PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 1, Pages 71–81
S 0002-9939(04)07447-7
Article electronically published on August 24, 2004

ON THE EVALUATION OF GENERALIZED WATSON INTEGRALS

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(Communicated by Jonathan M. Borwein)

ABSTRACT. The triple integrals

$$W_1(z_1) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\theta_3}{1-\frac{z_1}{3}(\cos\theta_1\cos\theta_2+\cos\theta_2\cos\theta_3+\cos\theta_3\cos\theta_1)}$$

and

$$W_2(z_2) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\theta_3}{1-\frac{z_2}{3}(\cos\theta_1+\cos\theta_2+\cos\theta_3)},$$

where z_1 and z_2 are complex variables in suitably defined cut planes, were first evaluated by Watson in 1939 for the special cases $z_1 = 1$ and $z_2 = 1$, respectively. In the present paper simple direct methods are used to prove that $\{W_j(z_j)\colon j = 1, 2\}$ can be expressed in terms of squares of complete elliptic integrals of the first kind for *general* values of z_1 and z_2. It is also shown that $W_1(z_1)$ and $W_2(z_2)$ are related by the transformation formula

$$W_2(z_2) = (1-z_1)^{1/2} W_1(z_1),$$

where

$$z_2^2 = -z_1\left(\frac{3+z_1}{1-z_1}\right).$$

Thus both of Watson's results for $\{W_j(1)\colon j = 1, 2\}$ are contained within a *single* formula for $W_1(z_1)$.

1. Introduction

In this paper we shall investigate the analytic properties of the two generalized Watson integrals

$$W_j(z_j) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\theta_3}{1-z_j\lambda_j(\theta_1,\theta_2,\theta_3)} \quad (j=1,2), \tag{1.1}$$

where

$$\lambda_1(\theta_1,\theta_2,\theta_3) = \tfrac{1}{3}(\cos\theta_1\cos\theta_2+\cos\theta_2\cos\theta_3+\cos\theta_3\cos\theta_1), \tag{1.2}$$

$$\lambda_2(\theta_1,\theta_2,\theta_3) = \tfrac{1}{3}(\cos\theta_1+\cos\theta_2+\cos\theta_3) \tag{1.3}$$

and z_1, z_2 are complex variables. The integral $W_1(z_1)$ defines a single-valued analytic function in the z_1 plane provided that cuts are made along the real axis from $z_1 = -\infty$ to $z_1 = -3$ and from $z_1 = +1$ to $z_1 = +\infty$. A similar property holds for the integral $W_2(z_2)$ provided that the cuts are made along the real axis

Received by the editors March 13, 2003.
2000 *Mathematics Subject Classification.* Primary 33-xx.

from $z_2 = -\infty$ to $z_2 = -1$ and from $z_2 = +1$ to $z_2 = +\infty$. The sets of points in the z_1 and z_2 cut planes will be denoted by $\mathcal{C}_1^-$ and $\mathcal{C}_2^-$ respectively.

An exact evaluation of the integral $W_1(z_1)$ was first carried out by Watson [W] for the special case $z_1 = 1$. In particular, he found that

$$W_1(1) = \frac{3\sqrt{3}}{4}\left[\frac{2}{\pi}K(k_3)\right]^2 = \frac{9}{2^{14/3}\pi^4}\left[\Gamma\left(\frac{1}{3}\right)\right]^6, \tag{1.4}$$

where $K(k)$ is the complete elliptic integral of the first kind with a modulus k and

$$k_3 = \frac{\sqrt{3}-1}{2\sqrt{2}} \tag{1.5}$$

is the third *singular value* (see Borwein and Borwein [B-B]). In the same paper Watson also proved that

$$W_2(1) = 3\left(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}\right)\left[\frac{2}{\pi}K(k_6)\right]^2, \tag{1.6}$$

where

$$k_6 = \left(2 - \sqrt{3}\right)\left(\sqrt{3} - \sqrt{2}\right) \tag{1.7}$$

is the sixth *singular value*. The evaluation of $W_2(1)$ in terms of the gamma function $\Gamma(x)$ can be carried out using an identity for $K(k_6)$ derived by Borwein and Zucker [B-Z]. We find that

$$W_2(1) = \frac{(\sqrt{3}-1)}{32\pi^3}\left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2. \tag{1.8}$$

A direct evaluation of the triple integral $W_1(z_1)$ was first carried out for *general* values of z_1 by Iwata [I] in terms of a product of two complete elliptic integrals of the first kind. A similar product formula for $W_2(z_2)$ was later derived by Joyce [J1], [J2] using an *indirect* method which involved differential equations and Heun function transformation theory (see Snow [S]). In more recent work, Joyce [J3], [J4] has shown that the product forms for $\{W_j(z_j)\colon j = 1, 2\}$ can both be written in terms of *squares* of complete elliptic integrals of the first kind.

