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CONTACT STRUCTURES ON ELLIPTIC 3-MANIFOLDS

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Abstract. We show that an oriented elliptic 3-manifold admits a universally tight positive contact structure if and only if the corresponding group of deck transformations on S^3 (after possibly conjugating by an isometry) preserves the standard contact structure.

We also relate universally tight contact structures on 3-manifolds covered by S^3 to the isomorphism $SO(4) = (SU(2) \times SU(2))/\pm 1$.

The main tool used is equivariant framings of 3-manifolds.

A contact structure ξ on a 3-dimensional manifold M is a smooth, totally non-integrable tangent plane field, i.e., a tangent plane field ξ locally of the form $\xi = ker(\alpha)$ for a 1-form α such that $\alpha \wedge d\alpha$ is everywhere non-degenerate. We shall assume that M is oriented. We say ξ is positive if the orientation on M agrees with that induced by the volume form $\alpha \wedge d\alpha$. Observe that the orientation of $\alpha \wedge d\alpha$ does not depend on the sign of α, and is thus determined by ξ (even though $\xi = ker(\alpha)$ only locally).

A central role in understanding 3-dimensional manifolds has been played by co-dimension-one structures – surfaces, foliations and laminations – in these manifolds. Without additional conditions, such structures always exist and are of not much consequence. However, the presence of *essential* co-dimension-one structures – *incompressible* surfaces, *taut* foliations and *essential* laminations, leads to deep topological consequences.

By the work of Eliashberg ([1], [2]), there is a similar dichotomy among contact structures between *tight* contact structures and *overtwisted* contact structures (we recall the definitions in the next section). Furthermore, there are deep connections between taut foliations and tight contact structures by the work of Eliashberg and Thurston [3], and more recently of Honda, Kazez and Matić ([9], [10]).

There is however one significant difference between contact structures and the other co-dimension-one structures – while one of the most basic consequences of the existence of other essential co-dimension-one structures in M is that the universal cover of M is $\mathbb{R}^3$ (this was, in fact, used to demonstrate the utility of essential laminations when they were introduced), one of the most basic examples of a tight contact structure is the *standard* contact structure on S^3 (which we recall in the next section). Several quotients of S^3 also admit tight contact structures.

Thus, tight contact structures clearly reveal a different aspect of 3-manifolds, at least in some cases. Our main goal here is to relate the existence of contact

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structures on elliptic manifolds (i.e., quotients of S^3 by a group of isometries) with tight universal covers to the isomorphism $SO(4) = (SU(2) \times SU(2))/\pm 1$, and more generally to spherical structures.

Our first main result is the following.

Theorem 0.1. *Suppose $M = S^3/G$ where G is a group of isometries of S^3. Then the oriented manifold M has a positive contact structure with tight universal cover if and only if G (after possibly conjugating by an isometry) leaves invariant the standard contact structure on S^3.*

It is easy to deduce, using the isomorphism, which finite groups $G \subset SO(4)$ preserve the standard contact structure on S^3. The main content of this paper is to show that the other elliptic manifolds do not admit a positive contact structure with tight universal cover. The main tool we use is *equivariant framings*, introduced in [5], and the methods of this paper are essentially a straightforward extension of the ones in that paper. An alternative approach to the results we obtain is to use Gompf's invariants for tangent plane fields (see [7]). We refer to [5] for the relation between these and equivariant framings.

The same methods yield the following stronger result.

Theorem 0.2. *Suppose M is a 3-manifold with universal cover S^3, and suppose $\pi_1(M) \cong G$ for some $G \subset SO(4)$ that acts freely on $S^3 \subset \mathbb{R}^4$, G not cyclic. Then if there are positive contact structures with tight universal cover for both orientations of M, then there are positive contact structures with tight universal cover for both orientations of S^3/G. Moreover, G preserves the standard contact structure on S^3 (after possibly conjugating by an isometry).*

Thus the restrictions to existence of positive contact structures with tight universal cover on elliptic manifolds are essentially homotopy theoretic.

Remark 0.3. An elliptic 3-manifold $N = S^3/G$ with G not cyclic is determined up to a (not necessarily orientation-preserving) isometry by its fundamental group. Thus, the above result says that if M admits contact structures for both orientations, so does the unique elliptic N with the same fundamental group.

