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COMPRESSIONS ON PARTIALLY ORDERED ABELIAN GROUPS

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ABSTRACT. If A is a C*-algebra and $p \in A$ is a self-adjoint idempotent, the mapping $a \mapsto pap$ is called a compression on A. We introduce effect-ordered rings as generalizations of unital C*-algebras and characterize compressions on these rings. The resulting characterization leads to a generalization of the notion of compression on partially ordered abelian groups with order units.

1. INTRODUCTION

Let A be a C*-algebra with unit 1, and let $p = p^2 = p^*$ be a projection in A. Then the mapping $J_p\colon A \to A$ defined by $J_p(a) := pap$ for all $a \in A$ is called the *compression* determined by p. If $d \in A$ and $a = J_p(d)$, then d is called a *dilation* of a [7]. By invoking various *dilation theorems*, one can dilate suitably structured families in A to more perspicuous structured families. For instance, by the Naimark (or Nagy-Naimark) dilation theorem [6, Chapter 2], a positive operator-valued measure on a Hilbert space can be dilated to a projection-valued measure.

In what follows, we denote by $G(A)$ the additive group of self-adjoint elements in the unital C*-algebra A. Then $G(A)$ can be organized into an archimedean partially ordered abelian group with positive cone $G(A)^+ := \{aa^* \mid a \in A\} = \{g^2 \mid g \in G(A)\}$, and the unit 1 is an order unit in $G(A)$. We denote by $P(A)$ the set of all projections $p \in A$. Then $0, 1 \in P(A) \subseteq G(A)$ and $P(A)$ acquires the structure of an orthomodular poset under the restriction of the partial order on $G(A)$. If $p \in P(A)$, then the compression J_p maps $G(A)$ into itself, and the restriction of J_p to $G(A)$, which we shall continue to denote by the symbol J_p, has the following properties: (1) $J_p\colon G(A) \to G(A)$ is an order-preserving endomorphism of the group $G(A)$, (2) $p = J_p(1) \leq 1$, (3) for all $e \in G(A)^+$, $e \leq p \Rightarrow J_p(e) = e$, and (4) J_p is idempotent. Our main result in this article is that, conversely, if $J\colon G(A) \to G(A)$ is an order-preserving endomorphism, $J(1) \leq 1$, and for all $e \in G(A)^+$, $e \leq J(1) \Rightarrow J(e) = e$, then $J = J_p$ where $p = J(1) \in P(A)$.

Properties (1)–(4) above make sense in any partially ordered abelian group with order unit, thus suggesting a generalized notion of compression that we shall call a *retraction* (Definition 2.1 below). As a consequence of Theorem 4.5 below, *every*

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retraction J on the partially ordered group $G(A)$ of self-adjoint elements of a C-algebra with unit* 1 *has the form $J = J_p$ where $p = J(1) \in P(A)$.*

A *partially ordered abelian group* is an additively written abelian group G with a distinguished subset G^+, called the *positive cone*, such that G^+ is closed under addition, $0 \in G^+$, and $g, -g \in G^+ \Rightarrow g = 0$. The positive cone determines a translation-invariant partial order $\leq$ on G according to $g \leq h \Leftrightarrow h - g \in G^+$ for $g, h \in G$, and in turn the partial order determines the positive cone according to $G^+ = \{g \in G \mid 0 \leq g\}$. If H is a subgroup of G, then H forms a partially ordered abelian group with the *induced* positive cone $H^+ := H \cap G^+$.

Let G be a partially ordered abelian group. An element $u \in G^+$ is called an *order unit* iff for every $g \in G$ there is a positive integer n such that $g \leq nu$. If $G = G^+ - G^+$, i.e., if G^+ generates G as a group, then G is said to be *directed*. If G has an order unit, then it is directed. A subgroup H of G is *order convex* iff $h_1, h_2 \in H, g \in G, h_1 \leq g \leq h_2 \Rightarrow g \in H$. The kernel of an order-preserving group homomorphism is order convex. A directed order convex subgroup of G is called an *ideal*. If, for $a, b \in G$, the condition $na \leq b$ holds for all positive integers n only if $a \leq 0$, then G is said to be *archimedean*. See [5, Chapters 1 and 2] for more details.