Our main aim in this paper is to give a *new* and *direct* procedure for evaluating $W_2(z_2)$. In this method a simple transformation formula is derived which enables one to determine the properties of $W_2(z_2)$ using the known results for $W_1(z_1)$. In Section 2 we shall also present a detailed *review* of the work of Iwata [I] and Joyce [J3], [J4] on the analytic properties of $W_1(z_1)$.

2. Analytic properties of $W_1(z_1)$

In this section we shall review the most important results for the triple integral $W_1(z_1)$.

2.1. **Iwata formula for** $W_1(z_1)$**.**

We begin by evaluating the integral $W_1(z_1)$ by following the method of Iwata [I]. In the first stage of the analysis we perform the integration over the variable θ_3 in (1.1), with $j = 1$. This procedure gives

$$W_1(z_1) = \frac{3}{z_1\pi^2}\int_0^\pi \frac{d\theta_1}{\sin\theta_1}\int_0^\pi \frac{d\theta_2}{[(a+\cos\theta_2)(b-\cos\theta_2)]^{1/2}}, \tag{2.1}$$

where

$$a = (3 + z_1 \cos\theta_1)/[z_1(1 - \cos\theta_1)], \tag{2.2}$$

$$b = (3 - z_1 \cos\theta_1)/[z_1(1 + \cos\theta_1)]. \tag{2.3}$$

We can now use the standard result

$$\int_0^\pi \frac{\mathrm{d}\theta_2}{[(a + \cos\theta_2)(b - \cos\theta_2)]^{1/2}} = \frac{2}{[(a+1)(b+1)]^{1/2}} K(k), \tag{2.4}$$

where

$$k^2 = \frac{2(a+b)}{(a+1)(b+1)}, \tag{2.5}$$

to write (2.1) in the form

$$W_1(z_1) = \frac{3}{3 + z_1} I(A_1, B_1), \tag{2.6}$$

where

$$A_1 \equiv A_1(z_1) = 2z_1(6 + z_1)/(3 + z_1)^2, \tag{2.7}$$

$$B_1 \equiv B_1(z_1) = 2z_1^2/(3 + z_1)^2 \tag{2.8}$$

and

$$I(A, B) \equiv \frac{2}{\pi^2} \int_0^\pi K\left(\sqrt{A + B\cos\psi}\right) \mathrm{d}\psi. \tag{2.9}$$

In general, the variables (A, B) in the definition (2.9) can be taken to be *independent* complex variables. However, it should be noted that the particular set of points $\{A_1(z_1), B_1(z_1) \colon z_1 \in \mathcal{C}_1^-\}$ is *restricted* to lie on the complex *rational* curve

$$(A_1 + B_1)^2 - 8B_1 = 0. \tag{2.10}$$

Next we apply the Gaussian hypergeometric series

$$\frac{2}{\pi} K(k) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(1)_n^2} k^{2n} \tag{2.11}$$

to the integrand in (2.9). In this manner, we obtain

$$I(A, B) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(1)_n^2} \Omega_n(A, B), \tag{2.12}$$

where

$$\Omega_n(A, B) = \frac{1}{\pi} \int_0^\pi (A + B\cos\psi)^n \mathrm{d}\psi \tag{2.13}$$

and $(a)_n$ denotes the Pochhammer symbol. The integral (2.13) can be readily evaluated using the method of residues. Hence, we obtain

$$\Omega_n(A, B) = (x_+)^n \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{x_-}{x_+}\right)^m, \tag{2.14}$$

where

$$x_\pm = \tfrac{1}{2}\left(A \pm \sqrt{A^2 - B^2}\right). \tag{2.15}$$

If (2.14) is substituted in (2.12) and the order of the resulting two summations is interchanged we find that it is possible to express $I(A, B)$ in the form

$$I(A,B) = F_4\left(\tfrac{1}{2}, \tfrac{1}{2}; 1, 1; x_+, x_-\right), \tag{2.16}$$

where $F_4(\alpha, \beta; \gamma, \gamma'; x, y)$ is the fourth Appell hypergeometric function in two variables x and y.

