ON A GENERALIZED CORONA PROBLEM
ON THE UNIT DISC

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Abstract. Let $g, f_1, \ldots, f_n \in H^\infty$. We give a sufficient condition on the size of a function $g$ in order for it to be in the ideal generated by $f_1, \ldots, f_n$. In particular, this improves Cegrell’s result on this problem.

Introduction

Let $D$ be the unit disc in the complex plane, and let $H^\infty = H^\infty(D)$ be the Banach algebra of bounded analytic functions on $D$. Carleson’s corona theorem says that the unit disc is dense in the space $M_{H^\infty}$ of maximal ideals of $H^\infty$ with the weak-star topology. This result is equivalent to the following fact: if we have functions $f_1, \ldots, f_n \in H^\infty$ such that

$$\sum_{j=1}^{n} |f_j(z)| \geq \delta > 0, \quad \forall z \in D,$$

then there exist solutions $g_1, \ldots, g_n \in H^\infty$ of the equation

$$\sum_{j=1}^{n} f_j g_j = 1.$$

In order to generalize this result, it is natural to ask if it is possible to replace the function $1$ by an arbitrary function $g \in H^\infty$; that is, one asks if the condition

$$|g(z)| \leq C \sum_{j=1}^{n} |f_j(z)|, \quad \forall z \in D,$$

implies that the function $g$ belongs to the ideal $I$ generated by $f_1, \ldots, f_n$. Condition (1) is clearly a necessary condition, but an example given by Rao (see [Ra]) shows that the answer is, in general, negative. Thus the following problem arises naturally.

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**Problem A.** Let $h$ be a positive continuous function on $[0, \infty)$ increasing in a neighbourhood of zero, and let $g, f_1, \ldots, f_n \in H^\infty$. For which functions $h$ does the condition
\begin{equation}
|g(z)| \leq h(|f_1(z)| + \cdots + |f_n(z)|), \quad \forall z \in \mathbb{D},
\end{equation}
imply that the function $g$ is in the ideal generated by $f_1, \ldots, f_n$?

For functions of the form $h(s) = s^\alpha$, with $\alpha \geq 1$, the problem is completely solved. For $1 \leq \alpha < 2$, a variation of Rao’s example shows that the answer is negative, and for $\alpha > 2$, work of Wolff, Cegrell and others gives an affirmative answer (see [Ce1], [Ga]). The problem for $\alpha = 2$ was an old question of Wolff, which remained open for twenty years. However, Treil (see [Tr]) has recently shown (using a connection with the best estimates of the solutions of the corona theorem) that the answer is, in general, negative.

In [Li], Lin gave an affirmative answer for this problem for the function
$$h(s) = s^2(-\log s)^{-(3/2+\varepsilon)}$$
with $\varepsilon > 0$, and in [Ce2] Cegrell established the following strongest known positive case for this problem.

**Theorem A (Cegrell).** Let $f_1, \ldots, f_n \in H^\infty$ with $|f(z)|^2 = \sum_{j=1}^n |f_j(z)|^2 > 0$, for all $z \in \mathbb{D}$. Then, Problem A has an affirmative answer for
$$h(s) = \frac{s^2}{(-\log s)^{3/2} (\log(-\log s))^{3/2} \log(-\log s)}.$$

Our main result below is an improvement of Cegrell’s theorem.

**Theorem 1.** Let $k : (0, 1) \to [0, \infty)$ be a nondecreasing bounded continuous function such that $k(x)/x$ is nonincreasing and
$$\int_0^1 \frac{k(x)}{x} |\log x| \, dx < +\infty,$$
and let $H(x) = \sqrt{k(x) \int_0^x \frac{k(x)}{s} \, ds}$. Furthermore, let $g, f_1, \ldots, f_n \in H^\infty$, where $0 < |f|^2 := \sum_{j=1}^n |f_j|^2 < 1$. Then the condition
$$|g| \leq |f|^2 H(|f|^2)$$
implies the existence of solutions $g_1, \ldots, g_n \in H^\infty$ of the equation
$$g = f_1 g_1 + \cdots + f_n g_n.$$

For example, if we take $k(x) = |\log x|^{-2} (\log |\log x|)^{-3/2}$, we see that Problem A has an affirmative answer for the function
$$h(s) = s^2(-\log s)^{-3/2} (\log(-\log s))^{-1},$$
and this clearly improves Cegrell’s result.

