PROJECTIVE SURFACES WITH MANY SKEW LINES

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Abstract. We give an example of a smooth surface $S_d \subset \mathbb{P}^3(\mathbb{C})$ of degree $d$ that contains $d\cdot(d-2)+2$ pairwise disjoint lines. In particular, our example shows that the degree in Miyaoka’s bound is sharp.

Up to now the maximal number of pairwise disjoint lines on a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3(\mathbb{C})$ was unknown. According to [4, p. 162] this number does not exceed $2 \cdot d \cdot (d-2)$.

Quartic surfaces with 16 skew lines are studied in [1], but it is not clear to what extent Miyaoka’s bound is sharp for $d \geq 5$. Here we give an example of a smooth surface $S_d \subset \mathbb{P}^3(\mathbb{C})$ of degree $d$ that contains $d \cdot (d-2)+2$ pairwise disjoint lines. All lines on $S_d$ form the following configuration:

```
\begin{array}{|c|c|c|c|}
\hline
& & & \cdots \\
\hline
& 2 \text{ skew lines} & & \\
\hline
& & \cdots & \\
\hline
& d \cdot (d-2)+2 \text{ pairwise disjoint lines} & & \\
\hline
\end{array}
```

Our example is inspired by the classical Klein quartic curve.

The equation of a quintic with 19 skew lines is given in [3, Example 2.3]. For $d \geq 6$ the surface $S_d$ contains the largest number of skew lines found on a smooth surface of degree $d$ in $\mathbb{P}^3(\mathbb{C})$. Let us mention that the Fermat surface $F_d$, i.e. the surface with $3d^2$ lines (the largest number known so far for $d \neq 4, 6, 8, 12, 20$), contains no family of $3d$ pairwise disjoint lines. The latter results from the description of configuration of lines on $F_d$ that can be found in [3].

Example. We define $S_d$ to be the surface given by the polynomial

$$s_d := x_0^{d-1} \cdot x_1 + x_1^{d-1} \cdot x_2 + x_2^{d-1} \cdot x_3 + x_3^{d-1} \cdot x_0$$

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where $d \geq 6$. One can easily check that $S_d$ is smooth. Let $L_1$ (resp. $L_2$) be the line $x_0 = x_2 = 0$ (resp. $x_1 = x_3 = 0$). We claim that
(a) $S_d$ contains $d \cdot (d - 2) + 2$ skew lines, each of which meets $L_1$ and $L_2$,
(b) the only lines on $S_d$ are $L_1$, $L_2$ and the above-mentioned skew lines.

**Proof of (a).** Fix $r_0, r_1 \in \mathbb{C}$. The line $(r_0 \lambda_0 : r_1 \lambda_1 : \lambda_0 : \lambda_1)$ lies on $S_d$ iff the polynomial
\[(r_0^{d-1} r_1 + 1) \lambda_0^{d-1} \lambda_1 + (r_1^{d-1} + r_0) \lambda_0 \lambda_1^{d-1}\]
vanishes identically. So the parameters $r_0, r_1$ satisfy the conditions
\[r_0 = (-r_1^{d-1}) \quad \text{and} \quad r_1^{(d-1)^2+1} = (-1)^d.\]
Let $L(r_1)$ be the line on $S_d$ that corresponds to $r_0 = (-r_1^{d-1})$. We are to show that, for $r_1 \neq r_1'$, the lines $L(r_1)$, $L(r_1')$ are disjoint. Suppose that $L(r_1)$, $L(r_1')$ meet in the point $(y_0 : y_1 : y_2 : y_3)$. If $y_3 \neq 0$, then the parametrization of the lines in question yields $r_1 = r_1'$ and they coincide. Otherwise we have $y_2 \neq 0$; so we get $r_1^{d-1} = (r_1')^{d-1}$. By (1) we have $r_1 = r_1'$.

**Proof of (b).** We claim that $L_2$ is the only line on $S_d$ that does not meet $L_1$. Indeed, let $C_1$ (resp. $C_2$) be the curve residual to the line $L_1$ in the intersection of $S_d$ with the plane $x_0 = 0$ (resp. $x_2 = 0$). We have the parametrizations
\[
\psi_1 : \mathbb{C} \ni a_1 \rightarrow (0 : a_1 : 1 : -a_1^{d-1}) \in C_1 \setminus \{(0 : 0 : 1)\},
\psi_2 : \mathbb{C} \ni a_2 \rightarrow (1 : -a_2^{d-1} : 0 : a_2) \in C_2 \setminus \{(0 : 1 : 0)\}.
\]
The line through the points $\psi_1(a_1)$, $\psi_2(a_2)$ lies on $S_d$ iff all coefficients of the polynomial $s_d(\lambda_1, \lambda_0, a_1 - \lambda_1 a_2^{d-1}, \lambda_0, \lambda_2 - \lambda_0 a_2^{d-1})$ vanish. Write down the coefficients of the terms $\lambda_0^{d-1} \lambda_1, \ldots, \lambda_0^{d-4} \lambda_1^4$ to see that $a_1$, $a_2$ satisfy the conditions
\[
\begin{align*}
- (d - 1) a_1^{d-2} a_2^{d-1} + a_2 + (-1)^{d-1} a_1^{(d-1)^2} &= 0, \\
\frac{d - 2}{2} a_1^{d-3} a_2^{2(d-1)} &= (-1)^{d-1} a_1^{(d-1)(d-2)} a_2, \\
\frac{d - 3}{3} a_1^{d-4} a_2^{3(d-1)} &= (-1)^{d-1} a_1^{(d-1)(d-3)} a_2^2, \\
\frac{d - 4}{4} a_1^{d-5} a_2^{4(d-1)} &= (-1)^{d-1} a_1^{(d-1)(d-4)} a_2^3.
\end{align*}
\]
By the equation (2) we have $a_1 = 0$ iff $a_2 = 0$. This solution corresponds to the line $L_2$. Dividing (3) by (4) and (4) by (5) one gets that $a_1 = a_2 = 0$ is the unique solution. Thus $L_2$ is the only line on $S_d$ that does not meet $L_1$.

The symmetry $(x_0 : x_1 : x_2 : x_3) \rightarrow (x_3 : x_0 : x_1 : x_2)$ interchanges the lines $L_1$, $L_2$. So the other lines on $S_d$ meet both $L_1$ and $L_2$. One can check (see the proof of (a)) that there are precisely $d \cdot (d - 2) + 2$ such lines.

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**References**


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