From another point of view, we conclude that if $M = S^3/G$ is an elliptic manifold, with $G \subset SO(4) = (SU(2) \times SU(2))/\pm 1$ not cyclic, then the image of G under projection onto the two factors is a *topological* property of M determined by contact structures on M.

There is an even closer connection between the isomorphism and *co-orientable* positive contact structures with tight universal cover whose Euler class is trivial on the quotient manifold as we see in the following theorem.

Theorem 0.4. *Suppose $M = S^3/G$ is an elliptic manifold and G is not a cyclic group. Then M admits a co-orientable positive contact structure with tight universal cover whose Euler class is trivial if and only if $G \subset SU(2) \times 1$.*

1. Preliminaries

We recall below basic definitions and results in contact geometry that we need. For details and motivations we refer to [1]. Henceforth let M denote a closed oriented 3-manifold.

Definition 1.1. A contact structure ξ on M is a smooth tangent plane field that is locally of the form $\xi = ker(\alpha)$ for a 1-form α such that $\alpha \wedge d\alpha$ is everywhere non-degenerate. We say that ξ is positive if the orientation on M agrees with that induced by $\alpha \wedge d\alpha$.

An important example is the *standard contact structure* on S^3.

Example 1.2. Consider the tangent plane field ξ on $S^3 \subset \mathbb{C}^2$ that is perpendicular to the vector field $V : S^3 \to TS^3$, $V : (z_1, z_2) \mapsto (\boldsymbol{i}z_1, \boldsymbol{i}z_2)$. This is a positive contact structure called the *standard* contact structure on S^3.

Definition 1.3. A contact structure ξ on M is said to be *overtwisted* if there is an embedded disc $D \subset M$ so that $TD|_{\partial D} = \xi|_{\partial D}$. A contact structure that is not overtwisted is said to be *tight*.

Definition 1.4. A contact structure ξ on M is said to be *universally tight* if the pullback of ξ to the universal cover of M is tight.

Universally tight contact structures are tight since any cover of an overtwisted contact structure is overtwisted. This follows since the disc D in Definition 1.3 lifts to any cover.

Eliashberg [2] has shown that there is a unique overtwisted contact structure representing each homotopy class of tangent plane fields on a manifold M. This is far from true in the case of tight contact structures, and their existence is still mysterious.

The standard contact structure on S^3 is tight. This is essentially the only tight contact structure on S^3 by the following result of Eliashberg [2].

Theorem 1.5 (Eliashberg)**.** *Any positive tight contact structure on S^3 is isotopic to the standard one.*

2. Elliptic 3-manifolds

An elegant classification of elliptic 3-manifolds is obtained by Hopf [11] (see Scott [12] for a very readable account) using the isomorphisms

$$SO(4) = (SU(2) \times SU(2))/\pm 1 \quad \text{and} \quad SO(3) = SU(2)/\pm 1.$$

We outline this in this section. This has a transparent connection with contact geometry, which we shall exploit to construct contact structures. For proofs we refer to [12].

2.1. **The exceptional isomorphisms.** Consider the quaternions $\mathbb{H} = \{\omega_1 + \boldsymbol{j}\omega_2 : \omega_1, \omega_2 \in \mathbb{C}\}$. The group $SU(2) = S^3$ can be identified with the set of unit quaternions. This acts on the quaternions isometrically by left multiplication and by right multiplication, giving a surjective map $SU(2) \times SU(2) \to SO(4)$ with kernel $\{1, -1\}$. This gives the isomorphism $\phi : (SU(2) \times SU(2))/\pm 1 \xrightarrow{\cong} SO(4)$. The isomorphism $SO(3) = SU(2)/\pm 1$ is obtained by considering the action of S^3 on itself by conjugation.

Notice that $\mathbb{H}$ has a complex structure induced by right multiplication by $\mathbb{C} \subset \mathbb{H}$. The image of $S^3 \times S^1$ gives complex linear maps since these commute with every element of $1 \times \mathbb{C}$.

2.2. **The classification.** An elliptic manifold is a quotient of S^3 by a finite subgroup $G \subset SO(4)$ that acts without fixed points on the unit sphere. The above isomorphisms can be used together with the classification of finite subgroups of $SO(3)$ to classify such groups.

Recall that the finite subgroups of $SO(3)$ are the cyclic groups, dihedral groups and the groups of symmetries of the tetrahedron, octahedron and icosahedron. The inverse images of these groups under the covering $SU(2) \to SO(3)$ give cyclic groups, quaternionic groups Q_{4n}, the binary tetrahedral group T, the binary octahedral group O and the binary icosahedral group I respectively.