For $1 \leq p < \infty$, let $H^p$ be the Hardy space of analytic functions in the unit disc such that
$$\|f\|_p^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$
It is well known that an analytic function $f$ belongs to $H^p$ if and only if the nontangential maximal function $Mf(e^{i\theta}) = \sup\{|f(z)| : z \in \Gamma(\theta)\}$ belongs to the usual Lebesgue space $L^p(\mathbb{T})$, where

$$\Gamma(\theta) := \{z \in \mathbb{D} : |e^{i\theta} - z| < (1 + \alpha)(1 - |z|)\}$$

is the Stolz angle with vertex at $e^{i\theta}$ and fixed aperture $\alpha > 0$ (the choice of $\alpha$ is irrelevant here), and $\mathbb{T}$ denotes the unit circle. Several $H^p$ versions of the corona theorem have been considered. Concretely, one is interested in conditions on functions $f_1, \ldots, f_n \in H^\infty$ such that the equation

$$1 = f_1 g_1 + \cdots + f_n g_n$$

has solutions $g_1, \ldots, g_n$ in $H^p$. If $|f|^2 = \sum_{j=1}^n |f_j|^2$ and $|g|^2 = \sum_{j=1}^n |g_j|^2$, it follows from (3) that $1 \leq |f||g|$, and hence $M(|f|^{-1}) \in L^p(\mathbb{T})$ is a necessary condition. We note that when $p = \infty$, this is the usual corona condition. However, for $1 \leq p < \infty$, this condition is far from being sufficient. In [ABN], it is shown that, for any $\varepsilon > 0$, the stronger condition $M(|f|^{-2+\varepsilon}) \in L^p(\mathbb{T})$ is not sufficient. Our next result is the $H^p$ version of Theorem 1.

**Theorem 2.** Let $k$ be as in Theorem 1, let $H(x) = \left(k(x) \int_0^x k(s)/s \, ds\right)^{1/2}$ and let $1 \leq p < \infty$. Given functions $g, f_1, \ldots, f_n \in H^\infty$, the condition

$$M\left(\frac{g}{|f|^2 H(|f|^2)}\right) \in L^p(\mathbb{T})$$

implies the existence of solutions $g_1, \ldots, g_n \in H^p$ of the equation

$$g = f_1 g_1 + \cdots + f_n g_n.$$

Finally, we want to remark that in both theorems, only the behavior of the function $k$, and hence $H$, near zero is essential.

1. **Carleson measures and the $\overline{\partial}$-equation**

Solutions of the $\overline{\partial}$-equation with boundary control will be of vital importance in the proofs of the main theorems, and Carleson measures play an important role in obtaining these solutions. We recall that a positive Borel measure $\mu$ on $\mathbb{D}$ is called a *Carleson measure* if there exists a constant $C$ such that

$$\int_\mathbb{D} |h|^2 \, d\mu \leq C\|h\|_2^2,$$

for every function $h$ in the Hardy space $H^2$. It is well known that Carleson measures are those positive measures $\mu$ for which there exists a constant $A$ such that

$$\mu(Q) \leq A l(Q)$$

for every Carleson square $Q$ defined by

$$Q = \{re^{i\theta} \in \mathbb{D} : 1 - r < l(Q), |\theta - \theta_0| < l(Q)\}.$$

Denote by $N(\mu) = \sup \{\mu(Q)/l(Q)\}$ the Carleson norm of $\mu$, where the supremum is taken over all Carleson squares $Q$. The operators $\partial$ and $\overline{\partial}$ are defined by

$$\partial f = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \overline{\partial} f = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

By the Cauchy-Riemann equations, a function $f$ is analytic if and only if $\overline{\partial} f = 0$. Recall that we can rewrite the Laplacian operator as $\Delta = 4\partial \overline{\partial}$. We need the
following result of T. Wolff on the existence of bounded solutions of the \( \overline{\partial} \)-equation (see, for example, [Ga] p. 322).

