BLOCKS OF CENTRAL $p$-GROUP EXTENSIONS

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Abstract. Let $G$ and $G'$ be finite groups that have a common central $p$-subgroup $Z$ for a prime number $p$, and let $\mathfrak{A}$ and $\mathfrak{A}'$ respectively be $p$-blocks of $G/Z$ and $G'/Z$ induced by $p$-blocks $A$ and $A'$ respectively of $G$ and $G'$, both of which have the same defect group. We prove that if $\mathfrak{A}$ and $\mathfrak{A}'$ are Morita equivalent via a certain special $(\mathfrak{A}, \mathfrak{A}')$-bimodule, then such a Morita equivalence lifts to a Morita equivalence between $A$ and $A'$.

0. Introduction

Let $G$ and $G'$ be finite groups, and let $p$ be a prime. Let $(\mathcal{O}, K, k)$ be a splitting $p$-modular system for all subgroups of $G$ and $G'$; that is, $\mathcal{O}$ is a complete discrete valuation ring of rank one with its quotient field $K$ of characteristic zero and with its residue field $k$ of characteristic $p$, and both $K$ and $k$ are splitting fields for all subgroups of $G$ and $G'$.

Let $A$ and $A'$, respectively, be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$ such that $A$ and $A'$ have a common defect group $P$. Suppose, moreover, that $P$ has a subgroup $Z$ satisfying $Z \subseteq Z(G) \cap Z(G')$, where $Z(G)$ is the center of $G$. Then, it is well known that the algebra $\overline{A}$, which is the image of $A$ via an epimorphism $\mathcal{O}G \to \mathcal{O}G' \to \mathfrak{A}'$, is again a block algebra of $\mathcal{O}G'$ with defect group $\overline{P} = P/Z$; see [3, Chap. 5, Theorems 8.10 and 8.11], for instance. Similarly, we get a block algebra $\overline{A}'$ of $\mathcal{O}G'$ with the same defect group $\overline{P}$, where $G' = G'/Z$. Then, one may ask the following natural question:

Question. If $\overline{A}$ and $\overline{A}'$ have common properties, then can we lift them to $A$ and $A'$?

There are several results concerning this question. [6, Corollary 1.12], [2, Theorem 7] and [9, Appendix A.4]. For example, Puig in [6] proves that, under a certain hypothesis, $A$ and $A'$ have isomorphic source algebras as interior $P$-algebras (we say that $A$ and $A'$ are Puig equivalent when these two block algebras are in this situation) if $\overline{A}$ and $\overline{A}'$ have isomorphic source algebras as interior $\overline{P}$-algebras. In their recent paper [11, 3.5] Usami and Nakabayashi show that, under a certain condition, if $A$ and $A'$ are the principal block algebras and if $\overline{A}$ and $\overline{A}'$ are Morita equivalent, then so are $A$ and $A'$. 

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The purpose of this note is to generalize the result by Usami and Nakabayashi to the case that $A$ and $A'$ are arbitrary block algebras. It is announced in a survey article of Rouquier [9, lines 11–12 of p. 143] that, if $P$ is abelian and if $A$ and $A'$ are the principal block algebras, then this is always the case; namely, $A$ and $A'$ are Morita (respectively Puig) equivalent whenever so are $\overline{A}$ and $\overline{A'}$ (note that in [9] he means that the same thing can be proved for non-principal block algebras; however it appears that the detailed proof has not yet shown up).

Throughout this note we use the following notation. First, we mean by a module a finitely generated right module unless stated otherwise. Let $R$ and $R'$ be rings. We denote by $1_R$ the unit element of $R$. For an $(R, R')$-bimodule $M$ we sometimes write $RM_{R'}$ to emphasize it. Similarly, for a left $R$-module $N$ and a right $R'$-module $N'$ we write also $RN$ and $N'_{R'}$. We denote by $\text{IBr}(G)$ the set of all non-isomorphic simple (irreducible) $kG$-modules. For a block algebra $A$ of $\mathcal{O}_G$, we set $\text{IBr}(A) = \{ S \in \text{IBr}(G) \mid S \text{ belongs to } A \}$. Let $X$ be a $kG$-module. We write $P(X)$ or $P_G(X)$ for the projective cover of $X$. For a projective indecomposable $kG$-module $P'$, we denote by $[P'|X]$ the multiplicity of $P'$ in the direct summands of $X$. For a $(kG, kG')$-bimodule $M$, we can consider $M$ as a right $k[G \times G']$-module as usual, namely, via $m \cdot (g, g') = g^{-1}mg'$ for $m \in M$, $g \in G$ and $g' \in G'$. We write $k_G$ for the trivial $kG$-module. For a $kG$-module $X'$ we write $X'|X$ when $X'$ is isomorphic to a direct summand of $X$ as a $kG$-module. Let $H$ be a subgroup of $G$, and let $Y$ be a $kH$-module. Then, we let $Y^G = Y^G_{\downarrow H}$ be the induced module of $Y$ from $H$ to $G$, namely, $Y^G = Y \otimes_{KH} kG$, and let $X^G_{\downarrow H} = X^G_{\downarrow H}$ be the restriction of $X$ from $G$ to $H$. We write $Z(G)$ for the center of $G$, and $\Delta G$ for the diagonal copy of $G$, namely, $\Delta G = \{(g, g) \in G \times G \mid g \in G\} \cong G$. For other notation and terminology, see the books of Alperin [1], Nagao-Tsushima [3] and Thévenaz [10].