Using the fact that G acts freely, one can deduce the following proposition (this is Theorem 4.10 of [12]), which is important for our purposes.

Proposition 2.1. *G is conjugate to the image in $SO(4)$ of a subgroup $\tilde{G}$ of $S^1 \times S^3$ or of a subgroup of $S^3 \times S^1$.*

Suppose now that G is a subgroup of $\phi(S^1 \times S^3)$ that acts freely on S^3. We have the following classification.

Theorem 2.2 (Hopf)**.** *Suppose $G \subset \phi(S^1 \times S^3)$ acts freely on S^3. Then one of the following holds:*

(1) *G is cyclic;*
(2) *G is the product of a quaternionic group Q_{4n} in $\phi(1 \times S^3)$ with a cyclic group of relatively prime order in $\phi(S^1 \times 1)$;*
(3) *G is the product of $T \subset \phi(1 \times S^3)$ with a cyclic group of relatively prime order in $\phi(S^1 \times 1)$;*
(4) *G is the product of $O \subset \phi(1 \times S^3)$ with a cyclic group of relatively prime order in $\phi(S^1 \times 1)$;*
(5) *G is the product of $I \subset \phi(1 \times S^3)$ with a cyclic group of relatively prime order in $\phi(S^1 \times 1)$;*
(6) *G is the quotient under ϕ of an index 2 diagonal subgroup (for the definition of diagonal subgroups see* [12]*, page* 453*) of $C_{2m} \times Q_{4n}$, where m is odd and n and m are relatively prime;*
(7) *G is an index 3 diagonal subgroup of $C_{3m} \times T$, where m is odd.*

2.3. **Reversing orientations.** Suppose M is an oriented elliptic manifold of the form S^3/G. Then it follows by using, for instance, the orientation-reversing anti-isomorphism

$$\psi : \omega_1 + \boldsymbol{j}\omega_2 \mapsto \omega_1 + \boldsymbol{j}\bar{\omega_2}$$

that the oriented manifold $-M$ (M with the opposite orientation) is obtained by switching the left and right components of $G \subset S^3 \times S^3$.

3. Quotient contact structures

We have seen that the standard contact structure ξ on $S^3 \subset \mathbb{C}^2$ is characterised by being perpendicular to the vector field $V : (z_1, z_2) \mapsto (\boldsymbol{i}z_1, \boldsymbol{i}z_2)$. We can immediately deduce that several elliptic manifolds have quotient contact structures.

Lemma 3.1. *Suppose $M = S^3/G$ with $G \subset \phi(S^3 \times S^1)$. Then M has a co-orientable quotient contact structure induced by ξ.*

Proof. Any $g \in S^3 \times S^1$ acts by complex linear maps and hence preserves V. $\square$

The following is immediate.

Theorem 3.2. *Any elliptic manifold M admits a universally tight contact structure for at least one orientation of M.*

Proof. Let $M = S^3/G$. If $G \subset \phi(S^3 \times S^1)$, then the above lemma shows that M has such a contact structure. Otherwise $G \subset \phi(S^1 \times S^3)$, and hence the manifold $-M$ obtained by reversing the orientation on M is a quotient of S^3 by a subgroup of $S^3 \times S^1$, and hence has a universally tight contact structure. □

Allowing deck transformations that reverse the co-orientation, we can construct more quotient contact structures.

Lemma 3.3. *Suppose $M = S^3/G$ with $G \subset \phi(S^3 \times (S^1 \cup \boldsymbol{j}S^1))$. Then M has a quotient contact structure induced by ξ.*

Proof. Any $g \in S^3 \times (S^1 \cup \boldsymbol{j}S^1)$ acts by complex linear or anti-linear maps and hence either preserves V or takes V to $-V$. In either case ξ is preserved. □

Theorem 3.4. *The manifolds in cases 1, 2 and 6 of Theorem 2.2 admit universally tight contact structures in each orientation.*

Proof. Let M be such a manifold. In case 1, after conjugation, $\pi_1(M) \subset \phi(S^1 \times S^1)$.