**Lemma 1.** Let \( G(z) \) be bounded and \( C^1 \) on the disc \( D \), and assume that the measures \( d\mu(z) = |G(z)|^2 \log \frac{1}{|z|} \, dx \, dy \) and \( d\sigma(z) = |\partial G(z)| \log \frac{1}{|z|} \, dx \, dy \) are Carleson measures. Then there exists a function \( u \in C(\overline{D}) \cap C^1(D) \) such that \( \overline{\partial} u = G \) and

\[
\|u\|_{L^\infty(T)} \leq C_1 N(\mu)^{1/2} + C_2 N(\sigma).
\]

We will also need an \( L^p \)-version of the Wolff criteria. The next lemma is a refinement of the version given in [ABN].

**Lemma 2.** Let \( 1 \leq p < \infty \), and let \( G \) be a \( C^1 \) function in \( D \) such that:

(a) \( G = \varphi \psi \), where \( M(\varphi) \in L^p(T) \), and \( |\psi(z)|^2 \log \frac{1}{|z|} \, dx \, dy \) is a Carleson measure;

(b) for every function \( k \in H^q \), where \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\int_D |k(z)||\partial G(z)| \log \frac{1}{|z|} \, dx \, dy \leq B\|k\|_q.
\]

Then there exists a function \( u \in C(\overline{D}) \cap C^1(D) \) such that \( \overline{\partial} u = G \) and

\[
\int_0^{2\pi} |u(e^{i\theta})|^p \, d\theta \leq C,
\]

where \( C \) depends only on the \( L^p \)-norm of \( M(\varphi) \), the constant \( B \) and the Carleson norm of the measure of (a).

**Proof.** Let \( q \) be the conjugate exponent of \( p \), \( 1 < q \leq \infty \). By duality,

\[
\inf \{ \|b\|_p : \overline{\partial} b = G \} = \sup \{ \frac{1}{2\pi} \int_0^{2\pi} |Fk(\theta)|_p \, k \in H^q, k(0) = 0, \|k\|_q \leq 1 \}
\]

where \( F \) is a priori solution, say the one given by the Cauchy kernel, which is continuous on \( \overline{D} \). By Green’s formula,

\[
\frac{1}{2\pi} \int_0^{2\pi} |Fk(\theta)|_p \, k \in H^q, k(0) = 0, \|k\|_q \leq 1 \}
\]

\[
\int_D \frac{1}{2\pi} \int_0^{2\pi} Fk(\theta) \Delta(Fk) \log \frac{1}{|z|} \, dx \, dy
\]

\[
= \frac{2}{\pi} \int_D \frac{k(z)G(z)}{|z|} \log \frac{1}{|z|} \, dx \, dy + \frac{2}{\pi} \int_D k(z)\partial G(z) \log \frac{1}{|z|} \, dx \, dy = I_1 + I_2.
\]

It is proved in [ABN] that if \( |\psi|^2 \log \frac{1}{|z|} \) is a Carleson measure with Carleson norm \( K \), then

\[
\int_D \frac{|k'(z)||\varphi(z)||\psi(z)|}{|z|} \log \frac{1}{|z|} \, dx \, dy \leq C \|k\|_{H^q} \|M\varphi\|_p K
\]

where \( C \) is an absolute constant. This implies the required bound for \( I_1 \), and the boundness of \( I_2 \) follows from condition (b).

The following lemma can be found in [Ni]. For completeness we will give a proof here.