In the final stage of the analysis Iwata simplified (2.16) by using the Bailey [B] identity

$$\begin{aligned} &F_4[\alpha, \beta; \gamma, \alpha + \beta - \gamma + 1; x(1-y), y(1-x)] \\ &\quad = {}_2F_1(\alpha, \beta; \gamma; x)\, {}_2F_1(\alpha, \beta; \alpha + \beta - \gamma + 1; y). \end{aligned} \tag{2.17}$$

This procedure yields the important *general* formula

$$I(A,B) = \left(\frac{2}{\pi}\right)^2 K(k_+)K(k_-), \tag{2.18}$$

where

$$k_\pm^2 \equiv k_\pm^2(A,B) = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{A^2 - B^2} - \tfrac{1}{2}\sqrt{(1-A)^2 - B^2}. \tag{2.19}$$

If we now substitute (2.18) in (2.6) and apply the relations (2.7) and (2.8) we obtain the required result:

$$W_1(z_1) = \frac{3}{3+z_1}\left(\frac{2}{\pi}\right)^2 K(k_+)K(k_-), \tag{2.20}$$

where

$$k_\pm^2 \equiv k_\pm^2(z_1) = \frac{1}{2} \pm \frac{2\sqrt{3}z_1}{(3+z_1)^{3/2}} - \frac{\sqrt{3}}{2}\frac{(3-z_1)(1-z_1)^{1/2}}{(3+z_1)^{3/2}}. \tag{2.21}$$

It can be shown that the formula (2.20) is valid provided that z_1 lies in a certain finite region $\mathcal{R}_1$ of the cut z_1 plane that includes the point $z_1 = 0$. The points z_1 on the boundary of $\mathcal{R}_1$ are associated with values of k_+^2 that lie in the interval $\frac{1}{2}(1 + \sqrt{2}) \le k_+^2 < \infty$. We can extend the range of validity of (2.20) across the boundary curve by constructing the analytic continuation of $K(k_+)$ onto the appropriate adjacent Riemann sheet.

For the special case $z_1 = 1$ we find that k_+ becomes the complementary modulus k'_- and $k_- = k_3$, where k_N denotes the Nth singular value for the modulus. From these results and (2.20) we obtain the expected Watson formula (1.4).

2.2. **Application of the cubic modular transformation.** The aim in this subsection is to show that the Iwata formula (2.20) can be expressed in terms of the *square* of just *one* complete elliptic integral $K(k_-)$. It is found from (2.21) that the algebraic functions $m_\pm = k_\pm^2(z_1)$ are both solutions of the polynomial equation

(2.22)
$$\begin{aligned} Q(m, z_1) \equiv (3+z_1)^4 m^4 - 2(3+z_1)^4 m^3 + 3(3+z_1)(9 + 21z_1 - 7z_1^2 + z_1^3)m^2 \\ - 2z_1(3+z_1)(18 - 15z_1 + z_1^2)m + z_1^4 = 0. \end{aligned}$$

If we determine the resultant of the two polynomials $Q(m_\pm, z_1)$, with respect to the variable z_1, we deduce that the moduli $k_\pm(z_1)$ satisfy the cubic modular equation

(Joyce [J4])

$$m_+^4 - 4m_-(33 - 96m_- + 64m_-^2)m_+^3 + 6m_-(64 - 127m_- + 64m_-^2)m_+^2 - 4m_-(64 - 96m_- + 33m_-^2)m_+ + m_-^4 = 0, \tag{2.23}$$

where $m_\pm = k_\pm^2(z_1)$. This surprising connection enables one to prove that

$$K(k_+) = M_3(z_1)K(k_-), \tag{2.24}$$

where the *multiplier* $M_3(z_1)$ is an *algebraic* function of z_1 (see Borwein and Borwein [B-B]). From the work of Joyce [J3] we also have the explicit formula

$$M_3(z_1) = \frac{\sqrt{3}}{(3+z_1)^{1/2}}(2 - \sqrt{1-z_1}). \tag{2.25}$$

The application of (2.24) and (2.25) to the Iwata formula (2.20) gives the alternative product form