An element $u \in G^+$ is *generative* iff every element $g \in G^+$ can be written as a finite sum $g = e_1 + e_2 + \cdots + e_n$ with $0 \leq e_i \leq u$ for $i = 1, 2, ..., n$ [2, Definition 3.2]. If G is directed, then a generative element in G^+ is automatically an order unit.

1.1. **Definition.** A *unital group* is a partially ordered abelian group G together with a specified generative order unit $u \in G^+$ called the *unit* in G [3, Definition 2.5].

1.2. **Example.** If A is a unital C*-algebra, then $G(A)$ is an archimedean unital group with unit 1. □

2. Retractions on partially ordered abelian groups with order units

Abstracting from properties (1)-(4) in Section 1 of a compression J_p on the group $G(A)$ of self-adjoint elements of a unital C*-algebra A, we formulate the following definition.

2.1. **Definition.** Let G be a partially ordered abelian group with order unit u. A mapping $J : G \to G$ is called a *retraction* on G iff:

(i) $J : G \to G$ is an order-preserving group endomorphism.
(ii) $J(u) \leq u$.
(iii) If $e \in G$ with $0 \leq e \leq J(u)$, then $J(e) = e$.
(iv) J is idempotent.

If J is a retraction on G, then $J(u)$ is called the *focus* of J.

2.2. **Lemma.** *If G is a unital group, then condition (iv) in Definition* 2.1 *is redundant.*

Proof. Suppose G is a unital group, i.e., u is a generative order unit in G^+, and let $J : G \to G$ satisfy conditions (i)-(iii) in Definition 2.1. Let $g \in G^+$. Since u is generative, we can write $g = \sum_{i=1}^{n} e_i$ with $0 \leq e_i \leq u$ for $i = 1, 2, ..., n$. Then, for $i = 1, 2, ..., n$, $0 \leq J(e_i) \leq J(u)$, whence $J(J(e_i)) = J(e_i)$, and it follows that $J(J(g)) = J(g)$. Because G has an order unit, it is directed; so $G = G^+ - G^+$, and it follows that every element $g \in G$ satisfies $J(J(g)) = J(g)$. □

For the remainder of this section, we assume that J is a retraction on the partially ordered abelian group G with order unit $u \in G^+$ and that $p := J(u)$ is the focus of J. Thus, by parts (i) and (ii) of Definition 2.1, we have $0 \le p \le u$.

2.3. **Lemma.** *Let $e, f, k \in G$ with $0 \le e, f \le p \le u$. Then:*
(i) $e + p \le f + u \Leftrightarrow e \le f$. *(ii) $e \le u - p \Rightarrow e = 0$.*
(iii) $2p \le u \Leftrightarrow p = 0$. *(iv) $e + f \le u \Rightarrow e + f \le p$.*
(v) $0 \le k \le u - p \Rightarrow J(k) = 0$.

Proof. Since $0 \le e, f, p \le p = J(u)$, Definition 2.1 (iii) implies that $J(e) = e$, $J(f) = f$, and $J(p) = p$. (i) If $e+p \le f+u$, then $e+p = J(e)+J(u) \le J(f)+J(u) = f + p$; so $e \le f$. Conversely, since $p \le u$, we have $e \le f \Rightarrow e + p \le f + u$. (ii) Follows from (i) with $f = 0$. (iii) Follows from (ii) with $e = p$. (iv) If $e + f \le u$, then $e + f = J(e) + J(f) \le J(u) = p$. (v) $0 \le k \le u - p \Rightarrow 0 \le J(k) \le p - J(p) = p - p = 0$. □

Suppose that $p \in P(A)$ is a projection in the unital C*-algebra A, and let $k \in G(A)$ with $0 \le k \le u$. By Lemma 3.2 (v), $k \le 1 - p \Rightarrow J_p(k) = 0$, but in this case, we also have the converse $J_p(k) = 0 \Rightarrow k \le 1 - p$. This observation motivates the following.

