) \). We have \( \gamma^k \circ \gamma_1 \circ \gamma^{-k}(0) = (c_1, B^k(-ka_1b + (c_2, \ldots, c_n))) \). Since \( B \) is contracting and the action of \( \Gamma \) is proper, we deduce that \( (c_2, \ldots, c_n) = b = 0 \). This will imply that the action of \( \Gamma \) preserves proper affine subspaces, which is impossible. See [FGH], Theorem 2.2.  

**Remark.** In contrast to the \( \text{Sim}(n, 0) \) affine manifolds (see [FG], Theorem 1), there exist compact \( \text{Sim}(n-1, 1) \) affine manifolds that are not radiant. Here is an example.

Endow \( \mathbb{R}^2 \) with the Lorentzian product \( (, ) \) such that \( (e_1, e_1) = (e_2, e_2) = 0 \) and \( (e_1, e_2) = 1 \).

Consider the subgroup \( \Gamma \) of \( \text{Aff}(\mathbb{R}^2) \) generated by the following transformations:

\[
\gamma_1(x, y) = (x + 1, y),
\]

\[
\gamma_2(x + y) = (x, 2y).
\]

The quotient of \( \mathbb{R} \times (\mathbb{R} - \{0\}) \) by \( \Gamma \) is a compact \( \text{Sim}(n-1, 1) \) affine manifold.

2. **On the universal cover of compact \( \text{Sim}(n-1, 1) \) affine manifolds**

In this part we are going to find properties of the universal cover of a closed radiant \( \text{Sim}(n-1, 1) \) affine manifold. We use the notion of discompacity defined by Carrière ([Ca], 2.2.1). Let us recall it.
We consider in \( \mathbb{R}^n \) the unit ball \( B_n \). The euclidean metric induces on closed subsets of \( \mathbb{R}^n \) the Hausdorff distance. Let \( G \) be a subgroup of \( \text{Gl}(n, \mathbb{R}) \), and let \( (g_p)_{p \in \mathbb{N}} \) be a sequence of elements of \( G \). The limit of the family \( (g_p(B_n) \cap B_n)_{p \in \mathbb{N}} \) converges in \( B_n \). It is a degenerated ellipsoid (see [Car]). The codimension of this ellipsoid is the discompactness \( d \) of the family \( (g_p)_{p \in \mathbb{N}} \). The discompactness of the group with respect to the euclidean metric is the smallest \( d \).

Obviously we cannot use the notion of discompactness in this form since the linear holonomy of our manifold may contain homotheties. Denote \( q : \mathbb{R}^n \to \mathbb{R}, x \to \langle x, x \rangle_L \). We can define in \( \mathbb{R}^n - q^{-1}(0) \) the metric

\[
(u, v) \mapsto \langle u, v \rangle = \frac{(u, v)_{\text{euc}}}{q(x)}
\]

where \( u \) and \( v \) are vectors of the tangent space at \( x \), and \( \langle \cdot, \cdot \rangle_{\text{euc}} \) is the euclidean scalar product.

**Theorem 2.1.** Let \( \hat{x} \) be an element of \( \hat{M} \), \( u \) and \( v \) elements of \( T_{\hat{x}} \hat{M} \), such that the geodesics \( c_1 : [0, 1] \to \hat{M}, t \to \exp_{\hat{x}}(tu) \), and the one \( c_2 : [0, 1] \to \hat{M}, t \to \exp_{\hat{x}}(tv) \) are defined. Suppose that the elements \( \exp_{\hat{x}}(u) \) and \( \exp_{\hat{x}}(v) \) cannot be joined by a geodesic, but for every \( t, t' < 1 \), there is a geodesic between \( \exp_{\hat{x}}(tu) \) and \( \exp_{\hat{x}}(t'v) \). Let \( c : [0, 1] \to \mathbb{R}^n, t \to \exp_{D(x)}(tv) \) be the geodesic between \( D(\exp_{\hat{x}}(u)) \) and \( D(\exp_{\hat{x}}(v)) \), and let \( U_z \) be the domain of definition of \( \exp_{\hat{x}} \). Consider the element \( t_0 \in [0, 1] \) such that for every \( t < t_0, \exp_{D(x)}(tv) \) is an element of \( D(\exp_{\hat{x}}(U_z)) \), but \( \exp_{D(x)}(tv)(t_0w) = y \) is not an element of \( D(\exp_{\hat{x}}(U_z)) \). Then \( y \) is an element of \( q^{-1}(0) \).