Next, let $\zeta_n \in \mathbb{C}$ denote a primitive nth root of unity. In case 2, $\pi_1(M)$ is the subgroup of $SO(4)$ that is the image of $(\boldsymbol{j}, 1) \in S^3 \times S^3$ and $(\zeta_{2n}, 1) \in S^3 \times S^3$, or the product of this with the subgroup generated by $(1, \zeta_m)$ for some m. In case 6, $\pi_1(M)$ is the subgroup of $SO(4)$ that is the image of $(\boldsymbol{j}, \zeta_{2k}) \in S^3 \times S^3$ and $(\zeta_{2n}, 1) \in S^3 \times S^3$, or the product of this with the subgroup generated by $(1, \zeta_m)$ for some m.

Thus for one orientation $\pi_1(M) \subset \phi(S^3 \times S^1)$, and for the other $\pi_1(M) \subset \phi(S^1 \times (S^1 \cup \boldsymbol{j}S^1))$. In either case we get a quotient contact structure. □

The main content of this paper is that there is no universally tight contact structure for the remaining elliptic manifolds.

4. Equivariant framings

The main tool we use is the *equivariant framing* of a 3-manifold of the form S^3/G introduced in [5]. We outline below the relevant concepts and results.

We use the fact that the tangent bundle of an oriented 3-manifold is trivialisable. Let $M = S^3/G$, where G is a finite group acting without fixed points on S^3 (not necessarily by isometries).

We define an invariant $\mathfrak{F}(M)$ of M with a given orientation, which we call the *equivariant framing* of M. Recall that the homotopy classes of trivialisations of the tangent bundle of S^3 are a torseur of $\mathbb{Z}$ (i.e., a set on which $\mathbb{Z}$ acts freely and transitively), which can moreover be canonically identified with $\mathbb{Z}$ by using the Lie group structure of S^3 as the unit quaternions and identifying a left-invariant framing with $0 \in \mathbb{Z}$.

Now, find a trivialisation τ of TM and pull it back to one of TS^3. Under the above identification, this gives an element $\mathfrak{F}(M, \tau) \in \mathbb{Z}$. This depends on τ, but we can see that its reduction modulo $|G|$, when $H_1(M, \mathbb{Z}/2\mathbb{Z}) = 0$ (in particular when $|G|$ is odd), and modulo $|G|/2$ otherwise, is well-defined by the following straightforward proposition.

Proposition 4.1 (see [5])**.** *Suppose $\tau_i, i = 1, 2$ are trivialisations of TM and $\pi^*(\tau_i)$ are their pullbacks under the covering map $\pi : S^3 \to M$. Then $\pi^*(\tau_1) - \pi^*(\tau_2)$ is divisible by $|G|$ when $H_1(M, \mathbb{Z}/2\mathbb{Z}) = 0$ and by $|G|/2$ when $H_1(M, \mathbb{Z}/2\mathbb{Z}) \neq 0$.*

□

Using this, we define the invariant $\mathfrak{F}(M)$.

Definition 4.2. Let $M = S^3/G$, where G is a finite group acting without fixed points on S^3. The equivariant framing $\mathfrak{F}(M) \in \mathbb{Z}/\langle G\rangle\mathbb{Z}$, where $\langle G\rangle = |G|$ when $H_1(M, \mathbb{Z}/2\mathbb{Z}) = 0$ and $\langle G\rangle = |G|/2$ when $H_1(M, \mathbb{Z}/2\mathbb{Z}) \neq 0$, is the equivalence class of the trivialisation of TS^3 obtained by pulling back a trivialisation of TM.

The above definition does not depend on the identification of S^3 with the universal cover of M, since two such identifications differ by an orientation-preserving self-homeomorphism of S^3, which is isotopic to the identity.

We shall need the following results regarding the framing invariant.

Proposition 4.3. *Suppose $M = S^3/G$. If $G \subset \phi(S^3 \times 1)$ (respectively $G \subset \phi(1 \times S^3)$), then $\mathfrak{F}(M) = 0$ (respectively $\mathfrak{F}(M) = 1$).*

Proof. If $G \subset \phi(S^3 \times 1)$, since the left-invariant trivialisation is preserved by the action of G, it gives a framing τ on the quotient M. The pullback of τ is then the left-invariant framing, i.e., the 0 element.

On the other hand, the right-invariant trivialisation is preserved by the action of a group $G \subset \phi(1 \times S^3)$, and hence gives a framing τ' on $M = S^3/G$ that pulls back to the right-invariant trivialisation on M.