$$W_1(z_1) = \frac{3\sqrt{3}}{(3+z_1)^{3/2}}(2 - \sqrt{1-z_1}) \left[\frac{2}{\pi}K(k_-)\right]^2, \tag{2.26}$$

where $k_- = k_-(z_1)$ is defined in (2.21). In the derivation of (2.26) it was necessary to assume that the variable z_1 lies in the restricted region $\mathcal{R}_1$ of the cut plane $\mathcal{C}_1^-$. It should be stressed, however, that the final result (2.26) can in fact be used to calculate $W_1(z_1)$ for *all* values of $z_1 \in \mathcal{C}_1^-$. We see, therefore, that the application of cubic modular theory leads to a *much improved* formula for $W_1(z_1)$. The evaluation of (2.26) for the case $z_1 = 1$ leads directly to the Watson result (1.4).

Further product forms for $W_1(z_1)$ may be derived by applying various ${}_2F_1$ transformation formulae to the complete elliptic integral $K(k_-)$ in (2.26). For example, it is found that

$$W_1(z_1) = \frac{3\sqrt{3}}{(3+z_1)^{3/2}}(2 - \sqrt{1-z_1}) \left[{}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; s_1\right)\right]^2, \tag{2.27}$$

where

$$s_1 \equiv s_1(z_1) = \frac{4z_1}{(3+z_1)^3}\left[(3-z_1) - 3\sqrt{1-z_1}\right]^2. \tag{2.28}$$

When $z_1 = 1$, we see that this result gives

$$W_1(1) = \frac{3\sqrt{3}}{4}\left[{}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; \frac{1}{4}\right)\right]^2. \tag{2.29}$$

2.3. **Parametric representations for** $W_1(z_1)$**.** A considerable reduction in the algebraic complexity of the Iwata formulae (2.20) and (2.21) can be achieved by applying the rational transformation

$$z_1 \equiv z_1(\xi_1) = \frac{12\xi_1(1+\xi_1)(1-3\xi_1)}{(1+3\xi_1^2)^2}. \tag{2.30}$$

In this manner, we obtain the ξ_1 parametric representation (Joyce [J4])

$$W_1(z_1) = \frac{(1+3\xi_1^2)^2}{(1-\xi_1)^2(1+3\xi_1)^2}\left(\frac{2}{\pi}\right)^2 K(k_+)K(k_-), \tag{2.31}$$

where

$$k_+^2 \equiv k_+^2(\xi_1) = \frac{16\xi_1}{(1-\xi_1)(1+3\xi_1)^3}, \tag{2.32}$$

$$k_-^2 \equiv k_-^2(\xi_1) = \frac{16\xi_1^3}{(1-\xi_1)^3(1+3\xi_1)}. \tag{2.33}$$

It is possible to derive *rational* parametric formulae for $z_1(\xi_1)$ *and* $k_\pm^2(\xi_1)$, because the equation $Q(m, z_1) = 0$ defines a complex curve that has a genus $g = 0$.

The application of the transformation (2.30) to equation (2.26) gives the further simplified representation

$$W_1(z_1) = \frac{(1+3\xi_1^2)^2}{(1-\xi_1)^3(1+3\xi_1)}\left[\frac{2}{\pi}K(k_-)\right]^2, \tag{2.34}$$

where $k_- = k_-(\xi_1)$ is defined in (2.33). Finally, we note that equation (2.30) can also be used to establish the inverse relation

$$\xi_1 \equiv \xi_1(z_1) = (1+\sqrt{1-z_1})^{-1}\left(-1+\sqrt{1+\frac{z_1}{3}}\right). \tag{2.35}$$

3. Analytic properties of $W_2(z_2)$

Watson [W] evaluated $W_2(1)$ by using a *direct* method which involved an inspired sequence of integrations and changes of variable. Unfortunately, it appears that Watson's analysis has never been generalized to deal with $W_2(z_2)$, when $z_2 \neq 1$. However, it was shown by Joyce [J1], [J2], that it is possible to evaluate $W_2(z_2)$ for general z_2 by following an alternative *indirect* procedure. In this method the third-order differential equations satisfied by $\{W_j(z_j)\colon j = 1, 2\}$ are used to express $W_1(z_1)$ and $W_2(z_2)$ in terms of squares of certain Heun functions. A connection between $W_1(z_1)$ and $W_2(z_2)$ is then established by applying various bilinear and quadratic transformations to the Heun functions (see Snow [S]). This connection enables one to derive an exact product form for $W_2(z_2)$ by making a simple algebraic substitution in the Iwata formula (2.20) for $W_1(z_1)$.