1. Main theorem

**Theorem.** Let $G$ and $G'$ be finite groups such that $G$ and $G'$ have a common subgroup $H$ satisfying $H \supseteq P \supseteq Z$ for a $p$-subgroup $P$ of $H$ and a central $p$-subgroup $Z$ of $G$ and $G'$; that is, $G = C_G(Z)$ and $G' = C_{G'}(Z)$. Let $A$ and $A'$, respectively, be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$ such that $P$ is a defect group of $A$ and $A'$. Set $\overline{G} = G/Z$, $\overline{G'} = G'/Z$, $\overline{P} = P/Z$ and $\overline{H} = H/Z$, and let $\pi : \mathcal{O}G \rightarrow \mathcal{O}\overline{G}$ and $\pi' : \mathcal{O}G' \rightarrow \mathcal{O}\overline{G'}$ be the canonical $\mathcal{O}$-algebra-epimorphisms induced by the canonical group-epimorphisms $G \rightarrow \overline{G}$ and $G' \rightarrow \overline{G'}$, respectively. Write $\overline{A} = \pi(A)$ and $\overline{A'} = \pi'(A')$. Then, it is well known that $\overline{A}$ and $\overline{A'}$, respectively, are again block algebras of $\mathcal{O}\overline{G}$ and $\mathcal{O}\overline{G'}$ such that $\overline{P}$ is a defect group of $\overline{A}$ and $\overline{A'}$ (see [3, Chap. 5, Theorems 8.10 and 8.11]).

Now, assume that

$$\overline{A}(\overline{A} \otimes_{\mathcal{O}\overline{H}} \overline{A'}) = \bigoplus_{i=1}^{m} X_i$$

is a decomposition of indecomposable right $\mathcal{O}[\overline{G} \times \overline{G'}]$-modules such that $X_1, \ldots, X_s$ are projective indecomposables and $X_{s+1}, \ldots, X_m$ are non-projective indecomposables, and set

$$\overline{M} = X_{s+1} \oplus \cdots \oplus X_m.$$

Similarly, assume that

$$\overline{A}(\overline{A} \otimes_{\mathcal{O}\overline{H}} \overline{A'}) = \bigoplus_{j=1}^{n} Y_j$$

is a decomposition of indecomposable right $\mathcal{O}[\overline{G} \times \overline{G'}]$-modules such that $Y_1, \ldots, Y_n$ are projective indecomposables and $Y_{n+1}, \ldots, Y_m$ are non-projective indecomposables, and set

$$\overline{M}' = Y_{n+1} \oplus \cdots \oplus Y_m.$$
is a decomposition of indecomposable right \( \mathcal{O}[G \times G'] \)-modules such that \( Y_1, \ldots, Y_t \) are indecomposables with vertex \( \Delta Z \) and \( Y_{t+1}, \ldots, Y_n \) are indecomposables whose vertices are not \( \Delta Z \), and set
\[
M = Y_{t+1} \oplus \cdots \oplus Y_n.
\]

If the \((\overline{A}, \overline{A}')\)-bimodule \( \overline{M} \) realizes a Morita equivalence between \( \overline{A} \) and \( \overline{A}' \) (so that \( \overline{M}_{\overline{G} \times \overline{G'}} \) is an indecomposable right \( \mathcal{O}[\overline{G} \times \overline{G}'] \)-module), then the \((A, A')\)-bimodule \( M \) realizes a Morita equivalence between \( A \) and \( A' \).