Thus, $\mathfrak{F}(M)$ is determined by the map $\psi : S^3 \to SO(3)$ that at a point $g \in S^3$ is the matrix of transition between the ordered basis $(\hat{u}, \hat{v}, \hat{w}) = (g\boldsymbol{i}, g\boldsymbol{j}, g\boldsymbol{k})$ of T_gS^3 corresponding to the left-invariant trivialisation, and the ordered basis $(\hat{u}', \hat{v}', \hat{w}') = (\boldsymbol{i}g, \boldsymbol{j}g, \boldsymbol{k}g)$ corresponding to the right-invariant trivialisation. More precisely, $F(M)$ is the degree of the lift of this map to the *comparison map* $\tilde{\psi} : S^3 \to S^3$.

Observe that $(\hat{u}', \hat{v}', \hat{w}') = g^{-1}(\hat{u}, \hat{v}, \hat{w})g$, and hence the map $\phi : S^3 \to SO(3)$ at the point g is the action by conjugation of g on the Lie algebra of S^3. Thus, ψ is the standard 2-fold covering map $\phi : S^3 \to SO(3)$ given by the adjoint action. It follows that $\tilde{\psi}$ is the identity map and hence has degree one. □

Corollary 4.4. *If $M = S^3/G$ with $G \subset \phi(S^3 \times 1)$ and $|G| > 2$, then no trivialisation of $-M$ pulls back to one isotopic to the left-invariant trivialisation of S^3.*

5. Nonexistence of contact structures

Suppose now that $M = S^3/G$ and ξ is a positive contact structure on M. We shall associate framings of M to certain contact structures. Note that the following proposition does not require G to act by isometries.

Proposition 5.1. *Let $M = S^3/G$ be a manifold with a positive, co-orientable contact structure ξ with trivial Euler class. Then there is a framing canonically associated to ξ. Furthermore, the pullback of this framing to any manifold that covers M is the framing induced by the pullback of the contact structure.*

Proof. Choose and fix a co-orientation for ξ. This induces an orientation on the plane-bundle given by the contact structure, which we identify with ξ.

Since the Euler class of ξ is trivial, there is a trivialisation of ξ as a vector bundle. Furthermore, two trivialisations differ by a map $M \to S^1$. Since $H^1(M) = 0$ (since $H_1(M) = G/[G, G]$ is finite), any such map is homotopic to a constant map.

Thus, there is a trivialisation (X_1, X_2) of ξ, canonical up to homotopy. This, together with a vector X_3 normal to ξ that is consistent with the co-orientation, gives a framing (X_1, X_2, X_3) of TM.

The homotopy class of this trivialisation does not depend on the choice of co-orientation since $(X_1, -X_2, -X_3)$ gives a trivialisation corresponding to the opposite co-orientation, and this is clearly homotopic to the trivialisation (X_1, X_2, X_3).

Since the trivialisation of ξ pulls back to give a trivialisation of the pullback to any cover of ξ, and the pullback of ξ is co-orientable and has trivial Euler class, the second claim follows. □

We can now prove our first nonexistence result. Let $\mathcal{P} = S^3/I$ be the Poincaré homology sphere with a fixed orientation, and let $-\mathcal{P}$ denote the same manifold with the opposite orientation. The Poincaré homology sphere is the quotient of S^3 by a group acting by left multiplication, and $-\mathcal{P}$ is the quotient of an action by right multiplication. We can now prove the following theorem.

Theorem 5.2 (Gompf, see also [5]). *The manifold $-\mathcal{P}$ does not have a universally tight positive contact structure.*

Proof. Let ξ be a positive contact structure on $-\mathcal{P}$. Since $-\mathcal{P}$ is a homology sphere, it follows that the first Stiefel-Whitney class of ξ vanishes, and hence ξ is co-orientable. Thus its Euler class is a well-defined element of $H^2(M) = 0$ and hence must vanish. Thus the hypotheses of Proposition 5.1 are satisfied.

Since $-\mathcal{P}$ is the quotient of S^3 by a group acting by right multiplication, it follows that any framing on $-\mathcal{P}$ pulls back to one homotopic to a framing invariant under right Lie multiplication, or one differing from this by $|\pi_1(\mathcal{P})|$ units (since $H_1(\mathcal{P}, \mathbb{Z}_2) = 0$). However, if $-\mathcal{P}$ had a universally tight positive contact structure ξ, then the pullback of ξ to S^3 is isotopic to the standard contact structure . Hence the framing associated to ξ pulls back to give the framing associated to left Lie multiplication. This contradicts Corollary 4.4. □

Remark 5.3. Etnyre and Honda [4] have shown that $-\mathcal{P}$ does not have a positive tight contact structure (and, in particular, does not have a positive universally tight contact structure).