2.4. **Definition.** The retraction $J \colon G \to G$ with focus $p = J(u)$ is a *compression* iff for all $k \in G$ with $0 \le k \le u$, $J(k) = 0 \Rightarrow k \le u - p$.

2.5. **Example.** Let $\mathbb{Z}$ be the additive group of integers with the standard positive cone $\mathbb{Z}^+$. Let $G := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with coordinatewise addition, partially ordered with the nonstandard cone $G^+ := \{(x, y, z) \in G \mid x, y, z \in \mathbb{Z}^+, x \le y + z\}$. Then G is an archimedean unital group with unit $u := (1, 1, 1)$. Define $J \colon G \to G$ by $J(x, y, z) := (0, 0, z)$ for all $(x, y, z) \in G$. Then J is a retraction with focus $p = J(u) = (0, 0, 1)$. But $k := (0, 1, 0)$ satisfies $(0, 0, 0) \le k \le u$ and $J(k) = (0, 0, 0)$; yet $k \not\le u - p = (1, 1, 0)$, and so J is not a compression. □

2.6. **Definition.** The retraction $J \colon G \to G$ is *direct* iff $g \in G^+ \Rightarrow J(g) \le g$.

2.7. **Example.** Let H and K be partially ordered abelian groups with order units v and w, respectively, and let $G := H \times K$ be organized into a partially ordered abelian group with coordinatewise operations, with positive cone $G^+ := H^+ \times K^+$, and with order unit $u := (v, w)$. Let $J, J' \colon G \to G$ be defined by $J(h, k) := (h, 0)$ and $J'(h, k) := (0, k)$ for all $(h, k) \in G$. Then both J and J' are direct compressions. □

We omit the straightforward proof of the following theorem.

2.8. **Theorem.** *Let $J \colon G \to G$ be a direct retraction, and define $J' \colon G \to G$ by $J'(g) := g - J(g)$ for all $g \in G$. Let $v := J(u)$, $w := J'(u)$, $H := J(G)$, $K := J'(G)$, and define $\Phi \colon H \times K \to G$ by $\Phi(h, k) := h + k$ for all $h \in H, k \in K$. Then: (i) J and J' are direct compressions. (ii) With the induced partial orders, H and K are partially ordered abelian groups, v is an order unit in H, and w is an order unit in K. (iii) $\Phi \colon H \times K \to G$ is an isomorphism of partially ordered abelian groups. (iv) H and K are ideals in G.*

An element $p \in G$ is said to be *characteristic* iff $0 \le p \le u$, the greatest lower bound $p \wedge (u - p)$ exists in G, and $p \wedge (u - p) = 0$ [5, Chapter 8].

2.9. **Theorem.** *Suppose G is an interpolation group* [5, Chapter 2]. *Then G is a unital group, the retraction $J\colon G \to G$ is a direct compression, and its focus $p = J(u)$ is a characteristic element of G. Conversely, if p is a characteristic element of G, there is a uniquely determined retraction $J\colon G \to G$ with $J(u) = p$.*

Proof. Suppose G is an interpolation group. By [5, Proposition 2.2 (b)], u is a generative order unit in G. By Lemma 2.3 (ii), if $e \in G$ with $0 \le e \le p, u-p$, then $e = 0$. We claim that the greatest lower bound $p \wedge (u-p)$ exists in G and equals 0. Clearly, $0 \le p, u-p$. Suppose $g \in G$ with $g \le p, u-p$. Then $g, 0 \le p, u-p$; so by interpolation there exists $e \in G$ with $g, 0 \le e \le p, u-p$. Thus, $e = 0$, so $g \le 0$, whence $p \wedge (u-p) = 0$, and it follows that p is a characteristic element of G. The remainder of the proof follows in a straightforward way from the development in [5, pp. 127-130]. □

As a consequence of Theorem 2.9 and [5, Theorem 8.7], the compressions on an interpolation group with order unit can be organized into a Boolean algebra.

3. Compressible groups

If A is a unital C*-algebra and $p \in P(A)$, then the compressions J_p and J_{1-p} on $G(A)$ are *quasicomplementary* in the following sense (cf. [1]).