Our main aim in the remainder of this section is to give a *new* and *direct* derivation of the Joyce product form for $W_2(z_2)$. In particular, the connection between $W_1(z_1)$ and $W_2(z_2)$ will be *simplified* and made more *transparent* by avoiding the use of differential equations and Heun functions.

3.1. **Connection between $W_1(z_1)$ and $W_2(z_2)$.** We begin by performing the integration over the variable θ_3 in (1.1), with $j = 2$. In this manner, it is found that

$$W_2(z_2) = \frac{3}{z_2\pi^2}\int_0^\pi \mathrm{d}\theta_1 \int_0^\pi \frac{\mathrm{d}\theta_2}{[(c-\cos\theta_2)(d-\cos\theta_2)]^{1/2}}, \tag{3.1}$$

where

$$c = (3/z_2) - \cos\theta_1 + 1, \tag{3.2}$$

$$d = (3/z_2) - \cos\theta_1 - 1. \tag{3.3}$$

If we now apply the standard result

$$\int_0^\pi \frac{\mathrm{d}\theta_2}{[(c-\cos\theta_2)(d-\cos\theta_2)]^{1/2}} = \frac{2}{[(c-1)(d+1)]^{1/2}}K(k), \tag{3.4}$$

where

$$k^2 = \frac{2(c-d)}{(c-1)(d+1)}, \tag{3.5}$$

to (3.1) we obtain the formula (Tikson [T])

$$W_2(z_2) = \frac{6}{\pi^2}\int_0^\pi \frac{1}{3 - z_2\cos\theta_1} K\left(\frac{2z_2}{3 - z_2\cos\theta_1}\right) \mathrm{d}\theta_1. \tag{3.6}$$

Next the integration variable in (3.6) is changed from θ_1 to ψ using the transformation

$$\cos\theta_1 = \frac{3\cos\psi + z_2}{3 + z_2\cos\psi}. \tag{3.7}$$

Hence, we find that

$$W_2(z_2) = \frac{3}{(9 - z_2^2)^{1/2}} J(C_2, D_2), \tag{3.8}$$

where

$$C_2 \equiv C_2(z_2) = 6z_2/(9 - z_2^2), \tag{3.9}$$

$$D_2 \equiv D_2(z_2) = 2z_2^2/(9 - z_2^2) \tag{3.10}$$

and

$$J(C, D) \equiv \frac{2}{\pi^2}\int_0^\pi K(C + D\cos\psi)\mathrm{d}\psi. \tag{3.11}$$

In general, the variables (C, D) in the definition (3.11) can be taken to be *independent* complex variables. However, it should be noted that the particular set of points $\{C_2(z_2), D_2(z_2)\colon z_2 \in \mathcal{C}_2^-\}$ is *restricted* to lie on the complex curve

$$C_2^2 - D_2^2 - 2D_2 = 0. \tag{3.12}$$

Recently, Joyce (to be published) has shown that

$$I(A, B) = (1 - A + B)^{-1/2} J(C, D), \tag{3.13}$$

where

$$C^2 = -\frac{(A - B)(1 - A - B)}{(1 - A + B)^2}, \tag{3.14}$$

$$D^2 = \frac{2B}{(1 - A + B)^2} \tag{3.15}$$

and the integrals $I(A, B)$ and $J(C, D)$ are defined in (2.9) and (3.11) respectively. This relation was derived by first applying the hypergeometric series (2.11) to the integrand in (3.11). Hence, one obtains a Kampé de Fériet series representation for $J(C, D)$ in the two variables C^2 and D^2 (see Srivastava and Karlsson [S-K]). It is then possible to establish the connection between $I(A, B)$ and $J(C, D)$ by applying two known transformation formulae to the Kampé de Fériet series. From equation (2.6) and the new general result (3.13) we see that

$$W_1(z_1) = \frac{3}{3 - z_1} J(C_1, D_1), \tag{3.16}$$

where

$$C_1^2 \equiv C_1^2(z_1) = -36z_1(1-z_1)(3+z_1)/(3-z_1)^4, \tag{3.17}$$

$$D_1^2 \equiv D_1^2(z_1) = 4z_1^2(3+z_1)^2/(3-z_1)^4. \tag{3.18}$$

It may be verified using (3.9), (3.10), (3.17) and (3.18) that the two sets of points

