PROJECTIVE SURFACES WITH MANY SKEW LINES

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Abstract. We give an example of a smooth surface \( S_d \subset \mathbb{P}_d(\mathbb{C}) \) of degree \( d \) that contains \( d \cdot (d - 2) + 2 \) pairwise disjoint lines. In particular, our example shows that the degree in Miyaoka’s bound is sharp.

Up to now the maximal number of pairwise disjoint lines on a smooth surface of degree \( d \geq 5 \) in \( \mathbb{P}_d(\mathbb{C}) \) was unknown. According to [4, p. 162] this number does not exceed
\[ 2 \cdot d \cdot (d - 2). \]
Quartic surfaces with 16 skew lines are studied in [1], but it is not clear to what extent Miyaoka’s bound is sharp for \( d \geq 5 \). Here we give an example of a smooth surface \( S_d \subset \mathbb{P}_d(\mathbb{C}) \) of degree \( d \) that contains
\[ d \cdot (d - 2) + 2 \]
pairwise disjoint lines. All lines on \( S_d \) form the following configuration:

\[ \ldots \]
\[ \ldots \]
\[ d \cdot (d - 2) + 2 \text{ pairwise disjoint lines} \]

Our example is inspired by the classical Klein quartic curve.
The equation of a quintic with 19 skew lines is given in [5, Example 2.3]. For \( d \geq 6 \) the surface \( S_d \) contains the largest number of skew lines found on a smooth surface of degree \( d \) in \( \mathbb{P}_d(\mathbb{C}) \). Let us mention that the Fermat surface \( F_d \), i.e. the surface with \( 3d^2 \) lines (the largest number known so far for \( d \neq 4, 6, 8, 12, 20 \)), contains no family of \( 3d \) pairwise disjoint lines. The latter results from the description of configuration of lines on \( F_d \) that can be found in [3].

Example. We define \( S_d \) to be the surface given by the polynomial
\[ s_d := x_0^{d-1} \cdot x_1 + x_1^{d-1} \cdot x_2 + x_2^{d-1} \cdot x_3 + x_3^{d-1} \cdot x_0, \]
where \( d \geq 6 \). One can easily check that \( S_d \) is smooth. Let \( L_1 \) (resp. \( L_2 \)) be the line \( x_0 = x_2 = 0 \) (resp. \( x_1 = x_3 = 0 \)). We claim that

(a) \( S_d \) contains \( d \cdot (d - 2) + 2 \) skew lines, each of which meets \( L_1 \) and \( L_2 \),

(b) the only lines on \( S_d \) are \( L_1, L_2 \) and the above-mentioned skew lines.

**Proof of** (a). Fix \( r_0, r_1 \in \mathbb{C} \). The line \( (r_0 \lambda_0 : r_1 \lambda_1 : \lambda_0 : \lambda_1) \) lies on \( S_d \) iff the polynomial

\[
(r_0^{d-1} r_1 + 1) \lambda_0^{d-1} \lambda_1 + (r_1^{d-1} + r_0) \lambda_0 \lambda_1^{d-1}
\]

vanishes identically. So the parameters \( r_0, r_1 \) satisfy the conditions

\[
(1) \quad r_0 = (-r_1^{d-1}) \quad \text{and} \quad r_1^{(d-1)^2+1} = (-1)^d.
\]

Let \( L(r_1) \) be the line on \( S_d \) that corresponds to \( r_0 = (-r_1^{d-1}) \). We are to show that, for \( r_1 \neq r_1' \), the lines \( L(r_1), L(r_1') \) are disjoint. Suppose that \( L(r_1), L(r_1') \) meet in the point \( (y_2 : y_1 : y_2 : y_3) \). If \( y_3 \neq 0 \), then the parametrization of the lines in question yields \( r_1 = r_1' \) and they coincide. Otherwise we have \( y_2 \neq 0 \); so we get \( r_1^{d-1} = (r_1')^{d-1} \). By (1) we have \( r_1 = r_1' \).

**Proof of** (b). We claim that \( L_2 \) is the only line on \( S_d \) that does not meet \( L_1 \). Indeed, let \( C_1 \) (resp. \( C_2 \)) be the curve residual to the line \( L_1 \) in the intersection of \( S_d \) with the plane \( x_0 = 0 \) (resp. \( x_2 = 0 \)). We have the parametrizations

\[
\begin{align*}
\psi_1 : C \ni a_1 & \mapsto (0 : a_1 : 1 : -a_1^{d-1}) \in C_1 \setminus \{(0 : 0 : 0 : 1)\}, \\
\psi_2 : C \ni a_2 & \mapsto (1 : -a_2^{d-1} : 0 : a_2) \in C_2 \setminus \{(0 : 1 : 0 : 0)\}.
\end{align*}
\]

The line through the points \( \psi_1(a_1), \psi_2(a_2) \) lies on \( S_d \) iff all coefficients of the polynomial \( s_d(\lambda_1, \lambda_0 a_1 - \lambda_1 a_2^{d-1}, \lambda_0, \lambda_1 a_2 - \lambda_0 a_2^{d-1}) \) vanish. Write down the coecients of the terms \( \lambda_0^{d-1} \lambda_1, \ldots, \lambda_0^{d-4} \lambda_2^4 \) to see that \( a_1, a_2 \) satisfy the conditions

\[
\begin{align*}
&\quad (2) \quad - (d - 1) a_1^{d-2} a_2^{d-1} + a_2 + (-1)^{d-1} a_1^{(d-1)^2} = 0, \\
&\quad (3) \quad \frac{d - 2}{2} a_1^{d-3} a_2^{2(d-1)} = (-1)^{d-1} a_1^{(d-1)(d-2)} a_2, \\
&\quad (4) \quad \frac{d - 3}{3} a_1^{d-4} a_2^{3(d-1)} = (-1)^{d-1} a_1^{(d-1)(d-3)} a_2^2, \\
&\quad (5) \quad \frac{d - 4}{4} a_1^{d-5} a_2^{4(d-1)} = (-1)^{d-1} a_1^{(d-1)(d-4)} a_2^3.
\end{align*}
\]

By the equation (2) we have \( a_1 = 0 \) iff \( a_2 = 0 \). This solution corresponds to the line \( L_2 \). Dividing (3) by (4) and (4) by (5) one gets that \( a_1 = a_2 = 0 \) is the unique solution. Thus \( L_2 \) is the only line on \( S_d \) that does not meet \( L_1 \).

The symmetry \( (x_0 : x_1 : x_2 : x_3) \to (x_3 : x_0 : x_1 : x_2) \) interchanges the lines \( L_1, L_2 \). So the other lines on \( S_d \) meet both \( L_1 \) and \( L_2 \). One can check (see the proof of \( (a) \)) that there are precisely \( d \cdot (d - 2) + 2 \) such lines.

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**References**


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