Remark 5.4. Gompf has proved the above results using methods that can be seen to be essentially the same as those of this paper. For details of the relation to his methods, see [5].

We next consider the manifold $M = S^3/G$, where $G \subset \phi(S^1 \times S^3)$ is as in case 3, 4 or 7 of Theorem 2.2.

Theorem 5.5. *Suppose $M = S^3/G$, where $G \subset \phi(S^1 \times S^3)$ is as in case 3, 4, 5 or 7 of Theorem 2.2. Then M does not admit a universally tight positive contact structure.*

Proof. The result has been proved in case 5. Suppose next that we are in one of the cases 3 or 7.

Let ξ be a positive contact structure on M. In cases 3 and 7, $\pi_1(M)$ is the product of a group with the presentation

$$\langle x, y, z; x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1\rangle$$

with a cyclic group of order n for some n that is odd and not divisible by 3. By abelianising, we see that $H_1(M) = \mathbb{Z}/3^k n\mathbb{Z}$, which has odd order. Hence $H^1(M, \mathbb{Z}/2\mathbb{Z}) = 0$. Hence the first Stiefel-Whitney class of ξ vanishes, and ξ is co-orientable. Thus its Euler class is a well-defined element of $H^2(M)$.

Let M' be a cover of M corresponding to the subgroup generated by an element $g \in G$ of order 4 (such an element exists in these cases). Then the pullback map $H^2(M) \to H^2(M') = \mathbb{Z}/4\mathbb{Z}$ vanishes since $H^2(M) = H_1(M)$ has odd order. Thus the hypotheses of Proposition 5.1 are satisfied by the pullback ξ' of ξ to M'.

On the other hand, M' is the quotient of S^3 by a subgroup acting by right multiplication, and hence an argument similar to the icosahedral case gives a contradiction.

Finally, a manifold M corresponding to case 4 has a cover N with fundamental group $T \subset \phi(S^1 \times S^3)$. Hence M does not admit a positive universally tight contact structure since this would pull back to give a universally tight positive contact structure on N, contradicting the above. □

We are now in a position to complete the proof of our main results.

Proof of Theorems 0.1, 0.2 and 0.4. We first prove Theorem 0.1. Suppose $M = S^3/G$ with $G \subset SO(4)$ as in the hypotheses. For one of the orientations of the manifold M, we have seen that $\pi_1(M)$ fixes the standard contact structure on S^3.

For the other orientation, in the cases 1, 2 and 6 of Theorem 2.2 we have seen that $\pi_1(M)$ fixes the standard contact structure. On the other hand, Theorem 5.5 shows that in cases 3, 4, 5 and 7, the quotient manifold with one of its orientations does not admit a positive universally tight contact structure. This exhausts all the cases, proving the result.

We next prove Theorem 0.4. Suppose $M = S^3/G$, $G \subset SU(2) \times 1$. Then G fixes the standard contact structure on S^3 and hence induces a universally tight positive contact structure ξ on M. Furthermore, the section $g \mapsto g\boldsymbol{i}$ of the standard contact structure is fixed by G and hence descends to a section of ξ. Thus ξ has trivial Euler class.

Conversely, suppose $M = S^3/G$ and $G \not\subset SU(2) \times 1$ is not cyclic, and M admits a universally tight positive contact structure. By considering cases, it follows that some cover of M is of the form S^3/H, $H \subset \phi(1 \times SU(2))$, and $|H| > 2$. By the hypothesis that the Euler class of ξ is trivial and Proposition 5.1, the pullback of ξ to S^3/H has a framing associated to it. As before, this lifts to a framing isotopic to the left-invariant framing of S^3 implying $\mathfrak{F}(S^3/H) = 0$, which gives a contradiction since $H \subset \phi(1 \times SU(2))$ and hence $\mathfrak{F}(S^3/H) = 1$.

Remark 5.6. In the case of cyclic groups G, the above proof still shows that under the usual hypotheses, $\mathfrak{F}(M) = 0$. For prime order cyclic groups, by the results of [5] we still can conclude that $G \subset \phi(S^3 \times 1)$. Tight contact structures on lens spaces (i.e., G cyclic) have been completely classified by Giroux [6] and Honda [8].