$$\{C_j^2(z_j), D_j^2(z_j)\colon z_j \in C_j^-\} \quad (j=1,2) \tag{3.19}$$

both lie on the *same* complex rational curve

$$(C_j^2 - D_j^2)^2 - 4D_j^2 = 0, \tag{3.20}$$

with $j = 1, 2$. It follows, therefore, that a *rational* function $z_2^2 = R(z_1)$ must exist that will give the identities

$$C_2^2\left(\sqrt{R(z_1)}\right) \equiv C_1^2(z_1) \text{ and } D_2^2\left(\sqrt{R(z_1)}\right) \equiv D_1^2(z_1). \tag{3.21}$$

We find that this transformation function is

$$z_2^2 = R(z_1) = -z_1(3+z_1)/(1-z_1). \tag{3.22}$$

The application of this result to (3.8) and (3.16) yields the required connection formula

$$W_2(z_2) = (1-z_1)^{1/2} W_1(z_1), \tag{3.23}$$

where $z_2 = z_2(z_1)$ is defined in (3.22).

The relations (3.22) and (3.23) with $z_1 = -1$ lead to a very *simple* derivation of the Watson result (1.6). In particular, we have

$$W_2(1) = \sqrt{2}W_1(-1), \tag{3.24}$$

while from (2.26) it is found that

$$W_1(-1) = \frac{6\sqrt{3}}{\pi^2}\left(\sqrt{2}-1\right)[K(k_-)]^2, \tag{3.25}$$

where

$$k_-^2 = -\tfrac{1}{2}\left(2\sqrt{3}-\sqrt{6}-1\right). \tag{3.26}$$

The application of the standard transformation formula

$$K(k) = \frac{1}{\sqrt{1-k^2}} K\left(\sqrt{\frac{-k^2}{1-k^2}}\right) \tag{3.27}$$

to (3.25) then gives

$$W_1(-1) = \frac{6\sqrt{3}}{\pi^2}\left(\sqrt{2}+1\right)\left(\sqrt{3}-1\right)^2\left(\sqrt{3}-\sqrt{2}\right)[K(k_6)]^2, \tag{3.28}$$

where the singular value k_6 is defined in (1.7). Finally, the substitution of (3.28) in (3.24) yields the expected Watson formula (1.6) for $W_2(1)$.

3.2. **Product forms for $W_2(z_2)$.** A general product form for $W_2(z_2)$ can now be derived by first substituting the formula (2.26) in (3.23). It is then possible to eliminate the variable z_1 by applying the inverse transformation

$$z_1 = z_1(z_2) = -\frac{1}{2}(3 - z_2^2) + \frac{1}{2}\sqrt{(1 - z_2^2)(9 - z_2^2)}. \tag{3.29}$$

After some algebraic manipulations it is found that

$$W_2(z_2) = \frac{6\sqrt{6}}{\pi^2} \frac{\left(\sqrt{9 - z_2^2} + \sqrt{1 - z_2^2}\right)^{1/2} \left(4 - \sqrt{9 - z_2^2} + \sqrt{1 - z_2^2}\right)}{\left(\sqrt{9 - z_2^2} + 3\sqrt{1 - z_2^2}\right)^{3/2}} [K(k_-)]^2, \tag{3.30}$$

where

$$\begin{aligned} k_-^2 \equiv k_-^2(z_2) = \frac{1}{2} &- \frac{\sqrt{3}\sqrt{9 - z_2^2}\,\left(\sqrt{9 - z_2^2} - \sqrt{1 - z_2^2}\right)^{1/2}}{\left(\sqrt{9 - z_2^2} + 3\sqrt{1 - z_2^2}\right)^{3/2}} \\ &+ \frac{4\sqrt{6}z_2^2\left(\sqrt{9 - z_2^2} + \sqrt{1 - z_2^2}\right)^{1/2}}{\left(\sqrt{9 - z_2^2} + 3\sqrt{1 - z_2^2}\right)^{5/2}}. \end{aligned} \tag{3.31}$$

By investigating the conformal mapping $z_2 \mapsto k_-^2(z_2)$ it can be shown that the formula (3.30) is valid for *all* values of $z_2 \in \mathcal{C}_2^-$.