3.1. **Definition.** Let G be a partially ordered abelian group with order unit u. Then the retractions J and I on G are said to be *quasicomplementary* iff, for all $g \in G^+$, $J(g) = g \Leftrightarrow I(g) = 0$ and $J(g) = 0 \Leftrightarrow I(g) = g$. If J and I are quasicomplementary retractions on G, we say that I is a *quasicomplement* of J and that J is a *quasicomplement* of I.

3.2. **Lemma.** *Let G be a unital group with unit u and unit interval E. Suppose that J and I are quasicomplementary retractions on G. Then:*

(i) If $p = J(u)$ is the focus of J, then the focus of I is $u - p$.
(ii) The images $J(G)$ and $I(G)$ of J and I are ideals in G.
(iii) J and I are compressions.

Proof. (i) We have $0 \le u-p$ with $J(u-p) = 0$, whence $I(u)-I(p) = I(u-p) = u-p$. But, $0 \le p$ with $J(p) = p$; so $I(p) = 0$, and it follows that $I(u) = u - p$.

(ii) Let $h \in H := J(G)$. Since G is directed, there exist $a, b \in G^+$ with $h = a-b$. Since J is idempotent, $h = J(h) = J(a) - J(b)$, and since J is order preserving, $J(a), J(b) \in H \cap G^+ = H^+$. Therefore, H is directed. Assume that $g \in G$ and $h \in H$ with $0 \le g \le h$. To prove that H is order convex in G, it is sufficient to show that $g \in H$. Since $0 \le h$ and $J(h) = h$, we have $0 \le I(g) \le I(h) = 0$, whence $I(g) = 0$. Therefore, $g = J(g) \in H$ and H is an ideal in G. By symmetry, $I(G)$ is an ideal in G.

(iii) Let $p = J(u)$; so $u - p = I(u)$ by (i). Suppose $e \in E$ with $J(e) = 0$. Then $I(e) = e$ and since $0 \le e \le u$, we have $0 \le e = I(e) \le I(u) = u - p$. Thus, J is a compression, and by symmetry, so is I. □

3.3. **Definition.** A *compressible group* is a unital group G such that every retraction on G is uniquely determined by its focus and every retraction on G has a quasicomplement. Let G be a compressible group with unit u. An element $p \in G$ is called a *projection* iff it is the focus $p = J(u)$ of a retraction J on G, and the set of all projections in G is denoted by $P(G)$. If $p \in P(G)$, we denote by J_p the unique retraction on G with focus p. Thus, $J_p(u) = p$.

3.4. **Lemma.** *Let G be a compressible group with unit u, and let $p \in P(G)$. Then every retraction on G is a compression, and the compression J_p has a unique quasicomplement, namely J_{u-p}.*

Proof. Assume the hypotheses. By Lemma 3.2 (iii), every retraction on G is a compression. Let I be a compression on G, and suppose that J_p is a quasicomplement of I. Then $0 \leq p, u-p \leq u$ and by Lemma 3.2 (i), I has focus $u-p$; so $u-p \in P(G)$ and $I = J_{u-p}$. □

The terminology "compressible group" is suggested by the notion that the compressions on such a group are particularly well-behaved. If G is an interpolation group with order unit as in Theorem 2.9, then G is a compressible group and $P(G)$ is a Boolean algebra. For a compressible group G it can be shown that $P(G)$ is always an orthomodular poset (OMP) and that every finite set of pairwise compatible elements of $P(G)$ is jointly compatible, i.e., is contained in a Boolean sub-OMP. In the next section we prove that, if A is a unital C*-algebra, then $G(A)$ is a compressible group. Therefore, compressible groups constitute a generalization of both interpolation groups with order unit and the additive group of self-adjoint elements of a unital C*-algebra.

4. Effect-ordered rings

The terminology in the following definition is suggested by the fact that, in certain approaches to the mathematical foundations of quantum mechanics, the self-adjoint elements between 0 and 1 in a unital C*-algebra are called *effects*.