**Lemma 3.** Let \( u \in C^2(\overline{D}) \) be a bounded subharmonic function. Then

\[
d\lambda(z) = \Delta u(z) \log \frac{1}{|z|} \, dx \, dy
\]

is a Carleson measure with Carleson norm bounded by \( 2\pi e\|u\|_\infty \).
Proof. By considering the function $b(z) = u(z) + \|u\|_{\infty}$ we can assume that our function $u$ is positive. Let $h \in H^2$. Then, for $t > 0$,

$$\Delta(|h|^2 e^{tu}) = 4|h'|^2 e^{tu} + |h|^2 \Delta e^{tu} + 8 \text{Re}(\bar{h}(|h|^2) \partial e^{tu})$$

$$= 4|h'|^2 e^{tu} + |h|^2 (4t^2 |\partial u|^2 + t \Delta u) e^{tu} + 8t e^{tu} \text{Re}(h \bar{\partial h} \partial u)$$

$$= t|h|^2 e^{tu} \Delta u + e^{tu} |2\partial h + 2th \partial u|^2$$

$$\geq t|h|^2 e^{tu} \Delta u \geq t|h|^2 \Delta u.$$  

Thus we have

$$\int_{\mathbb{D}} |h(z)|^2 \Delta u(z) \log \frac{1}{|z|} \, dx \, dy \leq \inf_{t > 0} \frac{1}{t} \int_{\mathbb{D}} \Delta(|h(z)|^2 e^{tu(z)}) \log \frac{1}{|z|} \, dx \, dy,$$

which, by Green’s formula, is bounded by

$$\inf_{t > 0} \frac{1}{t} \int_{\mathbb{D}} |h|^2 e^{tu} \leq \inf_{t > 0} \frac{2 \pi}{t} \|e^{tu}\|_{\infty} \|h\|_2^2 = 2 \pi e \|u\|_{\infty} \|h\|_2^2.$$  

(The last identity is obtained by computation of the minimum of $t^{-1} \|e^{tu}\|_{\infty}$, which is attained at the point $t_0 = 1/\|u\|_{\infty}$. Hence the measure $\lambda$ is a Carleson measure with Carleson norm bounded by $2 \pi e \|u\|_{\infty}$.)

Given functions $f_1, \ldots, f_n \in H^\infty$, we write $|f|^2 = \sum_{i=1}^n |f_i|^2$, and $|f'|^2 = \sum_{i=1}^n |f'_i|^2$. The next result is the key for the proof of Theorems 1 and 2.

**Lemma 4.** Let $k : (0, 1) \rightarrow [0, \infty)$ be a bounded continuous function such that

$$\int_0^1 \frac{k(\alpha)}{\alpha} \log \alpha \, d\alpha < \infty.$$  

Let $f_1, \ldots, f_n \in H^\infty$ with $0 < |f|^2 < 1$. Then the measures

(a) $\frac{|\partial(|f|^2)|^2}{|f|^4} \frac{k(|f|^2)}{\log \frac{1}{|z|}} \, dx \, dy$,

(b) $\frac{|f|^2 |f'|^2 - |\partial(|f|^2)|^2}{|f|^4} \left( \int_0^{|f|^2} \frac{k(s)}{s} \, ds \right) \frac{1}{\log \frac{1}{|z|}} \, dx \, dy$

are Carleson measures with Carleson norm bounded by $C \int_0^1 \frac{k(s)}{s} \log |s| \, ds$.

We note that when $n = 1$, part (b) is vacuous and part (a) is a known result that is also true for bounded analytic functions vanishing in $\mathbb{D}$ (see [ABN]).

**Proof.** Consider the function

$$U(z) = \log |f(z)| \int_0^{|f(z)|^2} \frac{k(s)}{s} \, ds + \int_0^{|f(z)|^2} \frac{k(s)}{s} \log s \, ds.$$  

It clearly satisfies

$$0 \leq U(z) \leq 2 \int_0^{|f(z)|^2} \frac{k(s)}{s} \log s \, ds,$$

and a computation gives

$$\frac{1}{4} \Delta U(z) = S(z),$$

where

$$S = \frac{|\partial(|f|^2)|^2}{|f|^4} k(|f|^2) + \frac{|f|^2 |f'|^2 - |\partial(|f|^2)|^2}{|f|^4} \left( \int_0^{|f|^2} \frac{k(s)}{s} \, ds \right).$$

Now, applying Lemma 3, we obtain the desired conclusion. $\square$
2. Proof of Theorem 1

The proof follows standard arguments with the use of Lemma 4 as a new ingredient. By a standard normal families argument, we can assume that the functions \( f_1, \ldots, f_n \) are analytic in a neighborhood of the closed unit disc. For \( j = 1, \ldots, n \) we define

\[
\varphi_j(z) = \frac{f_j(z)}{|f_j(z)|^2}.
\]