The rather complicated structure of (3.30) and (3.31) can be simplified by applying various ${}_2F_1$ transformation formulae to the complete elliptic integral $K(k_-)$ in (3.30). For example, Delves and Joyce [D-J] have shown that

$$W_2(z_2) = \frac{3\left(2 - \sqrt{1 - z_2^2}\right)}{3 + z_2^2} \left[{}_2F_1\left(\frac{1}{8}, \frac{3}{8}; 1; s_2\right)\right]^2, \tag{3.32}$$

where

$$s_2 \equiv s_2(z_2) = \frac{16z_2^2}{9(3 + z_2^2)^4} \left[(9 - 5z_2^2) - (9 - z_2^2)\sqrt{1 - z_2^2}\right]^2. \tag{3.33}$$

This result can also be used to calculate $W_2(z_2)$ at *any* point $z_2 \in C_2^-$. When $z_2 = 1$ we find that (3.32) gives

$$W_2(1) = \left(\frac{3}{2}\right) \left[{}_2F_1\left(\frac{1}{8}, \frac{3}{8}; 1; \frac{1}{9}\right)\right]^2. \tag{3.34}$$

Finally, we note that the Clausen product theorem (see [B-B], p. 188) can be used to write (3.34) in the alternative form

$$W_2(1) = \left(\frac{3}{2}\right) {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{9}\right). \tag{3.35}$$

From this result and (1.8) we can derive the striking identity

$$\sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1}{(48)^{2n}} = \frac{(\sqrt{3} - 1)}{48\pi^3} \left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2. \tag{3.36}$$

3.3. **Parametric representations for $W_2(z_2)$.** The algebraic complexity of the formulae (3.30) and (3.31) can also be much reduced by applying the rational transformation [J3], [J4]

$$z_2^2 = 36\xi_2^2(1-\xi_2^2)(1-9\xi_2^2)/(1-9\xi_2^4)^2. \tag{3.37}$$

In this manner, we obtain the ξ_2 parametric representation

$$W_2(z_2) = \frac{(1-9\xi_2^4)}{(1-\xi_2^2)^{3/2}(1-9\xi_2^2)^{1/2}}\left[\frac{2}{\pi}K(k_-)\right]^2, \tag{3.38}$$

where

$$k_-^2 \equiv k_-^2(\xi_2) = \frac{1}{2} - \frac{(1-6\xi_2^2-3\xi_2^4)}{2(1-\xi_2^2)^{3/2}(1-9\xi_2^2)^{1/2}}. \tag{3.39}$$

Next, the quadratic transformation formula

$${}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{1}{2}-\frac{(2-\tilde{k}^2)}{4\tilde{k}'}\right] = (\tilde{k}')^{1/2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\tilde{k}^2\right), \tag{3.40}$$

with

$$\tilde{k}^2 \equiv \tilde{k}^2(\xi_2) = \frac{16\xi_2^3}{(1-\xi_2)^3(1+3\xi_2)}, \tag{3.41}$$

is applied to (3.38). This procedure yields the further simple representation

$$W_2(z_2) = \frac{(1-9\xi_2^4)}{(1-\xi_2)^3(1+3\xi_2)}\left[\frac{2}{\pi}K(\tilde{k})\right]^2, \tag{3.42}$$

where $\tilde{k} = \tilde{k}(\xi_2)$ is defined in (3.41). It is also possible to express the parameter ξ_2 in terms of the variable z_2 using the inverse relation

$$\xi_2 \equiv \xi_2(z_2) = \left(1+\sqrt{1-z_2^2}\right)^{-1/2}\left(1-\sqrt{1-\frac{z_2^2}{9}}\right)^{1/2}. \tag{3.43}$$

Recently, Joyce [J5] has shown that the parametric formula (3.42) plays a crucial role in the exact evaluation of the general lattice Green function

$$W_2(l,m,n;z_2) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\cos l\theta_1\cos m\theta_2\cos n\theta_3}{1-z_2\lambda_2(\theta_1,\theta_2,\theta_3)}\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\theta_3, \tag{3.44}$$

where $\{l,m,n\}$ denotes a set of integers and $z_2 \in \mathcal{C}_2^-$.

Acknowledgements

We wish to thank our colleague Richard Delves for assistance in deriving various transformations involved in the evaluation of $W_2(z_2)$.

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