4.1. **Definition.** An *effect-ordered ring* is a ring A with unit 1 such that (1) under addition, A forms a partially ordered abelian group with positive cone A^+, (2) $1 \in A^+$, (3) the additive subgroup $G(A) := A^+ - A^+$ of A is a unital group with positive cone $G(A)^+ = A^+$ and unit 1, and (4) for all $a, b \in A^+$,

(i) $ab = ba \Rightarrow ab \in A^+$, (ii) $aba \in A^+$,
(iii) $aba = 0 \Rightarrow ab = ba = 0$, and (iv) $(a-b)^2 \in A^+$.

If A is an effect-ordered ring, we define $P(A) := \{p \in G(A) \mid p = p^2\}$, and for $p \in P(A)$, we define $J_p \colon G(A) \to G(A)$ by $J_p(g) := pgp$ for all $g \in G(A)$. If the unital group $G(A)$ is archimedean, we say that A is an *archimedean* effect-ordered ring.

If A is an effect-ordered ring and $E := \{e \in G(A) \mid 0 \leq e \leq 1\}$ is the set of effects in A, then $G(A)^+$ is the set of all finite linear combinations with nonnegative integer coefficients of elements of E. Thus, not only does the partial order on A determine the set E, but conversely, the set E determines the positive cone $G(A)^+ = A^+$, hence the partial order on A. This accounts for our terminology "effect-ordered ring."

4.2. **Example.** If A is a unital C*-algebra, then, with $A^+ := \{aa^* \mid a \in A\}$, A is an archimedean effect-ordered ring, $G(A)$ is the additive group of self-adjoint elements in A, $P(A)$ is the orthomodular poset of projections in A, and J_p is the compression determined by $p \in P(A)$.

4.3. **Example.** Let $\mathbb{Z}$ be the additive group of integers, ordered as usual, let X be a nonempty set, and let $\mathcal{B}$ be a field of subsets of X. Define $\mathcal{F}(X, \mathcal{B})$ to be the set of all bounded functions $f \colon X \to \mathbb{Z}$ such that $f^{-1}(z) \in \mathcal{B}$ for every $z \in \mathbb{Z}$, and organize $\mathcal{F}(X,\mathcal{B})$ into a ring with pointwise operations. The constant function $1(x) = 1$ for all $x \in X$ is a unit for the ring $\mathcal{F}(X,\mathcal{B})$, and with the positive cone $\mathcal{F}(X,\mathcal{B})^+ := \{f \in \mathcal{F}(X,\mathcal{B}) \mid f(X) \subseteq \mathbb{Z}^+\}$, $\mathcal{F}(X,\mathcal{B})$ is an archimedean effect-ordered ring. Furthermore, $G(\mathcal{F}(X,\mathcal{B})) = \mathcal{F}(X,\mathcal{B})$ is a lattice ordered (hence an interpolation) group and $P(\mathcal{F}(X,\mathcal{B}))$ is the Boolean algebra of all characteristic set functions χ_B of sets $B \in \mathcal{B}$.

By the Stone representation theorem, every Boolean algebra can be represented as the field $\mathcal{B}$ of compact open subsets of a totally-disconnected compact Hausdorff space X. Hence, as in Example 4.3, every Boolean algebra can be represented as $P(A)$ for an archimedean effect-ordered ring A.

Evidently, if A is an effect-ordered ring, then $0, 1 \in P(A) \subseteq G(A)^+$ and by Definition 4.1 (4, iv), $p \in P(A) \Rightarrow 1 - p \in P(A)$. In [4, Definition 6.1] a weaker version of an effect-ordered ring A, called an *effect ring*, is defined in which it is not assumed that 1 is a generative order unit in $G(A)$ and condition (4, iv) in Definition 4.1 is replaced by the weaker condition that, for all $p \in G(A)^+$, $p = p^2 \Rightarrow 1 - p \in G(A)^+$. Therefore, the properties of an effect ring developed in [4] hold as well for an effect-ordered ring A. For instance, by [4, Corollary 6.7], $P(A)$ is an orthomodular poset with $p \mapsto 1 - p$ as orthocomplementation.