**Proof.** First, we prove this over \( k \) instead of over \( \mathcal{O} \). Therefore, all block algebras \( A, A', \overline{A}, \overline{A}' \) and modules \( M \) and \( \overline{M} \) are over \( k \) instead of \( \mathcal{O} \) for a while. It is well known that we may consider \( \text{IBr}(A) = \text{IBr}(\overline{A}) \) and \( \text{IBr}(A') = \text{IBr}(\overline{A}') \). We can write
\[
(1) \quad (\overline{A} \otimes_{kH} \overline{A'})_{\overline{G} \times \overline{G'}} = \overline{M}_{\overline{G} \times \overline{G'}} \oplus \left( \bigoplus_{S \in \text{IBr}(\overline{A})} m(S, S') \times P_{\overline{G} \times \overline{G'}}(S \otimes_k S') \right)
\]
for non-negative integers \( m(S, S') \) since \( 1_{\overline{A}} \cdot 1_{\overline{A}'} = X_i \) for any \( i \). Since
\[
\overline{A} \otimes_{kH} \overline{A'} |_{kG \otimes_{kH} kG'} = k_{\overline{A}'} |_{\overline{G} \times \overline{G'}},
\]
we obtain that, for each \( S \) and \( S' \),
\[
(2) \quad m(S, S') = \left[ P_{\overline{G} \times \overline{G'}}(S \otimes_k S') \right]_{k_{\overline{A}'}}.
\]
Note that
\[
\{ P_{(G \times G')/\Delta Z}(S \otimes_k S') \mid S \in \text{IBr}(A), \ S' \in \text{IBr}(A') \}
\]
is the set of all trivial source \((p\text{-}\text{permutation}) k[G \times G']\)-modules with vertex \( \Delta Z \) in a block algebra \( A \otimes_k A' \) of \( k[G \times G'] \); see [3, Chap. 4, Problem 10]. Hence, by the definition of \( M \), we can write
\[
(3) \quad (A \otimes_{kH} A')_{G \times G'} = M_{G \times G'} \oplus \left( \bigoplus_{S \in \text{IBr}(A)} n(S, S') \times P_{(G \times G')/\Delta Z}(S \otimes_k S') \right)
\]
for non-negative integers \( n(S, S') \). Since
\[
(A \otimes_{kH} A')_{G \times G'} |_{kG(kG \otimes_{kH} kG')_{kG'}} = k_{\overline{A}'} |_{\overline{G} \times \overline{G'}} \cong k_{\overline{A}'} |_{\overline{G} \times \overline{G'}} \cong k_{\overline{A}'(G \times G')/\Delta Z},
\]
as right \( k[G \times G'] \)-modules, we know that, for each \( S \) and \( S' \),
\[
(4) \quad n(S, S') = \left[ P_{(G \times G')/\Delta Z}(S \otimes_k S') \right]_{k_{\overline{A}'(G \times G')/\Delta Z}}.
\]
Now, set
\[
\overline{N} = kG \otimes_{kG} A \otimes_{kH} A' \otimes_{kG'} kG'.
\]
Then, we get by (3) that
\[
\mathcal{N} = (k\mathcal{G} \otimes_{kG} M \otimes_{kG'} k\mathcal{G'}) + \bigoplus_{S \in \text{Ibr}(A)} n(S, S') \times \left( k\mathcal{G} \otimes_{kG} P_{(G \times G')/\Delta Z}(S \otimes_k S') \otimes_{kG'} k\mathcal{G'} \right)
\]
On the other hand,
\[
\mathcal{N} = (k\mathcal{G} \otimes_{kG} A) \otimes_{kH} (A' \otimes_{kG'} k\mathcal{G'})
\]
\[
= (k\mathcal{G} \otimes_{kG} A) \otimes_{kH} (k\mathcal{G'} \otimes_{kG'} k\mathcal{G'})
\]
since \( k\mathcal{G} \otimes_{kG} A = \bar{A} \)
\[
= k\mathcal{G} \otimes_{kG} \bar{A}
\]
\[
= k\mathcal{G} \otimes_{kG} \mathcal{G'} + \bigoplus_{S \in \text{Ibr}(A)} m(S, S') \times P_{\mathcal{G} \otimes \mathcal{G}'}(S \otimes_k S')
\]
by (1).