**Proof.** There is a geodesic \( \hat{c}_3 : [0, 1] \to \hat{M}, t \to \exp_{\hat{x}}(tb) \) such that \( y \) is an element of the adherence of \( D(\hat{c}_3([0, 1])) \) and such that \( D(\hat{c}_3) \) is contained in the convex hull of \( D(\hat{c}_1) \) and \( D(\hat{c}_2) \), where \( \hat{c}_1 \) and \( \hat{c}_2 \) are geodesics of \( \hat{M} \) respectively above \( c_1 \) and \( c_2 \). Set \( p(\hat{x}) = x \). The image \( c_3 \) of \( p(\hat{c}_3) \) is a maximal incomplete geodesic of \( M \). Since \( M \) is compact, there exists an element \( z \) of \( M \) such that the geodesic \( c_3 \) is recurrent in an affine chart \( U \) that contains \( z \). We deduce as in Carrière the existence of a family of ellipsoids \( s_p \) of \( \mathbb{R}^n \) whose centers are elements of \( D(\hat{c}_3) \), such that for each \( p, p' \), there is an element \( \gamma_{p, p'} \) of the holonomy such that \( \gamma_{p, p'}(s_p) = s_{p'} \) and the centers \( x_p \) of \( s_p \) go to \( y \).

Suppose that \( y \) is not an element of \( q^{-1}(0) \). Let \( z_p \) be an element of an ellipsoid \( s_p \), and let \( u_p, v_p \) be two vectors in its tangent space. Put \( \gamma_{p, p'} = \lambda_{p, p'} g_{p, p'} \) where \( g_{p, p'} \) is an element of \( O(n - 1, 1) \). We have

\[
\frac{\langle \gamma_{p, p'}(u_p), \gamma_{p, p'}(v_p) \rangle_{\text{euc}}}{q(\gamma_{p, p'}(x))} = \frac{\langle g_{p, p'}(u_p), g_{p, p'}(v_p) \rangle_{\text{euc}}}{q(x)},
\]

since the holonomy of \( M \) is supposed to be radiant.

The metrics \( \langle \cdot, \cdot \rangle_{\text{euc}} \) and \( \langle \cdot, \cdot \rangle' \) are equivalent in a neighborhood of \( y \) since \( q(y) \) is different from \( 0 \). We know that the discompactness of the family of \( g_p \) with respect to the riemannian metric \( \langle \cdot, \cdot \rangle_{\text{euc}} \) is 1. The family of ellipsoids \( s_p \) goes to an ellipsoid, or a codimension 1 degenerated ellipsoid centered in \( y \). We conclude as in Carrière that \( y \) must be an element of \( D(\exp_{\hat{x}}(U_z)) \). This is not possible; so \( q(y) = 0 \).

A similar result is given in [Gol].

**Corollary 2.2.** Let \( M \) be a compact radiant \( \text{Sim}(n - 1, 1) \) affine manifold, and let \( \hat{x}, u, \) and \( v \) be respectively elements of \( \hat{M} \) and \( T_{\hat{x}} \hat{M} \), such that \( \exp_{\hat{x}}(u) \) and \( \exp_{\hat{x}}(v) \)
are defined. If the convex hull $E$ of $(D(\hat{x}), D(exp_{\hat{x}}(u)), D(exp_{\hat{x}}(v)))$ is contained in a connected component of $\mathbb{R}^n - q^{-1}(0)$, then it is contained in $D(exp_{\hat{x}}(U_\hat{z}))$.

Proof. Suppose that $E$ is not contained in $D(exp_{\hat{x}}(U_\hat{z}))$. Let $y$ and $z$ be two elements of $E \cap D(exp_{\hat{x}}(U_\hat{z}))$ such that $y = D(exp_{\hat{x}}(u_1)), z = D(exp_{\hat{x}}(u_2))$, and for every $t_1, t_2 < 1$, $exp_{\hat{x}}$ is defined on the convex hull of $0, tu_1, tu_2$, but the elements $exp_{\hat{x}}(u_1)$ and $exp_{\hat{x}}(u_2)$ cannot be joined by a geodesic. Consider the geodesic $c : [0, 1] \to \mathbb{R}^n, t \to exp_{\hat{y}}(tw)$ between $y$ and $z$. There exists a real number $0 < t_0 < 1$, such that for $0 < t < t_0$, $exp_{\hat{y}}(tw)$ lies in $D(exp_{\hat{x}}(U_\hat{z}))$, but $exp_{\hat{y}}(t_0w)$ does not lie in $D(exp_{\hat{x}}(U_\hat{z}))$. We deduce from Theorem 2.1 that $exp_{\hat{y}}(t_0w)$ must lie in $q^{-1}(0)$. This is contrary to the hypothesis.