4.4. **Lemma.** *If A is an effect-ordered ring and $p \in P(A)$, then J_p is a compression on $G(A)$.*

Proof. It follows directly from Definition 4.1 that J_p is additive, order preserving, and $J_p(1) = p \leq 1$. Hence, by [4, Theorem 6.6 (iii)], J_p is a compression on $G(A)$. □

4.5. **Theorem.** *Let A be an archimedean effect-ordered ring, and let $J : G(A) \to G(A)$ be a retraction with $p := J(1)$. Then $p \in P(A)$ and $J = J_p$.*

Proof. Assume the hypotheses, and let $p' := 1 - p$. By Lemma 2.3 (ii), the infimum of p and p', calculated in $G(A)^+$, exists and equals 0, whence $p \in P(A)$ by [4, Theorem 6.8], and it follows that $p' \in P(A)$. By Lemma 2.3 (v), $0 \leq k \leq p' \Rightarrow J(k) = 0$ for all $k \in G(A)$. If $e \in G(A)$ with $0 \leq e \leq 1$, then $0 \leq p'ep' \leq p'$, whence, for $e \in G(A)$,

$$0 \leq e \leq 1 \Rightarrow J(p'ep') = 0. \tag{1}$$

Also, if $e \in G(A)$ with $0 \leq e \leq 1$, then $0 \leq pep \leq p$, whence

$$0 \leq e \leq 1 \Rightarrow J(pep) = pep. \tag{2}$$

We claim that

$$g \in G(A) \Rightarrow J(pgp) = pgp = J_p(g). \tag{3}$$

Indeed, since $G(A)$ is directed, we can write $g = x - y$ with $x, y \in G(A)^+$, and since 1 is generative, we can write $x = e_1 + e_2 + \cdots e_n$ with $0 \leq e_i \leq 1$ for $i = 1, 2, ..., n$. Thus, $pxp = pe_1p + pe_2p + \cdots + pe_np$ and, by (2), $J(pe_ip) = pe_ip$ for $i = 1, 2, ..., n$. Therefore, $J(pxp) = pxp$. Likewise, $J(pyp) = pyp$; so $J(pgp) = pgp$, proving (3). Arguing as we did to prove (3), and using (1), we find that

$$g \in G(A) \Rightarrow J(p'gp') = 0. \tag{4}$$

Now let $g \in G(A)$, let $b := pgp' + p'gp$, and consider the Peirce decomposition

$$g = pgp + b + p'gp'. \tag{5}$$

Since $g, pgp, p'gp' \in G(A)$, it follows from (5) that $b = g - pgp - p'gp' \in G(A)$. Evidently, $b = p'b + bp'$. Let n be an arbitrary integer. Then $b, np' \in G(A)$, and so by Definition 4.1 (4, iv),

$$0 \leq (b - np')^2 = b^2 - nb + n^2 p'. \tag{6}$$

Applying J to (6) and using the fact that $J(p') = 0$, we find that

$$nJ(b) \leq J(b^2) \text{ for all } n \in \mathbb{Z}. \tag{7}$$

Owing to the hypothesis that $G(A)$ is archimedean, the fact that (7) holds for positive $n \in \mathbb{Z}$ implies that $J(b) \leq 0$, and the fact that it holds for negative n implies that $-J(b) \leq 0$. Consequently, $J(b) = 0$. Therefore, by (5), (3), and (4), we have $J(g) = pgp = J_p(g)$. □

4.6. **Corollary.** *If A is an archimedean effect-ordered ring, then $G(A)$ is an archimedean compressible group, every retraction J on $G(A)$ is a compression of the form $J = J_p$ for $p = J(1) \in P(A)$, and $P(A) = \{J(1) \mid J$ is a compression on $G(A)\}$.*

4.7. **Corollary.** *If A is a unital C*-algebra, then the set $G(A)$ of self-adjoint elements in A is an archimedean compressible group with unit 1 and the compressions on $G(A)$ are the mappings J_p for $p = p^2 = p^* \in A$.*

By Corollary 4.7, Definition 2.1 does not overgeneralize the standard notion of a compression on a C*-algebra.

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