We see that the functions \( \varphi_j \) belong to \( C^\infty(D) \) and satisfy the equation \( \sum_{j=1}^n f_j \varphi_j = 1 \). For \( j, k = 1, \ldots, n \), let

\[
G_{jk} = \varphi_j \partial \varphi_k.
\]

Assume that for \( j, k = 1, \ldots, n \) we can solve the \( \partial \)-equations

\[
\partial u_{jk} = g G_{jk},
\]

with \( ||u_{jk}||_{L^\infty(T)} \leq M \). Then, for \( j = 1, \ldots, n \), the functions

\[
g_j = g \varphi_j + \sum_{k=1}^n (u_{jk} - u_{kj}) f_k
\]

are bounded, satisfy \( \partial g_j = 0 \) and so are analytic, and satisfy the equation

\[
g = \sum_{j=1}^n f_j g_j.
\]

It only remains to show that (8) has bounded solutions. To see this, we will use Lemma 1. Fix \( j, k \), and denote \( G_{jk} \) by \( G \). A computation gives

\[
G_{jk} = \frac{f_j}{|f|^2} \sum_{l \neq k} f_l (f_l f_k' - f_k f_l')
\]

and

\[
\sum_{j, k=1}^n |f_k' f_j - f_k f_j'|^2 = |f|^2 |f'|^2 - |\partial(|f|^2)|^2.
\]

By (6) and (7),

\[
|G| \leq 2 \left( \frac{|f|^2 |f'|^2 - |\partial(|f|^2)|^2}{|f|^4} \right)^{1/2}.
\]

Using (8) and our condition on the size of \( g \) we see that

\[
|gG|^2 \leq \frac{(|f|^2 |f'|^2 - |\partial(|f|^2)|^2)}{|f|^4} \left( \int_0^{|f|^2} \frac{k(s)}{s} ds \right) k(|f|^2),
\]

and because \( k \) is bounded,

\[
|g(z)G(z)|^2 \log \frac{1}{|z|} \, dx \, dy
\]

is a Carleson measure by Lemma 1.

We have \( \partial (gG) = g'G + g \partial G \), and since \( |g| \leq |f|^2 H(|f|^2) \), we have that the measure \( |g(z)\partial G(z)| \log \frac{1}{|z|} \, dx \, dy \) is a Carleson measure by the following result.

**Lemma 5.** The measure \( |f(z)|^2 H(|f(z)|^2) |\partial G(z)| \log \frac{1}{|z|} \, dx \, dy \) is a Carleson measure.
To prove this, we note that $k(|f|^2) \leq \int_0^{|f|^2} \frac{ks}{s} ds$, since the function $k(x)/x$ is nonincreasing. Also, a computation gives

$$|\partial G| \leq 2|G| \frac{|\partial(|f|^2)|}{|f|^2} + \frac{|f|^2|f'|^2 - |\partial(|f|^2)|^2}{|f|^4}.$$  

(9)

So, by (9) and then (8),

$$|f|^2 H(|f|^2) |\partial G| \leq 2H(|f|^2) |G| |\partial(|f|^2)| + \frac{(|f|^2|f'|^2 - |\partial(|f|^2)|^2)}{|f|^4} \int_0^{|f|^2} \frac{k(s)}{s} ds$$

$$\leq 2 \frac{|\partial(|f|^2)|^2}{|f|^4} k(|f|^2) + 2 \frac{|f|^2|f'|^2 - |\partial(|f|^2)|^2}{|f|^4} \left( \int_0^{|f|^2} \frac{k(s)}{s} ds \right),$$

and the result follows by Lemma 4.