More generally, we can determine the boundary of the image of the developing map of a compact radiant $Sim(n-1, 1)$ affine manifold. More precisely, we have the following proposition, which implies Theorem 2.

Proposition 2.3. Let $M$ be a compact radiant $Sim(n-1, 1)$ affine manifold whose developing map is injective. Then the boundary of $D(\hat{M})$ is contained in the union of $q^{-1}(0)$ and a hyperplane.

Proof. As in Car, p. 625, one can remark that elements of the boundary of $D(\hat{M})$ that are not elements of $q^{-1}(0)$ are limits of $(\gamma_n e)_{n \in \mathbb{N}}$, where $\gamma_n$ is an element of the holonomy and $e$ is an ellipsoid. We conclude that those elements are contained in at most two hyperplanes $H_1, H_2$. The case of two hyperplanes is impossible, since those hyperplanes are stable by the holonomy. The affine function $\alpha$ such that $\alpha(H_1) = 0$ and $\alpha(H_2) = 1$ will be invariant by the holonomy and so will define a differentiable function on $M$ without a maximum. (It is the same argument used in Car).

Proposition 2.4. Let $M$ be a compact radiant affine manifold. If the image of the developing map is a convex set contained in an open set of $\mathbb{R}^n - q^{-1}(0)$, then the developing map is injective.

Proof. Let $\hat{x}$ be an element of $\hat{M}$. For all elements $u$ and $v$ of $U_{\hat{x}}$, the convex hull of $D(\hat{x})$ is a subset of $D(\hat{M}) \cap (\mathbb{R}^n - q^{-1}(0))$, where $y = D(exp_{\hat{x}}(u))$ and $z = D(exp_{\hat{x}}(v))$. We deduce from Corollary 2.2 that $y$ and $z$ are elements of $D(U_{\hat{z}})$. This implies that $U_{\hat{z}}$ is a convex set. We can conclude the proof by using Kos.

A particular case of the situation of Corollary 2.4 is the following: endow a compact oriented surface $S$ of genus $> 2$ with a hyperbolic structure, and consider $q$, the Lorentzian form defined on $\mathbb{R}^3$, by $q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$. The hyperbolic structure can be defined by a representation of the fundamental group of $S$, $\pi_1(S) \to O(2, 1)$ such that the quotient of $H = q^{-1}(-1)$ by $\pi_1(S)$ is $S$. The quotient of $W = \{x : q(x)(0, x_3)\} = 0$ by the group generated by $\pi_1(S)$, and a homothety of ratio $0 < \lambda < 1$, is a compact $Sim(n-1, 1)$ affine manifold whose universal cover is $W$.

More generally we have

Corollary 2.5. Let $M$ be a radiant compact affine manifold such that the image of its developing map is contained in $W = \{x : q(x)(0, z)\}$. $M$ is the quotient of a connected component of $W - H$ by a discrete group of $Sim(n-1, 1)$, where $H$ is a hyperplane of $\mathbb{R}^n$. 

Proof. We remark that the interior of a connected component of $W - H$ is convex. This implies that the image of the developing map is a convex set. The result follows by using Propositions 2.3 and 2.4.

Let $M$ be a compact radiant $Sim(n - 1, 1)$ affine manifold. The foliation $D(\tilde{F}_{q})$ of $\mathbb{R}^n - \{0\}$ whose leaves are the submanifolds defined by $q = \text{constant}$ is invariant by the holonomy of $M$. Its pullback on $\tilde{M}$ defines a foliation $\tilde{F}_{q}$ of $\tilde{M}$, which gives rise to a foliation $\mathcal{F}_{q}$ of $M$. If $D(\tilde{N}) = D(\tilde{M}) \cap q^{-1}(0)$ is not empty, then $N = p(D^{-1}(D(\tilde{N})))$ is a compact submanifold of $M$. Note that the 1-parameter group $\phi_t$, generated by the radiant vector field, preserves the foliation $\mathcal{F}_{q}$, and is transverse to all the leaves but not to the connected components of $N$.

**Proposition 2.6.** If $N$ is empty, then $M$ is the total space of a bundle over $S^1$.

**Proof.** If $N$ is empty, then $\phi_t$ is transverse to the foliation $\mathcal{F}_{q}$. This implies that this foliation is a Lie foliation. We conclude the proof by using [God], Corollary 2.6, p. 154.

**Remark.** If the image of the developing map is contained in $\{x/q(x) < 0\}$, one may expect that the fibers of the previous fibration are hyperbolic manifolds.

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