To prove Theorem 0.2 we shall need the following proposition.

Proposition 5.7. *Suppose that an oriented manifold M of the form $M = S^3/\Gamma$, with Γ a finite group acting freely (but not necessarily isometrically) on S^3, has a framing that lifts to one isotopic to the left-invariant framing on S^3. Then $-M$ has a framing that lifts to one isotopic to the right-invariant framing on S^3.*

Proof. Let (X_1, X_2, X_3) be the framing on M that lifts to one isotopic to the left-invariant framing on S^3. Then $(-X_1, -X_2, -X_3)$ is a framing on $-M$ that lifts to a framing isotopic to a left-invariant framing on S^3 corresponding to the opposite orientation. Here the universal covering of $-M$ can be naturally identified with S^3 with the opposite orientation.

Composing the universal covering map of $-M$ with an orientation-reversing homeomorphism $\psi : S^3 \to S^3$, we get a covering map from S^3 with its usual orientation to $-M$ that respects orientations. The pullback $\mathfrak{F}$ of the framing on $-M$ is isotopic to the pullback under ψ of a left-invariant framing $(\hat{u}, \hat{v}, \hat{w})$ on S^3 corresponding to the opposite orientation. Since any two left-invariant framings corresponding to an orientation are equivalent, we can take $(\hat{u}, \hat{v}, \hat{w})_g = g(-\boldsymbol{i}, -\boldsymbol{j}, -\boldsymbol{k})$.

We use the orientation-reversing map $\psi : q \to \bar{q}$, where $q \to \bar{q}$ maps $\boldsymbol{i} \to -\boldsymbol{i}$, $\boldsymbol{j} \to -\boldsymbol{j}$, $\boldsymbol{k} \to -\boldsymbol{k}$ and $1 \to 1$ and is linear (over $\mathbb{R}$) on $\mathbb{H}$. This is an involution and is anti-linear over $\mathbb{H}$, i.e., $\overline{pq} = \bar{q}\bar{p}$.

Observe that $(\hat{u}, \hat{v}, \hat{w})_g = g(\bar{\boldsymbol{i}}, \bar{\boldsymbol{j}}, \bar{\boldsymbol{k}})$. Thus, since $\psi : q \to \bar{q}$ is an anti-linear involution, the pullback of $(\hat{u}, \hat{v}, \hat{w})_g = g(\bar{\boldsymbol{i}}, \bar{\boldsymbol{j}}, \bar{\boldsymbol{k}})$ under ψ at $\psi^{-1}(g) = \bar{g}$ is

$$(\boldsymbol{i}\bar{g}, \boldsymbol{j}\bar{g}, \boldsymbol{k}\bar{g}) = (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})\bar{g}.$$

This is precisely the right-invariant framing at this point corresponding to the usual orientation. □

We can now prove Theorem 0.2. Suppose $M = S^3/\Gamma$, with Γ a finite group acting freely (but not necessarily isometrically) on S^3. By hypothesis, $\Gamma \cong G$ as groups for some subgroup $G \subset SO(4)$, G not cyclic, that acts freely on S^3.

Suppose M admits a universally tight positive contact structure for each of its orientations. We shall show that G preserves the standard contact structure on S^3, and hence in particular S^3/G has a universally tight positive contact for each orientation. By the above, this is equivalent to showing that G is not in one of the cases 3, 4, 5 and 7.

We prove this by contradiction. Suppose G is in one of these cases. By replacing M by a two-fold cover in case 4 (the resulting M also satisfies the hypothesis as can be seen by pulling back contact structures on M), we can assume that we are in one of the cases 3, 5 and 7. Since M has a universally tight positive contact ξ, as before after passing to a cover N (with $|\pi_1(N)| \geq 4$) the pullback of ξ has trivial Euler class and hence a framing associated to it. This pulls back to a framing isotopic to a left-invariant framing on S^3 as ξ lifts to the standard contact structure on S^3. Thus, N has a framing that pulls back to the left-invariant framing of S^3.

But, since $-M$ also has a universally tight positive contact structure ξ', we can apply the same argument to $-M$. Thus, the pullback of ξ to $-N$ has associated with it a framing, and this pulls back to a framing isotopic to the left-invariant framing. By the above proposition, it follows that N has a framing that pulls back to the right-invariant framing of S^3. This contradicts Corollary 4.4. □

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