Take any \( S \in \text{Ibr}(A) = \text{Ibr}(\bar{A}) \) and \( S' \in \text{Ibr}(A') = \text{Ibr}(\mathcal{G'}) \). Then,
\[
n(S, S') = P_{(G \times G')/\Delta Z}(S \otimes_k S') \left( k\mathcal{G} \otimes_{kG} \mathcal{G'} \right)
\]
by (4)
\[
= P_{(\Delta H/\Delta Z)} \left( (S \otimes_k S') \downarrow_{\Delta H/\Delta Z} \right)
\]
by \([7, \text{Theorem 3}]\)
\[
= P_{k\Delta H} \left( (S \otimes_k S') \downarrow_{\Delta H/\Delta Z} \right)
\]
since \( \Delta H/\Delta Z \cong \Delta \mathcal{G}' \)
\[
= P_{k\Delta H} \left( (S \otimes_k S') \downarrow_{\Delta \mathcal{G}'} \right)
\]
by \([7, \text{Theorem 3}]\)
\[
= m(S, S')
\]
by (2).

Moreover, it follows that
\[
P_{\mathcal{G} \otimes \mathcal{G}'}(S \otimes_k S') = P_{(G \times G')/\Delta Z}(S \otimes_k S')
\]
\[
= P_{(G \times G')/\Delta Z}(S \otimes_k S') \otimes_{k[\Delta H/\Delta Z]} k[\mathcal{G} \times \mathcal{G}']
\]
by \([12, 2.1. \text{Proposition (a)}] \)
\[
= P_{(G \times G')/\Delta Z}(S \otimes_k S') \otimes_{k[\Delta H/\Delta Z]} k[\mathcal{G} \times \mathcal{G}']
\]
since there are canonical epimorphisms
\[
G \times G' \to (G \times G')/\Delta Z \to \mathcal{G} \times \mathcal{G}'
\]
\[
= k\mathcal{G} \otimes_{kG} P_{(G \times G')/\Delta Z}(S \otimes_k S') \otimes_{kG'} k\mathcal{G}'.
\]

Therefore, it follows from a theorem of Krull-Schmidt that
\[
\mathcal{N} = k\mathcal{G} \otimes_{kG} M_{G \times G'} \otimes_{kG'} k\mathcal{G}'.
\]

Now, by the hypothesis, \( \bar{A} \) and \( \mathcal{G'} \) are Morita equivalent via \( \mathcal{M} \). Thus, we get from (5) and a result of Rouquier \([8, \text{Lemma 10.2.11 and its proof}] \) that \( A \) and \( A' \) are Morita equivalent via \( A_M A' \).

Now, it follows that
\[
A_M A' = M_{G \times G'} \left( A \otimes_{kH} A' \right) kG \otimes_{kH} kG' = k_{\Delta H} kG \otimes_{kH} kG'.
\]
so that $M_{G \times G'}$ is a trivial source ($p$-permutation) $k[G \times G']$-module and is $\Delta H$-projective. Thus, $M_{G \times G'}$ has $\Delta P$ as a vertex since $A$ and $A'$ have $P$ as defect groups. Therefore, $A$ and $A'$ are Puig equivalent by a theorem of Puig (independently by Scott) [5, Remark 7.5]. That is, source algebras of $A$ and $A'$ are isomorphic as interior $P$-algebras. Hence, we get by a theorem of Puig [4, Lemma 7.8] (see [10, (38.7)Proposition and (38.8)Proposition]) that the Morita (Puig) equivalence between $A$ and $A'$ lifts from $k$ to $O$. We are done.

**Corollary.** Keep the notation and assumption as in the theorem. Suppose, moreover, that $G' = H \supseteq N_G(P)$, and let $B = A'$. If

$$1_A \mathcal{O} \Gamma_A 1_B = f \mathcal{O}(\mathcal{G} \times H, \Delta P)(\mathcal{G}) \oplus \text{(projective } \mathcal{O}(\mathcal{G} \times H, \Delta P) \text{-module)}$$

and if $\mathcal{M} = f \mathcal{O}(\mathcal{G} \times H, \Delta P)(\mathcal{G})$ realizes a Morita equivalence between $A$ and $B$, then $M = f \mathcal{O}(\mathcal{G} \times H, \Delta P)(A)$ realizes a Morita equivalence between $A$ and $B$, where $f \mathcal{O}(\mathcal{G} \times H, \Delta P)$ and $f \mathcal{O}(\mathcal{G} \times H, \Delta P)$ are the Green correspondences with respect to $(\mathcal{G} \times \mathcal{G}, \Delta P, G \times H)$ and $(G \times G, \Delta P, G \times H)$, respectively; see [3, Chap. 4, p. 276].

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