It remains to check that

$$|g'(z)G(z)| \log \frac{1}{|z|} dx dy$$

is a Carleson measure. Let $h \in H^2$. Then

$$\int_D |h(z)|^2 \left| \frac{g(z)}{f(z)} \right| \log \frac{1}{|z|} dx dy = \int_A + \int_{D \setminus A} = I_1 + I_2,$$

where $A = \{ z : |g(z)| \leq |f(z)|^5 \}$. For $z \in A$ we have

$$|g'(z)| \leq \frac{|g'(z)|^2}{|g(z)|} + \frac{|f(z)|^2}{|f(z)|}.$$

Since for $F \in H^\infty$, the measure $\frac{|F(z)|^2}{|f(z)|^2} \log \frac{1}{|z|} dx dy$ is Carleson (see [Ga], p. 327, or apply Lemma 4 with $k(x) = x^{1/2}$), we see that $I_1 \leq C_1 \| h \|_{H^2}^5$, by (4). To estimate $I_2$, let

$$B(|f|^2) = \frac{(|f|^2|f'|^2 - |\partial(|f|^2)|^2)}{|f|^4} \left( \int_0^{|f|^2} \frac{k(s)}{s} ds \right).$$

Since $k$ is nondecreasing, we see that

$$|g'(z)| \leq \frac{|g'(z)|^2}{|g(z)|} k(|f|^2) + B(|f|^2)
\leq \frac{|g'(z)|^2}{|g(z)|} \frac{|s|(|f|^2)}{|f|^2} + B(|f|^2)$$

in $D \setminus A$, where $s(x) = k(x^{1/5})$. One easily verifies that $s$ satisfies the condition $\int_0^1 \frac{s(x)}{x} |\log x| dx < \infty$. Then $I_2 \leq C_2 \| h \|_{H^2}^5$ by Lemma 4 and (4). Hence the measure

$$|g'(z)G(z)| \log \frac{1}{|z|} dx dy$$

is a Carleson measure. By Lemma 1 the proof is complete.

3. PROOF OF THEOREM 2

For $j, k = 1, \ldots, n$, let $G_{jk} = \phi_j \overline{\varphi_k}$, where $\varphi_j = \tilde{f}_j |f|^{-2}$. As in the proof of Theorem 1 it is sufficient to solve the $\bar{\partial}$-equations $\bar{\partial} u_{jk} = g G_{jk}$, with $\| u_{jk} \|_{L^p(\mathbb{T})} \leq M$. For this, we will make use of Lemma 2. Fix $j, k$, and for ease of notation, denote $G_{jk}$ by $G$. We can write $gG$ in the form $gG = \phi \psi_1$, where $\phi = g |f|^{-2} (H(|f|^2))^{-1}$ and $\psi_1 = |f|^2 H(|f|^2) G$. By hypothesis, $M(\phi) \in L^p(\mathbb{T})$, and the proof of Theorem 1 shows that

$$|\psi_1(z)|^2 \log \frac{1}{|z|} dx dy$$
is a Carleson measure. So condition (a) of Lemma [2] is satisfied. To check condition (b), let \( k \in H^q \), where \( 1/p + 1/q = 1 \). We have \( \partial (gG) = qG + g\partial G \), and we can write \( |g\partial G| \) as \( |\phi| \psi_2 \), where \( \psi_2 = |f|^2 H(|f|^2) |\partial G| \). By Lemma [5] the measure

\[
d\mu(z) = \psi_2(z) \log \frac{1}{|z|} \, dx \, dy
\]

is a Carleson measure. Then

\[
\int_D |k(z)| |(g\partial G)(z)| \log \frac{1}{|z|} \, dx \, dy \leq ( \int_D |\phi|^p \, d\mu )^{1/p} ( \int_D |k|^q \, d\mu )^{1/q} \leq C \|M(\phi)\|_{L^p(\mathbb{T})} \|k\|_{H^q},
\]

since, if \( \mu \) is a Carleson measure and \( M\psi \in L^p(\mathbb{T}) \), then \( \int_D |\psi|^p \, d\mu \leq \|M\psi\|_{L^p(\mathbb{T})}^p \) (see [Ga], p. 32). An argument similar to that in the proof of Theorem [1] shows that

\[
\int_D |k(z)| |(g'G)(z)| \log \frac{1}{|z|} \, dx \, dy \leq C \|k\|_{H^q},
\]

and condition (b) of Lemma [2] is satisfied. This completes the proof.

References


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